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Stabilization of a rotating flexible structure with shear force feedback controls*

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***Abstract*— This note deals with the stabilization problem of a rotating disk with a flexible beam attached to its center. Assuming that a torque control is applied on the disk and a shear force control is exerted at the free end of the beam, we end up with a nondissipative closed loop system. We prove that the system can be nonuniformly exponentially stabilized provided that the angular velocity of the disk is less than the square root of the first eigenvalue of the related self-adjoint positive definite operator. This result is illustrated by numerical examples.**

I. Introduction

The system under consideration consists of a disk where an elastic beam is attached to its center (see Figure 1). The disk rotates freely around its axis with a non uniform angular velocity and the motion of the beam is confined to a plane perpendicular to the disk. Assuming that a torque control is exerted on the disk and a boundary force control is applied at the free end of the beam, the global system is governed by the following nonlinear system [1]

$$\begin{cases} \rho y_{tt} + EI y_{xxxx} = \rho \omega^2(t) y, \ (x,t) \in (0,\ell) \times (0,\infty) \\ y(0,t) = y_x(0,t) = y_{xx}(\ell,t) = 0, \ t > 0 \\ EI y_{xxx}(\ell,t) = \alpha \Theta(t), \ t > 0 \\ \dfrac{d}{dt}\left\{ \omega(t) \left(I_d + \rho \displaystyle\int_0^\ell y^2(x,t)dx \right) \right\} = T(t), \ t > 0 \end{cases} \tag{1}$$

where the positive constants ℓ, EI, ρ and I_d are respectively the length of the beam, the flexural rigidity, the mass per unit length of the beam, and the disk's moment of inertia. Furthermore, y is the beam's displacement in the rotating plane and ω is the angular velocity of the disk. Finally, α is a positive feedback gain and $\Theta(t)$ is the force control acting on the free end of the beam whereas $T(t)$ is the torque control to be applied on the disk.

The stability and stabilization problem of the system (1) has been the subject of many research papers (see [3], [19], [20], [13], [14], [15], [11], [8], [4], [5] and the references therein). However, all the systems involved in the articles cited above have a decreasing energy.

The main contribution of this note is to show that the rotating flexible system is exponentially stabilized by means

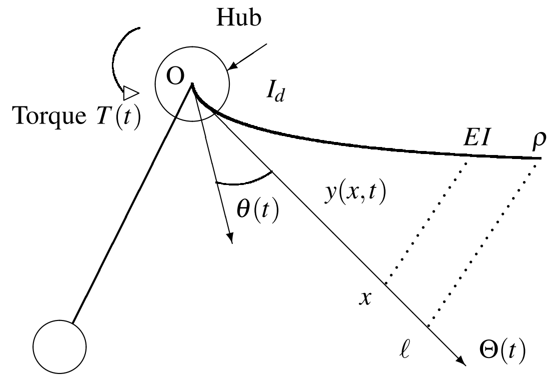

Fig. 1. The skeleton of the rotating flexible beam

of a control torque exerted on the disk, combined with only one shear force control applied at the free end of the beam, that is,

$$\begin{cases} \Theta(t) = y_{xt}(\ell,t), \\ T(t) = -\gamma(\omega(t) - \varpi), \text{ for each } \varpi \in \mathbb{R}, \end{cases} \tag{2}$$

where γ is a positive constant. Contrary to previous articles related to such a system, we end up with a nondissipative closed loop system. Despite this mathematical difficulty, it will be shown that for any desired angular velocity ϖ smaller than a critical one, the beam vibrations can be forced to decay exponentially to zero while the disk will rotates with the desired angular velocity ϖ. The key idea of the proof is use the Riesz basis approach to establish the exponential stability of a linear uncoupled system and then deduce the desired result for the nonlinearly coupled hybrid system (1)-(2). Note that for the linear system, the critical maximum angular velocity is shown to be less than the square root of the first eigenvalue of the related self-adjoint positive definite operator.

II. Preliminaries

For sake of simplicity and without loss of generality, assume that $EI = \rho = \ell = 1$ (in fact a simple change of variable will lead to unit physical parameters). Then, let

$$H_c^n = \left\{ f \in H^n(0,1); f(0) = f_x(0) = 0 \right\}, \text{ for } n = 2,3,\ldots \tag{1}$$

and consider the state space $\mathscr{X}$ defined by

$$\mathscr{X} = H_c^2 \times L^2(0,1) \times \mathbb{R} = \mathscr{H} \times \mathbb{R},$$

equipped with the following inner product:

$$\langle (y,z,\omega), (\tilde{y},\tilde{z},\tilde{\omega}) \rangle = \int_0^1 (y_{xx}\tilde{y}_{xx} + z\tilde{z})\, dx + \omega\tilde{\omega}. \tag{2}$$

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Next, let $z = y_t$, $\Phi = (y,z,\omega)$ and define on $\mathscr{X}$ the operators $\mathscr{A}$ and $\mathscr{P}$ by

$$\begin{aligned}\mathscr{D}(\mathscr{A}) = \Big\{&(y,z,\omega) \in H_c^4 \times H_c^2 \times \mathbb{R};\\ &y_{xx}(1) = 0,\, y_{xxx}(1) = \alpha z_x(1)\Big\} \qquad (3)\\ \mathscr{A}\Phi = \Big(&z, -y_{xxxx} + \varpi^2 y, 0\Big),\ \forall\, \Phi = (y,z,\omega) \in \mathscr{D}(\mathscr{A})\end{aligned}$$

and $\mathscr{P}$ is a nonlinear operator in $\mathscr{X}$ defined by, for $\Psi \in \mathscr{X}$

$$\mathscr{P}\Psi = \Big(0, (\omega^2 - \varpi^2)y, \frac{-\gamma(\omega - \varpi) - 2\,\omega < y, z >_{L^2(0,1)}}{I_d + \|y\|^2_{L^2(0,1)}}\Big). \tag{4}$$

Then, the closed loop system (1)-(2) can be formulated as a differential equation in $\mathscr{X}$ as follows

$$\Phi_t(t) = (\mathscr{A} + \mathscr{P})\,\Phi(t), \tag{5}$$

which, in turn, can written as

$$\Phi_t(t) = \left[\begin{pmatrix} A^{\varpi} & 0 \\ 0 & 0 \end{pmatrix} + \mathscr{P}\right]\Phi(t), \tag{6}$$

where A^{ϖ} is an unbounded linear operator defined by:

$$\begin{aligned}\mathscr{D}(A^{\varpi}) = \Big\{&\phi = (y,z) \in H_c^4 \times H_c^2;\\ &y_{xx}(1) = 0,\, y_{xxx}(1) = \alpha z_x(1)\Big\},\end{aligned} \tag{7}$$

and for $\phi \in \mathscr{D}(A^{\varpi})$,

$$A^{\varpi}\phi = \Big(z, -y_{xxxx} + \varpi^2 y\Big). \tag{8}$$

We shall consider, in the space $\mathscr{H} = H_c^2 \times L^2(0,1)$, the subsystem

$$\begin{cases} \phi_t(t) = A^{\varpi}\phi(t), \\ \phi(0) = \phi_0. \end{cases} \tag{9}$$

One can readily check that $\mathscr{H} = H_c^2 \times L^2(0,1)$, endowed with the inner product

$$\langle (y,z), (\tilde{y},\tilde{z})\rangle_{\mathscr{H}} = \int_0^1 \big(y_{xx}\tilde{y}_{xx} - \varpi^2 y\tilde{y} + z\tilde{z}\big)\,dx \tag{10}$$

is a Hilbert space provided that $|\varpi| < \sqrt{\mu_1}$, where μ_1 is the first eigenvalue of the self-adjoint positive operator $\mathbf{B}$ given by

$$\begin{cases} (\mathbf{B}\phi)(x) = \phi''''(x), \\ \mathscr{D}(\mathbf{B}) = \{\phi \in L^2(0,1) | B\phi \in L^2(0,1), \\ \qquad \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0\} \end{cases} \tag{11}$$

and $\sqrt{\mu_1} \simeq 3.516$ satisfies $1 + \cos(\sqrt[4]{\mu_1})\cosh(\sqrt[4]{\mu_1}) = 0$ (see Lemma 7). For $|\varpi| < \sqrt{\mu_1}$, we have

$$1 + \cos(\sqrt{|\varpi|})\cosh(\sqrt{|\varpi|}) > 0. \tag{12}$$

By Lemma 6, Theorem 1 and Remark 1 later, we show that $|\varpi^*| = \sqrt{\mu_1}$ is the critical maximum angular velocity so that for any $|\varpi| \geq \sqrt{\mu_1}$, the system will be unstable. So, in this paper, we always assume that $|\varpi| < \sqrt{\mu_1}$.

III. Well-posedness and stability of the uncoupled linear system (9)

In this section, a detailed analysis of the linear system (9) will be conducted.

Lemma 1: Assume that $|\varpi| < \sqrt{\mu_1}$. Let A^{ϖ} be defined by (7)-(8). Then $(A^{\varpi})^{-1}$ exists and is compact on $\mathscr{H}$. Hence, $\sigma(A^{\varpi})$, the spectrum of A^{ϖ}, consists of isolated eigenvalues of finite algebraic multiplicity only.

It is an easy task to check that the equation $A^{\varpi}(f,g) = \lambda(f,g)$, $(f,g) \in \mathscr{D}(A^{\varpi})$, is equivalent to write that $g = \lambda f$ and f satisfies the eigenvalue problem

$$\begin{cases} \lambda^2 f(x) + f''''(x) - \varpi^2 f(x) = 0, \quad 0 < x < 1, \\ f(0) = f'(0) = f''(1) = 0, \\ f'''(1) = \alpha\lambda f'(1). \end{cases} \tag{13}$$

Lemma 2: If $\alpha|\varpi| = 2$, then $\lambda = -|\varpi|$ is an eigenvalue of A^{ϖ} and $(x^3 - 3x^2, -|\varpi|(x^3 - 3x^2))$ is an associated eigenfunction of A^{ϖ}.

Let $\lambda \neq |\varpi|$. Then the fundamental solution of (13) is

$$f(x) = c_1 e^{a(\lambda)\sqrt{i}x} + c_2 e^{-a(\lambda)\sqrt{i}x} + c_3 e^{a(\lambda)i\sqrt{i}x} + c_4 e^{-a(\lambda)i\sqrt{i}x}, \tag{14}$$

where

$$a(\lambda) = \sqrt[4]{\lambda^2 - \varpi^2}. \tag{15}$$

Substituting (14) into the boundary conditions of (13), we have

$$\begin{cases} c_1 + c_2 + c_3 + c_4 = 0, \\ c_1 - c_2 + ic_3 - ic_4 = 0, \\ c_1 e^{a(\lambda)\sqrt{i}} + c_2 e^{-a(\lambda)\sqrt{i}} - c_3 e^{a(\lambda)i\sqrt{i}} - c_4 e^{-a(\lambda)i\sqrt{i}} = 0, \\ c_1\left[ia^2(\lambda) - \alpha\lambda\right] e^{a(\lambda)\sqrt{i}} - c_2\left[ia^2(\lambda) - \alpha\lambda\right] e^{-a(\lambda)\sqrt{i}} \\ \quad + c_3\left[a^2(\lambda) - i\alpha\lambda\right] e^{a(\lambda)i\sqrt{i}} \\ \quad - c_4\left[a^2(\lambda) - i\alpha\lambda\right] e^{-a(\lambda)i\sqrt{i}} = 0. \end{cases} \tag{16}$$

Clearly, λ is an eigenvalue of (13) if and only if λ is a zero of the determinant $G(\lambda)$ of the coefficient matrix of (16), where $G(\lambda)$ is given by

$$G(\lambda) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ e^{a(\lambda)\sqrt{i}} & e^{-a(\lambda)\sqrt{i}} & -e^{a(\lambda)i\sqrt{i}} & -e^{-a(\lambda)i\sqrt{i}} \\ G_1(\lambda) & G_2(\lambda) & G_3(\lambda) & G_4(\lambda). \end{vmatrix} \tag{17}$$

Here

$$\begin{cases} G_1(\lambda) = \left[ia^2(\lambda) - \alpha\lambda\right] e^{a(\lambda)\sqrt{i}}, \\ G_2(\lambda) = -\left[ia^2(\lambda) - \alpha\lambda\right] e^{-a(\lambda)\sqrt{i}}, \\ G_3(\lambda) = \left[a^2(\lambda) - i\alpha\lambda\right] e^{a(\lambda)i\sqrt{i}}, \\ G_4(\lambda) = -\left[a^2(\lambda) - i\alpha\lambda\right] e^{-a(\lambda)i\sqrt{i}}. \end{cases}$$

A straightforward computation leads us to the following result. We omit the details here.

Lemma 3: Let $G(\lambda)$ be given by (17). Then $G(\lambda) = -2G_*(\lambda)$, where

$$\begin{aligned} G_*(\lambda) = \ & 4a^2(\lambda) + \left[a^2(\lambda) - \alpha\lambda\right]\left[e^{\sqrt{2}a(\lambda)i} + e^{-\sqrt{2}a(\lambda)i}\right] \\ & + \left[a^2(\lambda) + \alpha\lambda\right]\left[e^{\sqrt{2}a(\lambda)} + e^{-\sqrt{2}a(\lambda)}\right]. \end{aligned} \tag{18}$$

Thus, for $\lambda \neq -|\varpi|$, $\lambda \in \sigma(A^{\varpi})$ if and only if $G_*(\lambda) = 0$.

Lemma 4: Let $\lambda \in \sigma(A^{\varpi})$. Then $\mathrm{Re}\lambda \neq 0$.

Lemma 5: Assume that $\alpha > 0$, and $|\varpi| < \sqrt{\mu_1}$. Let $\lambda = c + di \in \sigma(A^{\varpi})$, $c, d \in \mathbb{R}$. If $d = 0$, then $c < 0$.

Lemma 6: Assume that $\alpha > 0$, and $|\varpi| \geq \sqrt{\mu_1}$. Then there exists at least one non-negative real eigenvalue of A^{ϖ}.

Theorem 1: Assume that $\alpha > 0$, and $|\varpi| < \sqrt{\mu_1}$. If $\lambda \in \sigma(A^{\varpi})$, then $\mathrm{Re}(\lambda) < 0$.

Proof: Let $\lambda = a + bi$. When $b \neq 0$, from Lemma 9 in the Appendix, we have $\mathrm{Re}(\lambda) < 0$. When $b = 0$, it follows from Lemma 5 that $\mathrm{Re}(\lambda) < 0$. The desired result is then obtained. ■

Remark 1: Let $\alpha > 0$ and $\lambda \in \sigma(A^{\varpi})$. In the light of Lemma 6 and Theorem 1, we deduce that the assumption $|\varpi| < \sqrt{\mu_1}$ is indeed a sufficient and necessary condition for getting the important property $\mathrm{Re}(\lambda) < 0$.

Now we get the asymptotic eigenvalues and eigenfunctions of A^{ϖ} respectively.

Theorem 2: Assume that $\alpha > 0, \alpha \neq 1$, and $|\varpi| < \sqrt{\mu_1}$. Let A^{ϖ} be defined by (7)-(8). Then $\sigma(A^{\varpi}) = \{\lambda_n, \overline{\lambda_n}, n \in \mathbb{N}\}$ and as $n \to \infty$, λ_n has the following asymptotic expansion:

$$\lambda_n = \begin{cases} n\pi \ln \dfrac{\alpha-1}{\alpha+1} \\ \quad + \left[n^2\pi^2 - \frac{1}{4}\left(\ln \frac{\alpha-1}{\alpha+1}\right)^2\right] i + \mathscr{O}(n^{-3}), \text{ for } \alpha > 1, \\ (n - \dfrac{1}{2})\pi \ln \dfrac{1-\alpha}{\alpha+1} \\ \quad + \left[\left(n-\frac{1}{2}\right)^2 \pi^2 - \frac{1}{4}\left(\ln \frac{1-\alpha}{\alpha+1}\right)^2\right] i + \mathscr{O}(n^{-3}) \\ \quad \text{for } 0 < \alpha < 1. \end{cases} \tag{19}$$

Hence, we have

$$\mathrm{Re}(\lambda_n) \to -\infty, \quad \text{as } n \to \infty. \tag{20}$$

Theorem 3: Assume that $\alpha > 0, \alpha \neq 1$, and $|\varpi| < \sqrt{\mu_1}$. Let $\sigma(A^{\varpi}) = \{\lambda_n, \overline{\lambda_n}, n \in \mathbb{N}\}$ be the eigenvalues of A^{ϖ} and let λ_n be given by (19). Then the corresponding eigenfunctions $\{(f_n, \lambda_n f_n), (\overline{f_n}, \overline{\lambda}_n \overline{f}_n), n \in \mathbb{N}\}$ have the following asymptotic expansion: as $n \to \infty$,

$$\begin{cases} \lambda_n f_n(x) = (1+i)e^{\rho_n\sqrt{i}(1+x)} + (1-i)e^{\rho_n\sqrt{i}(1-x)} \\ \qquad -2e^{\rho_n\sqrt{i}}e^{\rho_n i\sqrt{i}x} + (1-i)e^{\rho_n i\sqrt{i}(1-x)} + \mathscr{O}(n^{-3}), \\ f_n''(x) = -(1-i)e^{\rho_n\sqrt{i}(1+x)} + (1+i)e^{\rho_n\sqrt{i}(1-x)} \\ \qquad +2ie^{\rho_n\sqrt{i}}e^{\rho_n i\sqrt{i}x} - (1+i)e^{\rho_n i\sqrt{i}(1-x)} + \mathscr{O}(n^{-3}), \end{cases} \tag{21}$$

where ρ_n is given by

$$\rho_n = \begin{cases} \dfrac{1}{\sqrt{2}(1+i)}\left[2n\pi i + \ln \dfrac{\alpha-1}{\alpha+1}\right] + \mathscr{O}(n^{-4}) \\ \quad \text{for } \alpha > 1, \\ \dfrac{1}{\sqrt{2}(1+i)}\left[(2n-1)\pi i + \ln \dfrac{1-\alpha}{\alpha+1}\right] + \mathscr{O}(n^{-4}) \\ \quad \text{for } \alpha < 1. \end{cases} \tag{22}$$

Furthermore, $(f_n, \lambda_n f_n)$ are approximately normalized in the sense that there exists positive constants c_1 and c_2 independent on n, such that

$$c_1 \leq \|f_n''\|_{L^2(0,1)}, \ \|\lambda_n f_n\|_{L^2(0,1)} \leq c_2. \tag{23}$$

Now we present the Riesz basis property for the linear system (9).

Theorem 4: Assume that $\alpha > 0, \alpha \neq 1$, and $|\varpi| < \sqrt{\mu_1}$. Let A^{ϖ} be defined by (7)-(8). Then there is a set of generalized eigenfunctions which form a Riesz basis in $\mathscr{H}$ and each eigenvalue of A^{ϖ} with sufficiently large moduli is algebraically simple.

Proof: In order to show the Riesz basis, we introduce another operator defined $\mathscr{H}$:

$$\begin{cases} A^0(y,z) = \Big(z, -y_{xxxx}\Big), \ \forall (y,z) \in \mathscr{D}(A^0), \\ \mathscr{D}(A^0) = \Big\{(y,z) \in H_c^4 \times H_c^2;\ y_{xx}(1) = 0, \\ \qquad\qquad y_{xxx}(1) = \alpha z_x(1)\Big\}. \end{cases} \tag{24}$$

It is found that A^0 is the system operator with compact resolvent for the linear system with $\varpi = 0$ and this operator has been intensively studied in [10]. Indeed, it has been shown in [10, Theorem 4.1] that the generalized eigenfunctions $\{(f_{n0}, \lambda_{n0} f_{n0}), (\overline{f_{n0}}, \overline{\lambda}_{n0} \overline{f}_{n0}), n \in \mathbb{N}\}$ of A^0, where λ_{n0} denotes the eigenvalues of A^0, form a Riesz basis in $\mathscr{H}$. and in [10, Proposition 3.1] that each eigenvalue λ_{n0} of A^0 with sufficiently large moduli is algebraically simple. Furthermore, similar arguments and computations for A^{ϖ} will lead us to claim that λ_{n0} and $(f_{n0}, \lambda_{n0} f_{n0})$ has the same asymptotic expressions (19) and (21) respectively. So, there is an $N > 0$ large enough such that

$$\sum_{n \geq N} \|(f_n, \lambda_n f_n) - (f_{n0}, \lambda_{n0} f_{n0})\|_{\mathscr{H}}^2 = \sum_{n \geq N} \mathscr{O}(n^{-6}) < \infty.$$

The same is true for their conjugates. By using the extended Bari's theorem [9, Theorem 6.3], we have that the generalized eigenfunctions $\{(f_n, \lambda_n f_n), (\overline{f}_n, \overline{\lambda}_n \overline{f}_n), n \in \mathbb{N}\}$ of A^{ϖ} form a Riesz basis in $\mathscr{H}$ and each eigenvalue of A^{ϖ} with sufficiently large moduli is algebraically simple. The proof is complete. ■

Now we present the exponential stability and Gevrey regularity for linear system (9). First, we establish the exponential stability for (9).

Theorem 5: Assume that $\alpha > 0, \alpha \neq 1$, and $|\varpi| < \sqrt{\mu_1}$. Let A^{ϖ} be defined by (7)-(8). Then A^{ϖ} generates a C_0-semigroup $e^{A^{\varpi}t}$ in $\mathscr{H}$ and the semigroup $e^{A^{\varpi}t}$ satisfies spectrum-determined growth condition: $S(A^{\varpi}) = \omega(A^{\varpi})$, where $S(A^{\varpi}) = \sup_{\lambda \in \sigma(A^{\varpi})} \mathrm{Re}(\lambda)$ is the spectral bound and $\omega(A^{\varpi})$ is the growth order of the semigroup $e^{A^{\varpi}t}$. Furthermore, $e^{A^{\varpi}t}$ is exponentially stable in $\mathscr{H}$.

Proof: The C_0-semigroup and spectrum-determined growth condition follow directly from the Riesz basis property of A^{ϖ} claimed by Theorem 4. So we only need to verify the exponential stability, which can be proved by the distributions of eigenvalues of A^{ϖ}. In fact, for $\lambda \in \sigma(A^{\varpi})$, from Theorem 1, we have $\mathrm{Re}(\lambda) < 0$ and from (20), we have $\mathrm{Re}(\lambda) \to -\infty$ as $|\lambda| \to \infty$. So the exponential stability is achieved because the spectrum-determined growth condition holds. The proof is complete. ■

To end this section, we show that C_0-semigroup $e^{A^{\varpi}t}$ generated by A^{ϖ} is of a Gevrey class δ with any $\delta > 2$, which is the semigroup class between the differentiable semigroup class and the analytical ones. The Gevrey regularity gives some estimates for the semigroup. Recently, it has been shown in [18] that the interconnected Schrödinger and heat equations have the Gevrey regularity with $\delta > 2$.

Definition 1: ([2], [17]) A C_0-semigroup $S(t)$ in a Hilbert space H is of a Gevrey class $\delta > 1$ for $t > t_0$ if $S(t)$ is infinitely differentiable for $t > t_0$ and for every compact subset $K \subset (t_0, \infty)$ and each $\theta > 0$, there is a constant $C = C(K, \theta)$ such that

$$\|S^{(n)}(t)\| \leq C\theta^n (n!)^{\delta},\ \forall t \in K,\ n = 0, 1, 2, \ldots.$$

Clearly, if $\delta = 1$, then $S(t)$ is analytic.

In order to get the Gevrey regularity of the system (9), we need the following theorem established in [16, theorem 13].

Theorem 6: Let A be an infinitesimal generator of a C_0-semigroup e^{At} and let A be a Riesz-spectral operator in a Hilbert space, that is, the generalized eigenfunctions of A form a Riesz basis in the Hilbert space. Then the following assertions are equivalent:

1) e^{At} is of Gevrey class $\delta \geq 1$ for $t > 0$.
2) There is $b > 0$, $a \in \mathbb{R}$ such that

$$\sigma(A) \subset \left\{ \lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq a - b|\mathrm{Im}\lambda|^{1/\delta} \right\}. \tag{25}$$

Now we are in a position to establish the Gevrey regularity for the system (9).

Theorem 7: Assume that $\alpha > 0, \alpha \neq 1$, and $|\varpi| < \sqrt{\mu_1}$. Let A^{ϖ} be defined by (7)-(8). Then the semigroup $e^{A^{\varpi}t}$, generated by A^{ϖ}, is of a Gevrey class $\delta > 2$ with $t_0 = 0$.

Proof: From (19) and (20), it is found that (25) is satisfied if we take $\delta = 2$, $b = |\ln r|$ and $a > 0$ large enough. Therefore, the semigroup $e^{A^{\varpi}t}$, generated by A^{ϖ}, is of a Gevrey class $\delta > 2$ with $t_0 = 0$. ■

Remark 2: Note that when $\varpi = 0$, it is shown in [10] that the corresponding system operator A^0 generates a differentiable C_0-semigroup and the regularity for $e^{A^0 t}$ is extended to be a Gevrey class $\delta > 2$ with $t_0 = 0$ in [2].

IV. THE CLOSED LOOP SYSTEM (1)-(2)

In this section, using the same arguments as in [20] (see also [5]), one can show the main result

Theorem 8: Assume that each desired angular velocity ϖ satisfies $|\varpi| < \sqrt{\mu_1}$ and the feedback gain $\alpha \neq 1$. Then, for each initial data $\Phi_0 \in \mathscr{D}(\mathscr{A})$, the corresponding solution $\Phi(t)$ of the closed loop system (6) exponentially tends to the equilibrium point $(0_{\mathscr{H}}, \varpi)$ in $\mathscr{X}$ as $t \to \infty$.

V. NUMERICAL SIMULATIONS

The objective of this section is to illustrate the main result stated and proved in the previous sections thorough some numerical examples. For convenience, assume that $\omega(t) = \varpi > 0$ in the closed loop system (5) whose energy-norm is

$$E_0(t) = -\frac{1}{2}\varpi^2 \int_0^1 y^2 dx + \frac{1}{2}\int_0^l (y_t^2 + y_{xx}^2) dx.$$

Throughout this section, we will assume the initial displacement $y_0(x) = 0$ and the initial velocity $y_1(x) = x$. By using MATLAB we shall consider the situation depending on the values of the feedback gain $\alpha = 0.7$.

Examining the Figures 2-4, it can be seen that the system energy decays in a short time as long as $\varpi < \sqrt{\mu_1} \simeq 3.5160$. This agrees with Theorem 8. Furthermore, the spectrum of the system lies in the left-half plane (see Figures 5-7).

In turn, if $\varpi = 4 > \sqrt{\mu_1}$, we notice that the spectrum gets at least one eigenvalue with nonnegative real part (see Figures 8 and 9), which ties in with the statement of Lemma 6.

Appendix

In this appendix, we present and show some lemmas which are needed for the proof of Theorem 1. The idea in this Appendix is coming from [12]. Integrating the first equation of (13) over $(1, x)$ twice yields

$$f''(x) - \alpha\lambda f'(1)(x-1) + \left(\lambda^2 - \varpi^2\right)\int_1^x (x-\tau) f(\tau) d\tau = 0.$$

Replacing x by $1 - x$ yields

$$\begin{aligned} f''(1-x) &+ \alpha\lambda f'(1)x \\ &+ \left(\lambda^2 - \varpi^2\right)\int_1^{1-x} (1-x-\tau) f(\tau) d\tau = 0. \end{aligned}$$

Let

$$\phi(x) := \int_1^{1-x} (1-x-\tau) f(\tau) d\tau.$$

Then $\phi''(x) = f(1-x)$ and so $\phi \neq 0$. A direct computation shows that ϕ satisfies

$$\begin{cases} \lambda^2\phi(x) - \alpha x\lambda\phi'''(0) + \phi''''(x) - \varpi^2\phi(x) = 0, \\ \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0 \end{cases} \tag{26}$$

and the eigenvalue problem (13) is equivalent to (26). Hence, $\lambda \in \sigma(A^{\varpi})$ if and only if there exist $\phi \neq 0 \in H^2(0,1)$ and $\lambda \in \mathbb{C}$ satisfying (26). Moreover, (26) can be written as

$$(\lambda^2 - \varpi^2)\phi(x) - \alpha x\lambda\phi'''(0) + \mathbf{B}\phi(x) = 0 \tag{27}$$

where $\mathbf{B}$ is a linear operator in $L^2(0,1)$ defined by (11). The main properties of $\mathbf{B}$ are stated in the following lemma [12]

Lemma 7: Let $\mathbf{B}$ be given by (11) and let $\{(\mu_n, \varphi_n(x))\}_{n=1}^{\infty}$ be the eigenpairs of $\mathbf{B}$. Then $\mathbf{B}$ is a self-adjoint and positive definite operator. Moreover, we have the following properties of the pairs $(\mu_n, \varphi_n(x))$ for $n \geq 1$:

1) $\mu_n = \beta_n^4$, with β_n satisfying $1 + \cos(\beta_n)\cosh(\beta_n) = 0$ and $\beta_n = \mathscr{O}(n) > 0$;

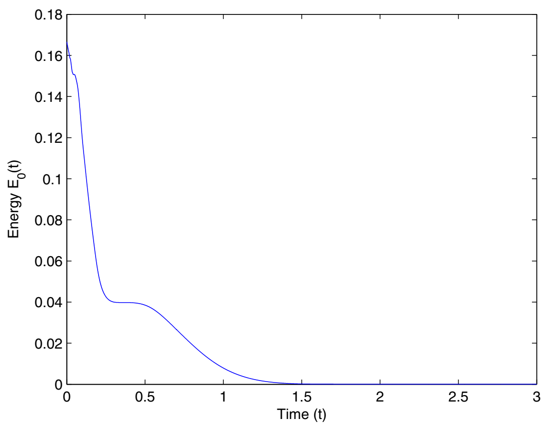

Fig. 2. Energy with $\varpi = 2$ and $\alpha = 0.7$

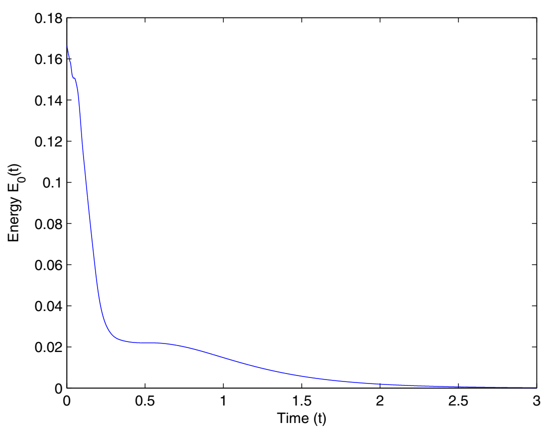

Fig. 3. Energy with $\varpi = 3$ and $\alpha = 0.7$

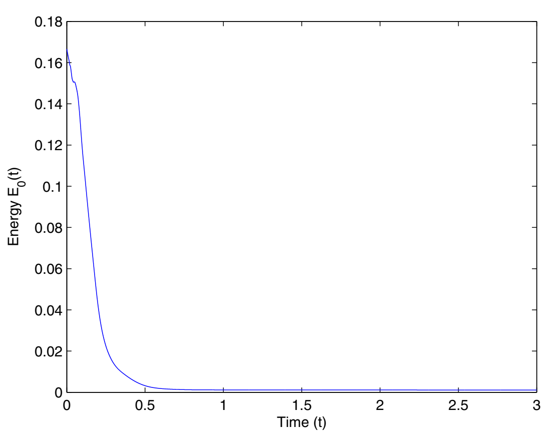

Fig. 4. Energy with $\varpi = 3.5$ and $\alpha = 0.7$

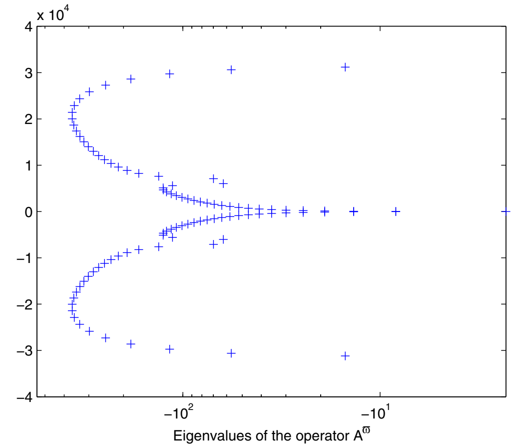

Fig. 5. Spectrum with $\varpi = 2$ and $\alpha = 0.7$

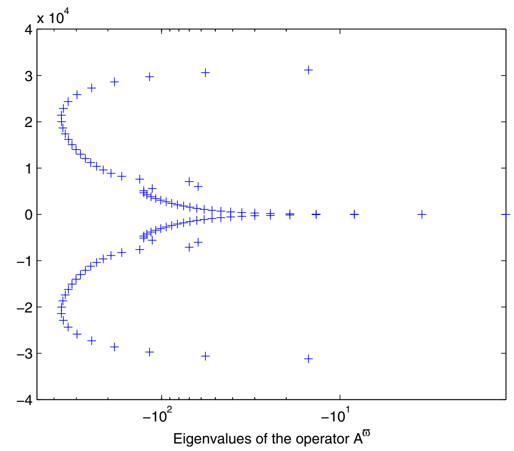

Fig. 6. Spectrum with $\varpi = 3$ and $\alpha = 0.7$

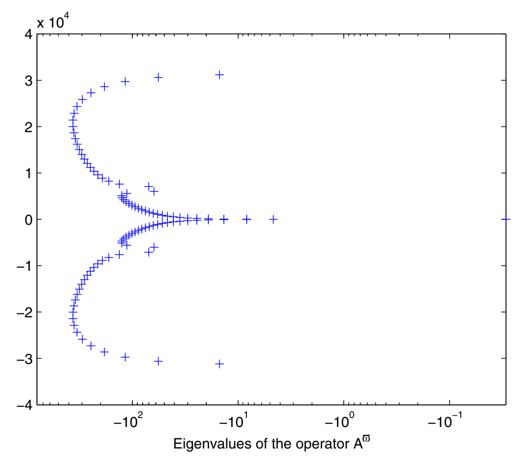

Fig. 7. Spectrum with $\varpi = 3.5$ and $\alpha = 0.7$

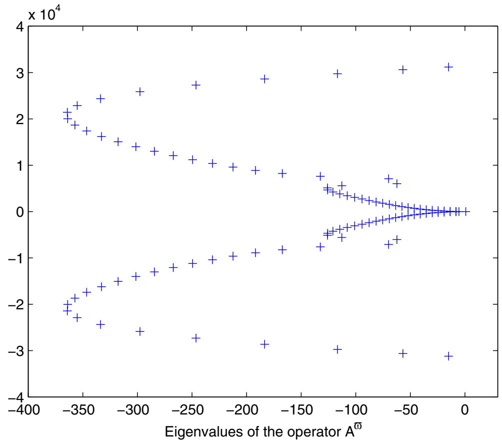

Fig. 8. Spectrum with $\varpi = 4$ and $\alpha = 0.7$

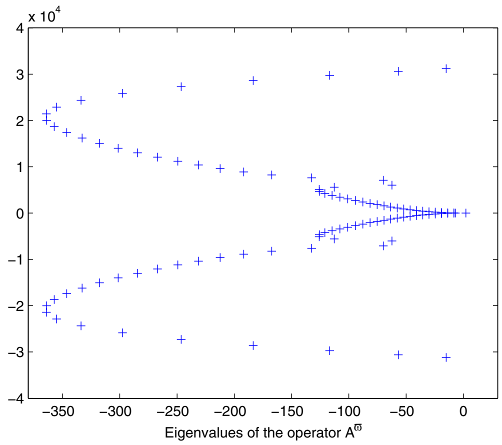

Fig. 9. Spectrum with $\varpi = 5$ and $\alpha = 0.7$

2) $\{\varphi_n(x)\}_{n=1}^{\infty}$ forms an orthogonal basis on $L^2(0,1)$ and $\varphi_n(x)$ has the following asymptotic expression:

$$\begin{aligned}\varphi_n(x) = & -\frac{1+\gamma_n}{2}\exp(\beta_n x) - \frac{1-\gamma_n}{2}\exp(-\beta_n x) \\ & +\gamma_n \sin(\beta_n x) + \cos(\beta_n x)\end{aligned}$$

where

$$\gamma_n = -\frac{\exp(\beta_n) - \sin(\beta_n) + \cos(\beta_n)}{\exp(\beta_n) + \sin(\beta_n) + \cos(\beta_n)} \to -1, \text{ as } n \to \infty$$

and $\gamma_n < 0$ for every n;

3) Each $x \in L^2(0,1)$ can be expressed as follows:

$$x = \sum_{n=1}^{\infty} b_n \varphi_n(x) \tag{28}$$

where

$$\begin{cases} b_n = -\dfrac{2}{\beta_n^2}\|\varphi_n\|^{-2}, & \beta_n^2 = -\dfrac{1}{2}\varphi_n''(0), \\ \beta_n^3 = -\dfrac{1}{2\gamma_n}\varphi_n'''(0), & \|\varphi_n\| = \mathscr{O}(1). \end{cases} \tag{29}$$

Our next result is

Lemma 8: Let $\lambda \in \sigma(A^{\varpi})$ with $\mathrm{Im}(\lambda) \neq 0$ and let $\alpha > 0$. Then, λ satisfies the following equation involving the entire function

$$F(\lambda) = 1 + \frac{1}{2}\alpha\lambda + 4\alpha \sum_{n=1}^{\infty} \frac{\lambda(\lambda^2 - \varpi^2)}{\lambda^2 - \varpi^2 + \mu_n}\gamma_n \beta_n^{-3}\|\phi_n\|^{-2} = 0. \tag{30}$$

Lemma 9: Let λ with $\mathrm{Im}(\lambda) \neq 0$ be a zero of $F(\lambda)$, and let $\alpha > 0$. Then $\mathrm{Re}(\lambda) < 0$.

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