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Semidefinite relaxations for optimal control problems with oscillation and concentration effects

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Abstract— **Converging hierarchies of finite-dimensional convex semidefinite relaxations have been proposed for state-constrained optimal control problems featuring oscillation phenonema, using the notion of relaxed control or Young measure, interpreting the control as the conditional of an occupation measure for the systems trajectories. These semidefinite relaxations were later on extended to optimal control problems depending linearly on the control input and typically featuring concentration phenomena, interpreting the control as a measure of time with a discrete singular component modeling discontinuities or jumps of the state trajectories. In this contribution, we use measures introduced originally by DiPerna and Majda in the partial differential equations literature to model simultaneously, and in a unified framework, possible oscillation and concentration effects of the optimal control policy. We show that hierarchies of semidefinite relaxations can also be constructed to deal numerically with optimal control problems with nonconvex polynomial extended vector field (i.e. Lagrangian and dynamics) and nonconvex semi-algebraic state constraints.**

Index Terms— **optimal control; relaxed controls; impulsive controls; convex relaxations; semidefinite programming**

We consider nonconvex optimal control problems with polynomial or rational extended vector fields (i.e. Lagrangian and dynamics) and semialgebraic state constraints, under the assumption that the control law, as a function of time, belongs to the Lebesgue space L^p with a given finite exponent $p \in [1, \infty)$. On the one hand, nonconvexity of the Lagrangian in the control variable implies that the optimal control signal can feature oscillation effects, also sometimes called chattering. It means that there may exist a minimizing sequence of deterministic control functions that tends weakly to a probability measure, the so-called Young measure of the calculus of variations and partial differential equations (PDE) iterature, see e.g. [13], [14]. On the other hand, under our standing assumption that the control signal belongs to a Lebesgue space with finite exponent, the optimal control signal can feature concentration effects. It means that the control can be unbounded for vanishingly small amounts of time, or that the control tends weakly to a discrete singular measure of time, corresponding to impulsive controls and jumps, or discontinuities of the state trajectories, see e.g. [12], [15].

Oscillating but bounded controls and Young measures can be accomodated for with a converging hierarchy of finite-dimensional convex semidefinite programming (SDP) problems via the introduction of occupation measures. The key idea is to reformulate the nonconvex nonlinear optimal control problem as a convex linear programming problem in the cone of nonnegative Borel measures. The control is then understood as the probability measure which is the conditional (of time and/or space) of the occupation measure w.r.t. the control variables, see [10]. Concentration effects can also be addressed with SDP hierarchies by interpreting the control as a measure of time, namely the marginal w.r.t. time of the occupation measure. In particular, this allows for impulsive controls and state trajectory discontinuities. This is the approach developed in [1], as an extension to [10], under the restrictive assumption that the extended vector field depends linearly on the control variables.

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In this contribution, we generalize further this SDP hierarchy framework to accomodate in a unified fashion both oscillation and concentration effects in minimizing control sequences, allowing a nonlinear (but polynomial) dependence of the extended vector field on the control variables, while still being able to account for explicit semi-algebraic state constraints. The key technical ingredient in this context are DiPerna-Majda measures, originally introduced in the PDE literature to model phenomena in mathematical physics [2] and later on exploited in the context of calculus of variations and optimal control theory [14], [9].

The theory of DiPerna-Majda measures generalizes the notion of Young measures which is a well-known tool to cope with oscillations in optimal control and/or variational problems. The main difference consists in the fact that the DiPerna-Majda measures can capture, besides oscillations, also concentrations. Having a bounded sequence of controls in some L^p space with a given finite exponent $p \in [1, \infty)$, we immediately see that the sequence of moduli of the controls in the p-th power is bounded only in L^1. As this space is non-reflexive, it is not always possible to find a subsequence which would converge weakly to some element of L^1. Instead, we can describe this limit as a Radon measure which can be very general. In particular, it may have also a singular part which records point concentrations (think e.g. about an impulse). Clearly, the space of controls values cannot be only the m-dimensional Euclidean space but we must replace it by its suitable compactification. This allows us to "see" points at infinity. For example, an impulse is modeled as an instantaneous release of a finite amount of energy which means that the power is infinite. Distribution of values of the sequence of controls is recorded in a probability measure supported on the compactification. Although we work here with probability measures the whole concept is fully deterministic. DiPerna-Majda measures proved to be a very powerful tool to study not only optimal control problems but also weak lower semicontinuity and/or continuity of variational problems under various differential constraints, see e.g. [4], [6], [3], [8], [7].

We present the method on a variation of the smeared impulse example of [9]. Consider the following optimal control problem

$$\begin{aligned} J = \inf \quad & \int_0^1 \left(\frac{u^2}{1+u^4} + (y-t)^2 \right) dt \\ \text{s.t.} \quad & \dot{y} = u, \quad u \in L^1([0,1];\mathbb{R}). \end{aligned} \tag{1}$$

The cost favors solutions with an average control value of 1, but realized by alternating between null and very large instantaneous control values. The optimal control thus escapes L^1 even though it is uniformely bounded. To capture such a limiting behavior, DiPerna and Majda introduced [2] a generalization of Young measures, which allow to relax (1) as [9]:

$$\begin{aligned} J_R = \inf \quad & \int_0^1 \int_{\gamma\mathbb{R}} \frac{\frac{s^2}{1+s^4} + (y-t)^2}{1+|s|} \nu(ds|t)\, \sigma(dt) \\ \text{s.t.} \quad & y(dt) = \int_{\gamma\mathbb{R}} \frac{s}{1+|s|}\, \nu(ds|t)\, \sigma(dt), \end{aligned} \tag{2}$$

where dynamics are interpreted in a suitable weak sense. Here, σ is a Radon measure on the time interval and $\nu(ds|t)$ is a family of probability measures defined on the (two point-) compactified positive line $\gamma\mathbb{R}$. Compare this with traditional Young measures, where ν would be supported on a compact set and σ would simply be the Lebesgue measure dt. For this particular example, the (unique) optimal solution is [9]:

$$\sigma^*(dt) = 2\, dt, \;\; \nu^*(ds|t) = \frac{1}{2}\delta_0(ds) + \frac{1}{2}\delta_{+\infty}(ds), \tag{3}$$

where δ_z is the Dirac measure concentrated at z. This solution captures the expected limiting behavior, in the sense that there indeed exists a minimizing sequence $\{u_k\} \subset L^1([0,1])$ such that

$$\lim_{k\to\infty} v(t, u_k) = \int_0^1 \int_{\gamma\mathbb{R}} \frac{v(t,s)}{1+|s|}\, \nu(ds|t)\, \sigma(dt) \tag{4}$$

for all appropriate test functions $v(t,u)$.

We propose to recover this solution numerically along the lines of [1]: by defining an occupation measure μ that also encodes trajectory $y(t)$. As such, the problem is recast, for all appropriately defined test functions $v(t,y)$, as

$$\begin{aligned} J_R = \inf \quad & \int \frac{\frac{s^2}{1+s^4} + (r-t)^2}{1+|s|}\mu \\ \text{s.t.} \quad & v(1,y(1)) - v(0,y(0)) \\ & = \int \left(\frac{\partial v}{\partial t}\frac{1}{1+|s|} + \frac{\partial v}{\partial y}\frac{s}{1+|s|} \right)\mu \end{aligned} \tag{5}$$

where $\mu(dr\,ds\,dt)$ is a Radon measure supported on $\mathbb{R}\times[0,1]\times[0,1]$. If the solution is unique, ν and σ are obviously recovered as marginals of μ, while the trajectory is found by looking at the state marginal .

Linear program (5) is an instance of a generalized problem of moments, which can be solved by a hierarchy of well-known *moment relaxations* [11]. After a few algebraic manipulations, not shown for conciseness, (5) can be easily coded and solved through the GloptiPoly software [5]. Indeed, at the third relaxation, the moments returned by GloptiPoly can certifiably be shown to be those of optimal measures (3).

Note that contrary to many numerical approaches, the proposed framework can handle semialgebraic state constraints seamlessly, through constraints on the support of the occupation measure. In our hierarchy of relaxations, this translates into semidefinite constraints, which preserve the convexity of the relaxations.

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