21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Some Synthesis Aspects of Controlling Interconnected Linear Systems

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Abstract— **This paper addresses some synthesis problems that arise in the control of large scale networks of linear systems. For homogeneous networks of identical SISO systems we characterize the rational transfer functions of the networks in terms of associated Galois groups. We then study the problem of synthesizing broadcast open loop controls for networks from knowledge of local controls for the node systems. The difficulty in doing so is both a function of the size of the network and the complexity of the interconnection patterns. We shall focus on two points: The characterization of controllability and the computation of open loop controls that steer the system from rest to an arbitrary prescribed state. Explicit formulas for the open loop controls are derived for parallel connection of N linear state space systems.**

Keywords. System Identification, Network Control Systems, Linear Systems, Controllability, Galois Group.

AMS subject classification: 93B05, 93B25, 93B28.

I. INTRODUCTION

The task of controlling large-scale networks of interconnected dynamical systems poses fascinating challenges, both to the mathematical foundations as well as to systems engineering implementations. We consider interconnected systems that are described by a finite number of finite-dimensional linear systems Σ_i, the so-called node systems, whose inputs and outputs are interconnected through constant coupling parameters. While inputs to the network are thought of being broadcasted to the node systems, the output of the network is defined as a weighted sum of the outputs of the node systems. Thus the network defines an input/output system with a rational transfer function. In this paper we study synthesis problems for interconnected systems by focussing on the following two questions:

- How can one characterize the class of rational transfer functions of a network of N node systems? Given such a transfer function, how many realizations as network transfer functions exist?
- How can one synthesize N local controls into a global input function that steers the network to a desired global state?

Of course, the first question seems closely related to the classical synthesis problem for electrical circuits, as studied by O. Brune in 1931 as well as R. Bott and R.J. Duffin in 1949, and with more recent contributions by [9] and [15]. However, for interconnected systems the situation is different, insofar that in circuit analysis the edges of a graph are defined by the dynamical elements while the nodes correspond to the interconnecting junctions. In contrast, in the synthesis problem for interconnected systems the linear dynamical node systems represent the nodes of a graph while the edges of the graph define the interconnection parameters. Therefore the transfer function of an interconnection of passive systems does not need to be passive. Nevertheless, investigating such realization questions for interconnected systems still makes sense. We will show that the question posed is equivalent to a problem in Galois theory and thus can be tackled using algebraic-geometric methods. Proceeding in a different direction, we consider open loop control problems for networks of systems. This aspect of our work is very much related to the study of flatness based control, now however for a network of linear systems rather than for single, unstructured systems. While earlier work by e.g. [11] has emphasized the problem of computing flat outputs for higher order systems, we are not aware of any extensions of such work to interconnected systems. A natural question to start with is that of characterizing controllability of networks. This is done in Section 4 where we derive a Hautus-type condition for characterizing controllability of networks. We then consider the simplest possible interconnection structure, namely that of parallel connection. We show, by brute force computations using doubly coprime factorizations of polynomial matrices, how to compute from any given local controls for the node systems a global input function to the network that steers the zero state to a prescribed state. This process is closely related to the Chinese remainder theorem. Due to space limitations, we focus on the study of parallel connections. Similar results on series interconnections, as well as full proofs of the presented results, can be found in [5] on which the present paper is based.

We now describe a state space formulation of the situation we are interested in and introduce our subsequent notation. Consider N discrete-time linear systems to which we refer as **node systems** Σ_i, which are given by:

$$x_i(t+1) = \alpha_i x_i(t) + \beta_i v_i(t) \quad (1)$$

$$w_i(t) = \gamma_i x_i(t), \; i = 1, \ldots, N. \quad (2)$$

Here $\alpha_i \in \mathbb{F}^{n_i \times n_i}$, $\beta_i \in \mathbb{F}^{n_i \times m_i}$ and $\gamma_i \in \mathbb{F}^{p_i \times n_i}$ are the associated system matrices and $\mathbb{F}$ denotes an arbitrary field. We assume that each system is controllable and observable. To interconnect the node systems we use static coupling laws

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ISBN: 978-90-367-6321-9

as:

$$v_i(t) = \sum_{j=1}^{N} A_{ij} w_j(t) + B_i u(t) \in \mathbb{F}^{m_i},$$

with $A_{ij} \in \mathbb{F}^{m_i \times p_j}$ and $B_i \in \mathbb{F}^{m_i \times m}$, although more complex dynamic interconnections laws are possible. The interconnected output is given by:

$$y(t) = \sum_{i=1}^{N} C_i w_i(t) + Du(t) \quad \text{with} \quad C_i \in \mathbb{F}^{p \times p_i},\ i = 1, \ldots, N.$$

To express the closed loop system in compact matrix form, define $\overline{n} := n_1 + \cdots + n_N$, $\overline{m} := m_1 + \cdots + m_N$, $\overline{p} := p_1 + \cdots + p_N$. Moreover, we define $A := (A_{ij})_{ij} \in \mathbb{F}^{\overline{m} \times \overline{p}}$, $B := \operatorname{col}(B_1, \ldots, B_N) \in \mathbb{F}^{\overline{n} \times m}$, $C := (C_1, \ldots, C_N) \in \mathbb{F}^{p \times \overline{n}}$, $D \in \mathbb{F}^{p \times m}$ and $\alpha := \operatorname{diag}(\alpha_1, \ldots, \alpha_N) \in \mathbb{F}^{\overline{n} \times \overline{n}}$, $\beta := \operatorname{diag}(\beta_1, \ldots, \beta_N) \in \mathbb{F}^{\overline{n} \times \overline{m}}$, $\gamma := \operatorname{diag}(\gamma_1, \ldots, \gamma_N) \in \mathbb{F}^{\overline{p} \times \overline{n}}$, and $x := \operatorname{col}(x_1, \ldots, x_N) \in \mathbb{F}^{\overline{n}}$. Thus the global state space representation of the node systems Σ_i is given by:

$$\begin{aligned} x(t+1) &= \alpha x(t) + \beta v(t) \\ w(t) &= \gamma x(t) \end{aligned}$$

and the **interconnection** is given as

$$\begin{aligned} v(t) &= Aw(t) + Bu(t) \\ y(t) &= Cw(t) + Du(t) \end{aligned}$$

Here $u(t)$ is the external input and $y(t)$ the external output of the network. Thus the network dynamics has the state space form:

$$\begin{aligned} x(t+1) &= \mathcal{A}x(t) + \mathcal{B}u(t) \\ y(t) &= \mathcal{C}x(t) + Du(t), \end{aligned}$$

with $\mathcal{A} \in \mathbb{F}^{\overline{n} \times \overline{n}}$, $\mathcal{B} \in \mathbb{F}^{\overline{n} \times m}$, $\mathcal{C} \in \mathbb{F}^{p \times \overline{n}}$ and

$$\mathcal{A} := \alpha + \beta A \gamma, \quad \mathcal{B} := \beta B, \quad \mathcal{C} := C\gamma. \tag{3}$$

It is convenient to describe the interconnected systems in terms of the transfer functions of the node systems. The i-th node transfer function is given in state space form as well as using coprime polynomial matrix factorizations as

$$\begin{aligned} G_i(z) &= \gamma_i(zI - \alpha_i)^{-1}\beta_i \\ &= Q_i(z)^{-1}P_i(z) = \overline{P}_i(z)\overline{Q}_i(z)^{-1} \\ &= V_i(z)T_i(z)^{-1}U_i(z) + W_i(z). \end{aligned}$$

Here we assume that all the representations are minimal in the sense that $\delta(G_i) = \deg\det Q_i = \deg\det \overline{Q}_i = \deg\det T_i$. This is equivalent to left coprimeness of $Q_i(z), P_i(z)$, the right coprimeness of $\overline{P}_i(z), \overline{Q}_i(z)$, as well as to left coprimeness of $T_i(z), U_i(z)$ and right coprimeness of $T_i(z), V_i(z)$. Let $V(z) = \operatorname{diag}(V_1(z), \ldots, V_N(z))$, $T(z) = \operatorname{diag}(T_1(z), \ldots, T_N(z))$, $U(z) = \operatorname{diag}(U_1(z), \ldots, U_N(z))$, $W(z) = \operatorname{diag}(W_1(z), \ldots, W_N(z))$, and similarly for $P(z)$, $Q(z)$, etc. We define the **node transfer function** as

$$G(z) := \operatorname{diag}(G_1(z), \ldots, G_N(z)) = \gamma(zI - \alpha)^{-1}\beta.$$

The global **network transfer function** then is defined as $\Phi(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} + D$ or, explicitly, $\Phi(z) = C\gamma(zI - \alpha - \beta A\gamma)^{-1}\beta B + D$. A straightforward computation shows the following explicit expressions for the network transfer function:

$$\begin{aligned} \Phi(z) &= \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} + D \\ &= C(Q(z) - P(z)A)^{-1}P(z)B + D \\ &= C\overline{P}(z)(\overline{Q}(z) - A\overline{P}(z))^{-1}B + D \\ &= CV(z)(T(z) - U(z)AV(z))^{-1}U(z)B + D. \end{aligned}$$

A network is called **homogeneous**, if the node systems $G_1(z) = \cdots = G_N(z) = g(z)$ are identical scalar rational functions which are strictly proper. Thus the network transfer function of a homogeneous network is proper rational.

II. NETWORK TRANSFER FUNCTION CHARACTERIZATIONS

In this section we review some classical, as well as more recent, material on Galois theory of rational functions which sheds light on the question which transfer functions arise by interconnecting a finite number of identical SISO node systems. Thus let $g(z) \in \mathbb{F}(z)$ denote an arbitrary strictly proper transfer function of McMillan degree n and let

$$\phi(z) = C(zI - A)^{-1}B + D \tag{4}$$

denote the transfer function that is associated with the network interconnection parameters. Then for $h(z) = 1/g(z)$ the network transfer function is

$$\Phi(z) = C(h(z)I - A)^{-1}B + D. \tag{5}$$

Thus $\Phi = \phi \circ h$ is the composition of the two rational functions $\phi(z)$ and $h(z)$; this observation is due to [7]. Let $B(\Phi), B(\phi), B(g)$ denote the Bezoutian matrices, defined by coprime factorizations of Φ, ϕ, g, respectively. This leads to

Theorem 2.1 ([8]): The transfer function $\Phi(z) = \phi \circ h$ of a homogeneous network is a rational function of McMillan degree

$$\delta(\Phi) = \delta(\phi)\delta(g). \tag{6}$$

If $\phi(z) = \phi(z)^\top$ is a symmetric $m \times m$-transfer function, then $B(\Phi)$ is congruent to the Kronecker product $B(\phi) \otimes B(g)$. In particular, the Cauchy-Maslov index of Φ satisfies

$$CI(\Phi) = CI(\phi) \cdot CI(g). \tag{7}$$

The preceding result leads to a very simple characterization of controllability and observability for homogeneous networks due to [7]. See also our subsequent Theorem 4.1 for a more general result.

In a series of publications [13], [14] J. F. Ritt posed and partially solved the following problems:

- Which scalar rational functions are compositions of rational functions?
- In how many ways a given rational function can be expressed as a composition of rational functions.

Ritt solved this problem for complex polynomials and derived an abstract Galois theoretic characterization in the rational function case. In the polynomial case, Ritt succeeded to prove a Jordan-Hölder type decomposition result, i.e. that any two decompositions of a polynomial

$$\phi_1 \circ \cdots \circ \phi_r = \psi_1 \circ \cdots \circ \psi_s \tag{8}$$

by indecomposable polynomials ϕ_i, ψ_j contain the same number of polynomials $r = s$ and the degrees of the polynomials in one decompositions are the same as in the other (up to permutations). Moreover, Ritt showed how to pass from one decomposition to another one by a simple process involving Tschebychev polynomials. The situation becomes much more complicated for rational functions and is yet not fully understood. Every complex rational function $\Phi(z)$ defines a unique branched covering transformation $\Phi : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ on the Riemann sphere. Thus the monodromy group of the covering exists and coincides with Galois group of the finite field extension $\mathbb{C}(\Phi(z)) \subset \mathbb{C}(z)$. We refer to [1] for some early investigations on this Galois group and associated Lie groups of feedback systems. The following result is proven in [13].

Proposition 2.1 (Ritt(1922)): A complex rational function $\Phi(z) \in \mathbb{C}(z)$ is the composition of two rational functions if and only if the Galois group of $\Phi(z)$ is primitive. In particular, a strictly proper real rational is the transfer function $\Phi(z)$ of a homogeneous network with identical strictly proper node transfer function if and only if the Galois group of Φ is primitive.

Of course, a full characterization which primitive groups occur as Galois groups of rational functions is a difficult question and refers to the so-called inverse problem of Galois theory. Sufficient conditions though are known. For instance, it has been shown in [10] that symmetric groups S_n, cyclic groups C_n, alternating groups A_n and dihedral groups D_n all occur as Galois groups of rational functions. Moreover, it is conjectured that all finite simple groups occur as Galois groups of rational functions, while non-cyclic solvable groups typically do not arise as such. We mention another interesting geometric characterization of network transfer functions.

Theorem 2.2 ([6],[12]): A complex rational function $\Phi(z)$ is the transfer function of a homogeneous network of identical SISO transfer functions $g(z) = p(z)/q(z)$ if and only if

$$\Phi(z_1(k)) = \Phi(z_2(k))$$

holds for any two branches $z_i(k)$ of the root locus curve $q(z) - kp(z) = 0$. Assume that all critical values of a rational function $\Phi(z)$ are simple. Then $\Phi(z)$ is a network transfer function if and only if the complex polynomial

$$f(x, y) = \frac{p(x)q(y) - p(y)q(x)}{x - y}$$

is reducible.

Having stated conditions that characterize the transfer functions for a homogeneous network we next give conditions for uniqueness of representation $\Phi = \phi \circ h$. Obviously, any relation $\phi_1 \circ h = \phi_2 \circ h$ among rational functions implies $\phi_1 = \phi_2$. However, this is not true for relations of the form $\phi \circ h_1 = \phi \circ h_2$, as the example $g_1(z) = 1/z, g_2(z) = -1/z$ and $\phi(z) = 1/z^2$ shows.

Theorem 2.3 ([12]): Assume that $\phi(z) \in \mathbb{C}(z)$ has McMillan degree $n \geq 4$ and assume that the breakaway values of ϕ are all simple. Then for any rational functions $h(z), \tilde{h}(z) \in \mathbb{C}(z)$ we have

$$\phi \circ h = \phi \circ \tilde{h} \implies h = \tilde{h}.$$

This next result contains an obvious implication on uniqueness of network representations in the presence of symmetries.

Proposition 2.2: Assume that $\phi, h, \tilde{h}$ are non constant real rational functions which satisfy $\phi \circ h = \phi \circ \tilde{h}$. If $\phi(h(z)) = \phi(h(-z))$ holds for all z then one of the following two conditions is satisfied

(i) $\tilde{h}(z) = h(-z)$.
(ii) $\phi(-z) = \phi(z)$ and $\tilde{h}(-z) = -h(z)$.

Proof. Consider the real rational function $r(z) = \tilde{h}(z)/h(-z)$. Then $\tilde{h}(z) = r(z)h(-z)$ and $\phi(\tilde{h}(z)) = \phi(h(-z))$. Thus for any fixed complex number ζ we obtain $\phi(r(\zeta)h(-\zeta)) = \phi(h(-\zeta))$. Thus for all integers n we get

$$\phi(r(\zeta)^n h(-\zeta)) = \phi(h(-\zeta)). \tag{9}$$

Since a rational function ϕ has only a finite number of preimages we conclude that $r(\zeta)^n = 1$ holds for n suitable. Choose any real number ζ. Then $r(\zeta)$ is real. Thus either $r(\zeta) = 1$ or $r(\zeta) = -1$. This implies the identity of rational functions $r(z) = 1$ or $r(z) = -1$. In the first case thus $\tilde{h}(z) = h(-z)$ while in the second case we obtain $\tilde{h}(-z) = -h(z)$. This completes the proof. ∎

III. CONTROLLABILITY AND OPEN LOOP CONTROL

Throughout this paper we focus on discrete-time systems although, with minor modifications, our results hold true also for continuous time systems. We work over an arbitrary field $\mathbb{F}$. For background on the subsequent material we refer to [5]. Let $\mathbb{F}((z^{-1}))^m$ denote the vector space of truncated Laurent series in z^{-1}, i.e., the elements of $\mathbb{F}((z^{-1}))^m$ are of the form $f(z) = \sum_{j=-\infty}^{n_f} f_j z^j = f_-(z) + f_+(z)$, with $f_-(z) = \sum_{j=-\infty}^{-1} f_j z^j$ and $f_+(z) = \sum_{j=0}^{n_f} f_j z^j$. There are two basic projection maps that we consider, i.e., $\pi_- : \mathbb{F}((z^{-1}))^m \longrightarrow z^{-1}\mathbb{F}[[z^{-1}]]^m$ and $\pi_+ : \mathbb{F}((z))^m \longrightarrow \mathbb{F}[z]^m$, defined by $\pi_\pm f = f_\pm$. Given any nonsingular polynomial matrix $D(z) \in \mathbb{F}^{m \times m}$, we introduce the projection operator in $\mathbb{F}[z]^m$ as $\pi_D : \mathbb{F}[z]^m \longrightarrow \mathbb{F}[z]^m$ by

$$\pi_D f = D\pi_-(D^{-1}f), \qquad f(z) \in \mathbb{F}[z]^m. \tag{10}$$

We define the **polynomial model** induced by $D(z)$ as the image space $X_D = \operatorname{Im} \pi_D$ of the π_D and the **shift operator** $S_D : X_D \longrightarrow X_D$ by $S_D f = \pi_D z f$. The shift operator defines a natural $\mathbb{F}[z]$-module structure on X_D, defined as $p \cdot f(z) = \pi_D(p(z)f(z))$ for any polynomial $p(z) \in \mathbb{F}[z]$ and any $f(z) \in X_D$. Since clearly $\operatorname{Ker} \pi_D = D\mathbb{F}[z]^m$, we have the isomorphism $X_D \simeq \mathbb{F}[z]^m / D\mathbb{F}[z]^m$.

The study of homomorphisms between polynomial models and the characterization of isomorphisms and their inversion is an all important tool for the computation of open loop controls. For the inversion of isomorphisms we need the concept of unimodular embedding.

Theorem 3.1: Let $N(z) \in \mathbb{F}[z]^{p\times m}$, $D(z) \in \mathbb{F}[z]^{p\times p}$ be right coprime and $\overline{N}(z) \in \mathbb{F}[z]^{p\times m}$, $\overline{D}(z) \in \mathbb{F}[z]^{m\times m}$ be left coprime satisfying the **intertwining relation**:

$$D(z)\overline{N}(z) = N(z)\overline{D}(z).$$

Then there exist **doubly coprime unimodular embeddings** of it, i.e., there exist matrices $X(z) \in \mathbb{F}[z]^{m\times p}, \overline{X}(z) \in \mathbb{F}[z]^{p\times m}, Y(z) \in \mathbb{F}[z]^{m\times m}, \overline{Y}(z) \in \mathbb{F}[z]^{p\times p}$ with

$$\begin{pmatrix} Y(z) & X(z) \\ -N(z) & D(z) \end{pmatrix}\begin{pmatrix} \overline{D}(z) & -\overline{X}(z) \\ \overline{N}(z) & \overline{Y}(z) \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_p \end{pmatrix}.$$

The following theorem characterizes model homomorphisms and establishes a formula for their inverses.

Theorem 3.2: Let $\overline{D}(z) \in \mathbb{F}[z]^{m\times m}$ and $D(z) \in \mathbb{F}[z]^{p\times p}$ be nonsingular. Then

1) $Z : X_{\overline{D}} \longrightarrow X_D$ is an $\mathbb{F}[z]$-homomorphism if and only if there exist $\overline{N}(z), N(z) \in \mathbb{F}[z]^{p\times m}$ such that $N(z)\overline{D}(z) = D(z)\overline{N}(z)$ and $Zf = \pi_D(Nf)$.
2) In terms of the unimodular embedding above, the inverse homomorphism $Z^{-1} : X_D \longrightarrow X_{\overline{D}}$ is given by $Z^{-1}g = \pi_{\overline{D}}(\overline{X}g)$ for $g(z) \in X_D$.

Given a $p \times m$ proper rational function $G(z) = \sum_{i=0}^{\infty} \frac{G_i}{z^i}$ with representation

$$G(z) = V(z)T(z)^{-1}U(z) + W(z). \tag{11}$$

Let X_T denote the polynomial model of $T(z)$ with shift operator S_T. Then, the state space system with $A : X_T \longrightarrow X_T$, $B : \mathbb{F}^m \longrightarrow X_T$, $C : X_T \longrightarrow \mathbb{F}^p$ and $D : \mathbb{F}^m \longrightarrow \mathbb{F}^p$, defined by

$$\Sigma_{VT^{-1}U+W} := \left\{ \begin{array}{rcll} Af &=& S_T f, & f \in X_T \\ B\xi &=& \pi_T(U\xi), & \xi \in \mathbb{F}^m \\ Cf &=& (VT^{-1}f)_{-1}, & f \in X_T \\ D &=& G_0 & \end{array} \right. \tag{12}$$

is called the **shift realization** of (11). Fundamental properties of the shift realization are summarized in the following theorem.

Theorem 3.3: The shift realization (A, B, C, D) has the following properties:

1) The transfer function of (A, B, C, D) is $G(z)$.
2) The reachability and observability maps of the realization (12) are respectively given by
$$\mathcal{R}u = \pi_T(Uu),\ u(z) \in \mathbb{F}[z]^m$$
$$\mathcal{O}f = \pi_-(VT^{-1}f),\ f(z) \in X_T.$$
3) The realization is observable if and only if $V(z)$ and $T(z)$ are right coprime and reachable if and only if $T(z)$ and $U(z)$ are left coprime.
4) If both coprimeness conditions are satisfied, then for the McMillan degree $\delta(G)$ we have
$$\delta(G) = \deg\det T(z). \tag{13}$$

A. Open loop control

We present now a polynomial approach to the terminal state problem from the point of view of inverting the reachability map. This leads directly to the problem of unimodular embedding and hence, indirectly, to the study of flat outputs. We consider a system given in state space form as

$$x(t+1) = Ax(t) + Bu(t), \quad t = 0, 1, 2, \ldots .$$

We assume that $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ is reachable. An input sequence $u_0, u_1, \ldots u_{T-1}$ steers the zero state into a desired state $x(T) = \xi \in \mathbb{F}^n$ at time T if and only if

$$x(T) = \sum_{k=0}^{T-1} A^k B u_{T-1-k}.$$

We refer to the polynomial

$$u(z) = \sum_{k=0}^{T-1} u_{T-1-k} z^k \in \mathbb{F}[z]^m$$

as the **input polynomial** for ξ. Note that the input vectors $u_0, \ldots, u_{T-1}$ appear in reverse order as the coefficients of the input polynomial. Reachability of (A, B) is equivalent that such an input polynomial always exists. We can identify $\mathbb{F}^n$, endowed with the $\mathbb{F}[z]$-module structure induced by A, with the polynomial model X_{zI-A}. The **reachability map** then is the $\mathbb{F}[z]$-module homomorphism given as

$$\mathcal{R}_{(A,B)} : \mathbb{F}[z]^m \longrightarrow X_{zI-A}$$
$$\mathcal{R}_{(A,B)}u(z) = \pi_{zI-A} B \sum_{i=0}^{s} u_i z^i = \sum_{i=0}^{s} A^i B u_i.$$

We have the following representation for the kernel of the reachability map.

Lemma 3.1: Let $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ be a reachable pair and let

$$(zI - A)^{-1}B = \overline{H}(z)\overline{D}(z)^{-1} \tag{14}$$

be coprime factorizations, with $\overline{D}(z) \in \mathbb{F}[z]^{m\times m}, \overline{H}(z) \in \mathbb{F}[z]^{n\times m}$. The reachability map $\mathcal{R}_{(A,B)}$ is an $\mathbb{F}[z]$-homomorphism, its kernel is a submodule of $\mathbb{F}[z]^m$, hence has the representation

$$\operatorname{Ker} \mathcal{R}_{(A,B)} = \overline{D}\mathbb{F}[z]^m.$$

Clearly, to compute controls that steer to a state ξ is equivalent to invert the reachability map $\mathcal{R}_{(A,B)}$. To get an invertible map, we factor out the kernel. Denote by $\mathcal{R}$ the **reduced reachability map**, namely the map induced by $\mathcal{R}_{(A,B)}$ on $\mathbb{F}[z]^m/\overline{D}\mathbb{F}[z]^m$, which we identify with the polynomial model $X_{\overline{D}}$. Thus $\mathcal{R} : X_{\overline{D}} \longrightarrow X_{zI-A}$ is given by

$$\begin{array}{l} \mathcal{R} : X_{\overline{D}} \longrightarrow X_{zI-A} \\ \quad \mathcal{R}u = \pi_{zI-A}Bu, \quad u(z) \in X_{\overline{D}}. \end{array} \tag{15}$$

By the assumption of reachability, $\mathcal{R}$ is surjective, whereas by construction it is injective, hence $\mathcal{R}$ is an isomorphism. So, it turns out that computing the minimal control sequence is achieved by inverting the map $\mathcal{R}$. Using our results on inverting model homomorphisms, this is best achieved by use of embedding intertwining maps in doubly coprime factorizations.

Theorem 3.4: Let $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ be a reachable pair and let

$$(zI - A)^{-1}B = \overline{H}(z)\overline{D}(z)^{-1} \quad (16)$$

be coprime factorizations, with $\overline{D}(z) \in \mathbb{F}[z]^{m\times m}, \overline{H}(z) \in \mathbb{F}[z]^{n\times m}$. Then

1) The intertwining relation $B\overline{D}(z) = (zI - A)\overline{H}(z)$ can be embedded in the following doubly coprime factorization
$$\begin{pmatrix} Y(z) & X(z) \\ -B & zI-A \end{pmatrix}\begin{pmatrix} \overline{D}(z) & -\overline{X}(z) \\ \overline{H}(z) & \overline{Y}(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$
2) There exists a unique such embedding for which X is constant. In that case, the pair (X, A) is an observable pair, $\overline{D}(z)^{-1}\overline{X}(z)$ is strictly proper and the columns of $\overline{X}$ form a basis for the polynomial model $X_{\overline{D}}$.
3) Given a state $\xi \in \mathbb{F}^n$, there exists a unique minimal degree input poylnomial $\mathbf{u}_{\min}(z)$ that steers the system from the zero state to ξ and it is given by the coefficients of $\mathbf{u}_{\min}(z) = \mathcal{R}^{-1}\xi$. Specifically, in terms of the above doubly coprime factorization, we have
$$\mathbf{u}_{\min}(z) = \mathcal{R}^{-1}\xi = \pi_{\overline{D}}\overline{X}\xi.$$
4) An arbitrary solution $\mathbf{u}_*(z)$ to the steering problem is given by $\mathbf{u}_*(z) = \mathbf{u}_{\min}(z) + \overline{D}(z)g(z)$, with $g(z) \in \mathbb{F}[z]^m$.

IV. THE SYNTHESIS OF CONTROLS

A. *Controllability of Parallel Connections*

The study of interconnected systems is not new. It started with the work by Gilbert on controllability and observability for generic classes of one-dimensional linear systems in parallel, series and feedback interconnections. Complete characterizations for arbitrary pairs of multivariable linear systems were subsequently obtained in [2] for series and feedback interconnections and, in a short note, in [3] for parallel interconnections. The interconnection structures of most complex systems are however not of the parallel or series connection type. Thus one needs to extend the controllability analysis from the standard interconnections to more complex ones, where the interconnection patterns between the node systems are described by arbitrary weighted directed graphs. This is done elsewhere, and the beginnings of such a general theory are outlined in [5]. In this section we develop the controllability analysis of parallel connections. However, in contrast to the early work [3], we approach this problem using a controllability result for arbitrary interconnected systems.

Theorem 4.1 ([5]): The following statements are equivalent:

1) The system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is controllable.
2) $Q(z) - P(z)A, P(z)B$ are left coprime
3) $\overline{Q}(z) - A\overline{P}(z), B$ are left coprime.
4) $T(z) - U(z)AV(z), U(z)B$ are left coprime.

As mentioned in the introduction, this result presents a far reaching generalization of the controllability result by [7] for homogeneous networks. We next deduce a characterization of reachability for the parallel connection of N linear systems

$$\begin{aligned} x_1(t+1) &= A_1x_1(t) + B_1u(t) \\ &\vdots \\ x_N(t+1) &= A_Nx_N(t) + B_Nu(t). \end{aligned} \quad (17)$$

Thus the parallel connection is given by the pair $(\mathcal{A}, \mathcal{B})$ where

$$\mathcal{A} := \mathrm{diag}\,(A_1, \ldots, A_N),\ \mathcal{B} = \mathrm{col}\,(B_1, \ldots, B_N). \quad (18)$$

Assume that $(A_i, B_i) \in \mathbb{F}^{n_i\times n_i} \times \mathbb{F}^{n_i\times m}, i = 1, \ldots, N$, are reachable pairs. Consider any right coprime factorization

$$\hat{H}_i(z)\overline{D}_i(z)^{-1} = (zI - A_i)^{-1}B_i \quad (19)$$

of $(zI - A_i)^{-1}B_i$. Then the intertwining relation $B_i\overline{D}_i(z) = (zI - A_i)\hat{H}_i(z)$ can be embedded in a doubly coprime factorization as

$$\left(\begin{array}{c|c} Y_i(z) & X_i(z) \\ \hline -B_i & zI - A_i \end{array}\right)\left(\begin{array}{c|c} \overline{D}_i(z) & -\overline{X}_i(z) \\ \hline \hat{H}_i(z) & \overline{Y}_i(z) \end{array}\right) = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right).$$

Define $\overline{D}(z) := l.c.r.m.\{\overline{D}_i(z)|i = 1, \ldots, N\}$, $\overline{L}_i(z) := \overline{D}_i^{-1}(z)\overline{D}(z)$, $L_i(z) := l.c.r.m.\{\overline{D}_j(z)|j \neq i\}$, $D_i(z) := L_i(z)^{-1}\overline{D}(z)$. Our next result extends the earlier analysis in [3] for $N = 2$ to an arbitrary number of subsystems.

Theorem 4.2: The following statements are equivalent:

1) The parallel connection pair $(\mathcal{A}, \mathcal{B})$ reachable.
2) $zI - \mathcal{A}, \mathcal{B}$ are left coprime.
3) The reduced reachability map $\mathcal{R} : X_{\overline{D}} \longrightarrow X_{zI-\mathcal{A}}$, defined by $\mathcal{R}u = \pi_{zI-\mathcal{A}}\mathcal{B}u,\ u(z) \in X_{\overline{D}}$, is an $\mathbb{F}[z]$-isomorphism.
4) The polynomial matrices $\overline{D}_i(z),\ i = 1, \ldots, N$, are mutually left coprime.
5) The polynomial matrices $D_i(z),\ i = 1, \ldots, N$, are mutually right coprime.
6) We have the direct sum representation
$$X_{\overline{D}} = L_1X_{D_1} \oplus \cdots \oplus L_NX_{D_N}.$$
7) The polynomial matrix
$$\begin{pmatrix} -\overline{D}_1(z) & \overline{D}_2(z) & & \\ & \ddots & \ddots & \\ & & -\overline{D}_{N-1}(z) & \overline{D}_N(z) \end{pmatrix}$$
is left prime.

We point out that analogous characterizations of controllability for series connections are derived in [5].

B. Open Loop Control for Parallel Connections

We derive next explicit formula for open loop control of parallel interconnections of N systems. The extension to parallel and series interconnections of an arbitrary number of systems is possible, but is not reported here and we refer to our forthcoming book [5] for further details. We start with a simplifed situation by considering an arbitrary number N of parallel connected reachable SISO systems

$$\begin{aligned} x_1(t+1) &= A_1 x_1(t) + b_1 u(t) \\ &\vdots \\ x_N(t+1) &= A_N x_N(t) + b_N u(t). \end{aligned} \tag{20}$$

Assume that for each of the N local subsystems (A_j, b_j) we are given local control sequences $\mathbf{v}_j$ that steer the zero state to a desired terminal state ξ_j. How can one compute from such local controls a single global input sequence $\mathbf{u}$ that steers all subsystems simultaneously to the desired terminal states? Ideally, one would like to obtain a formula such as $\mathbf{u} = \sum_{j=1}^{N} f_j \mathbf{v}_j$ that expresses the desired control as weighted sum of local controls, where the weights f_j are suitable filter operators that act on the respective local input. To answer this question we need just a little bit more notation.

Let $q_j(z) = \det(zI - A_j)$ denote the characteristic polynomial of A_j and define $\hat{q}_j(z) = \prod_{i \neq j} q_i(z)$. Assume that the pairs (A_j, b_j) are reachable for each $j = 1, \ldots, N$. Reachability of the parallel connection (20) then is equivalent to coprimeness of $q_j, \hat{q}_j$ for each $j = 1, \ldots, N$. Let the polynomials $c_j(z), d_j(z)$ denote the unique solutions of the Bezout equation

$$c_j(z) q_j(z) + d_j(z) \hat{q}_j(z) = 1, \quad j = 1, \ldots, N$$

with degrees $\deg d_j < n_j$. Proceeding as before, we store the input sequences $v_0, \ldots, v_{M-1}$ for controlling (20) as coefficients of the associated input polynomial $\mathbf{u}(z) = \sum_{j=0}^{M-1} v_{M-j-1} z^j$. Our basic control result for (20) is stated as follows.

Theorem 4.3: Assume that (20) is reachable. Given any local state vectors $\xi_1, \ldots, \xi_N$ and arbitrary input polynomials $\mathbf{v}_1(z), \ldots, \mathbf{v}_N(z) \in \mathbb{R}[z]$ for the local systems satisfying

$$\mathbf{v}_1(A_1) b_1 = \xi_1, \ \ldots, \ \mathbf{v}_N(A_N) b_N = \xi_N.$$

Then the input polynomial for (20)

$$\mathbf{u}(z) = \sum_{j=1}^{N} d_j(z) \hat{q}_j(z) \mathbf{v}_j(z) \tag{21}$$

satisfies $\mathbf{u}(A_j) b_j = \xi_j$ for all j.

Proof. From the Bezout equation we obtain $d_k(A_k)\hat{q}_k(A_k) = I$ and $d_j(A_k)\hat{q}_j(A_k) = 0$ for $j \neq k$. This implies

$$\mathbf{u}(A_k) b_k = \sum_{j=1}^{N} d_j(A_k) \hat{q}_j(A_k) \mathbf{v}_j(A_k) b_k = \mathbf{v}_k(A_k) b_k = \xi_k.$$

∎

Let $(A_i, B_i) \in \mathbb{F}^{n_i \times n_i} \times \mathbb{F}^{n_i \times m}, i = 1, \ldots, N$, be reachable pairs. Assume that $\hat{H}_i(z)\overline{D}_i(z)^{-1}$ is a right coprime factorization of $(zI - A_i)^{-1} B_i$. Assume further that the intertwining relation $B_i \overline{D}_i(z) = (zI - A_i)\hat{H}_i(z)$ is embedded in the doubly coprime factorization

$$\left(\begin{array}{c|c} Y_i(z) & X_i(z) \\ \hline -B_i & zI - A_i \end{array} \right) \left(\begin{array}{c|c} \overline{D}_i(z) & -\overline{X}_i(z) \\ \hline \hat{H}_i(z) & \overline{Y}_i(z) \end{array} \right) = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array} \right).$$

The parallel connection system is given by the pair $(\mathcal{A}, \mathcal{B})$, $\mathcal{A} := \mathrm{diag}\,(A_1, \ldots, A_N)$, $\mathcal{B} = \mathrm{col}\,(B_1, \ldots, B_N)$. Define $\overline{D}(z) := l.c.r.m.\{\overline{D}_i(z) | i = 1, \ldots, N\}$, $\overline{L}_i(z) := \overline{D}_i^{-1}(z)\overline{D}(z)$, $L_i(z) := l.c.r.m.\{\overline{D}_j(z) | j \neq i\}$, $D_i(z) := L_i(z)^{-1}\overline{D}(z)$, where $\overline{D}(z) = \overline{D}_i(z)\overline{L}_i(z) = L_i(z) D_i(z)$. Defining $\overline{H}(z) := \mathrm{col}\,(\overline{H}_1(z), \ldots, \overline{H}_N(z))$, where $\overline{H}_i := \hat{H}_i(z)\overline{L}_i(z)$, we have the intertwining relation

$$\mathcal{B}\overline{D}(z) = (zI - \mathcal{A})\overline{H}(z).$$

We now state our main result on computing open loop controls for parallel interconnections of a finite number of multi-input linear systems. The case of series connection can be treated as well, but at the expense of a more elaborate analysis. We also like to stress that our analysis is very closely related to the problem of computing flat outputs for higher order systems; see e.g. [11]. Thus our subsequent Theorem can be seen as a first result towards explicit formulas for flat outputs of networks of linear systems.

Theorem 4.4: 1. There exist doubly coprime factorizations

$$\begin{pmatrix} \overline{E}_i(z) & F_i(z) \\ -L_i(z) & \overline{D}_i(z) \end{pmatrix} \begin{pmatrix} D_i(z) & -\overline{F}_i(z) \\ \overline{L}_i(z) & E_i(z) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

with $F_i(z)\overline{D}_i(z)^{-1}$ and $D_i(z)^{-1}\overline{F}_i(z)$ are strictly proper.

2. A doubly coprime embedding of the intertwining relation $\mathcal{B}\overline{D}(z) = (zI - \mathcal{A})\overline{H}(z)$ is given by

$$\begin{pmatrix} \overline{D}(z) & -\overline{\Xi}(z) \\ \overline{H}(z) & \overline{\Theta}(z) \end{pmatrix} \begin{pmatrix} \Theta(z) & \Xi(z) \\ -\mathcal{B} & zI - \mathcal{A} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

where

$$\begin{aligned} \overline{\Xi}_i &= L_i \overline{F}_i \overline{X}_i \\ \overline{\Theta}_{ii} &= \overline{Y}_i + \hat{H}_i E_i \overline{X}_i \\ \overline{\Theta}_{ij} &= -\hat{H}_i \overline{D}_i^{-1} L_j \overline{F}_j \overline{X}_j \\ \Xi_j &= F_j X_j \end{aligned}$$

3. The map $\mathcal{R}^{-1} : X_{zI-\mathcal{A}} \longrightarrow X_{\overline{D}}$, is given, for $\xi = \mathrm{col}\,(\xi_1, \ldots, \xi_N) \in X_{zI-\mathcal{A}}$, by

$$\mathbf{u}(z) = \mathcal{R}^{-1}\xi = \sum_{i=1}^{N} L_i(z) \pi_{D_i} \overline{F}_i \pi_{\overline{D}_j} \overline{X}_j \xi_j. \tag{22}$$

Thus $\mathbf{u}(z)$ is the minimal steering controller to the state ξ. The proof of Theorem 4.4 is by a long sequence of highly nontrivial computations that are omitted.

We illustrate Theorem 4.3 by constructing open loop controls for the parallel connection of N scalar discrete-time harmonic oscillators

$$x_k(t+2) + \omega_k^2 x_k(t) = u(t), \quad k = 1, \dots, N.$$

The node transfer functions are $g_k(z) = p_k(z)/q_k(z)$ with $p_k(z) = 1, q_k(z) = z^2 + \omega_k^2$. Consider any desired local state vectors $\xi_k = (\xi_{1,k}, \xi_{2,k})^\top \in \mathbb{R}^2$ for $k = 1, \dots, N$ to which we want to steer the system to. From the Bezout equation $c_k(z)(z^2 + \omega_k^2) + d_k(z)\hat{q}_k(z)$ we get $d_k(z) = \prod_{j \neq k} \left(\omega_j^2 - \omega_k^2\right)^{-1}, \quad k = 1, \dots, N$. Thus the desired open loop control sequence $u_0, \dots, u_{2N-1}$ with $\mathbf{u}(z) = \sum_{j=0}^{2N-1} u_{2N-j-1} z^j$ is given as

$$\mathbf{u}(z) = \sum_{k=1}^{N} \prod_{j \neq k} \frac{z^2 - \omega_k^2}{\omega_j^2 - \omega_k^2} (\xi_{1,k} + z\xi_{2,k})$$

V. CONCLUSIONS

We considered two different synthesis problems for networks of interconnected linear systems. A characterization of all rational transfer functions of homogeneous networks is given in terms of the associated Galois group and root locus invariants, such as the breakaway values. The extension to multivariable networks or heterogeneous networks is an open problem. Concerning the synthesis problem of open loop controls in networks we developed an explicit formula in terms of local open loop controls and explicit computations of doubly coprime factorizations. These results are of interest, as the formula for open loop control based on doubly coprime factorizations is very closely related to the computation of flat outputs for higher order linear input/output systems. Similar expressions can be derived for open loop control of series connections. The task of computing open loop controls for more general interconnection schemes, such as paths, cycles or circulant structures is left as an open problem.

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