21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

A SURVEY OF A NEW ν-METRIC IN CONTROL THEORY

AMOL SASANE

We recall the general stabilization problem in control theory. Suppose that R is a commutative integral domain with identity (thought of as the class of stable transfer functions) and let $\mathbb{F}(R)$ denote the field of fractions of R (thought of as the set of unstable plants). The stabilization problem is then the following: given an unstable plant transfer function $P \in (\mathbb{F}(R))^{p\times m}$, find a stabilizing controller transfer function $C \in (\mathbb{F}(R))^{m\times p}$ such that

$$H(P,C) := \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \in R^{(p+m)\times(p+m)}.$$

Robust stabilization goes one step further; in many practical situations one knows that the plant is merely an approximation of reality and therefore one wishes that the controller C not only stabilizes the nominal plant P, but also all plants $\widetilde{P}$ sufficiently close to P. A metric which emerged from the need to define closeness of plants, is the so-called ν-metric, introduced by Vinnicombe in [5], where it was shown that stabilizability is a robust property of the plant with respect to the ν-metric. However, there R was essentially taken to be the set of rational functions without poles in the closed unit disk.

In [1] the ν-metric of Vinnicombe was extended in an abstract manner, in order to cover the case when R is a ring of stable transfer functions of possibly infinite-dimensional systems. In particular, the set-up for defining the abstract ν-metric was as follows:

(A1) R is a commutative integral domain with identity.
(A2) S is a unital commutative semisimple complex Banach algebra with an involution $\cdot^*$, such that $R \subset S$.
(A3) With inv S denoting the invertible elements of S, there exists a map $\iota : \text{inv } S \to G$, where $(G, \star)$ is an Abelian group with identity denoted by $\circ$, and ι satisfies:
  (I1) $\iota(ab) = \iota(a) \star \iota(b)$ for all $a, b \in \text{inv } S$,
  (I2) $\iota(a^*) = -\iota(a)$ for all $a \in \text{inv } S$,
  (I3) ι is locally constant, that is, ι is continuous when G is equipped with the discrete topology.
(A4) $x \in R \bigcap \text{inv } S$ is invertible as an element of R if and only if $\iota(x) = \circ$.

In [1], it was shown that the abstract ν-metric defined in the above framework (recalled below), is a metric on the class of all stabilizable plants, and moreover, that stabilizability is a robust property of the plant.

Let $\cdot^*$ denote the involution in the Banach algebra, mentioned in (A2). For $F \in S^{p\times m}$, the notation $F^* \in S^{m\times p}$ denotes the matrix given by $(F^*)_{ij} = (F_{ji})^*$ for $1 \le i \le p$ and $1 \le j \le m$. Here M_{ij} is used to denote the entry in the ith row and jth column of a matrix M.

Given a matrix $P \in (\mathbb{F}(R))^{p\times m}$, a factorization $P = ND^{-1}$, where N and D are matrices with entries from R, is called a *right coprime factorization of* P if there exist matrices X, Y

Key words and phrases. partial differential equations, delay differential equations, control theory, transfer functions, feedback, stabilization problem, robust control.

ISBN: 978-90-367-6321-9

with entries from R, such that $XN + YD = I_m$. If in addition, $N^*N + D^*D = I_m$, then the right coprime factorization is referred to as a *normalized right coprime factorization of* P.

Given a matrix $P \in (\mathbb{F}(R))^{p\times m}$, a factorization $P = \widetilde{D}^{-1}\widetilde{N}$, where $\widetilde{D}$ and $\widetilde{N}$ are matrices with entries from R, is called a *left coprime factorization of* P if there exist matrices $\widetilde{X}$, $\widetilde{Y}$ with entries from R, such that $\widetilde{N}\widetilde{X} + \widetilde{D}\widetilde{Y} = I_p$. If in addition, $\widetilde{N}\widetilde{N}^* + \widetilde{D}\widetilde{D}^* = I_p$, then the left coprime factorization is referred to as a *normalized left coprime factorization of* P.

Let $\mathbb{S}(R, p, m)$ denote the set of all elements $P \in (\mathbb{F}(R))^{p\times m}$ that possess normalized right- and left coprime factorizations. For $P \in \mathbb{S}(R, p, m)$, with factorizations $P = \widetilde{D}^{-1}\widetilde{N} = ND^{-1}$, G and $\widetilde{G}$ are defined by $G := \begin{bmatrix} N \\ D \end{bmatrix}$ and $\widetilde{G} := \begin{bmatrix} -\widetilde{D} & \widetilde{N} \end{bmatrix}$. Further, we will define a norm on matrices with entries in S using the Gelfand transform. Let $\mathfrak{M}(S)$ denote the maximal ideal space of the Banach algebra S. For a matrix $M \in S^{p\times m}$, we define $\|M\|_{S,\infty} = \max_{\varphi\in\mathfrak{M}(S)} \|\mathbf{M}(\varphi)\|$, where $\mathbf{M}$ denotes the entry-wise Gelfand transform of M, and $\|\cdot\|$ denotes the induced operator norm from $\mathbb{C}^m$ to $\mathbb{C}^p$. (For the sake of concreteness, we assume that $\mathbb{C}^m$ and $\mathbb{C}^p$ are both equipped with the usual Euclidean 2-norm.) We will now present the abstract ν-metric from [1]. For $P_1,\ P_2 \in \mathbb{S}(R, p, m)$, with normalized left/right coprime factorizations $P_1 = N_1D_1^{-1} = \widetilde{D}_1^{-1}\widetilde{N}_1$ and $P_2 = N_2D_2^{-1} = \widetilde{D}_2^{-1}\widetilde{N}_2$, the ν-metric d_ν is given by

$$d_\nu(P_1, P_2) = \begin{cases} \|\widetilde{G}_2G_1\|_{S,\infty} & \text{if } \det(G_1^*G_2) \in \text{inv } (S) \text{ and } \iota(\det(G_1^*G_2)) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 0.1. [1] *d_ν is a metric on $\mathbb{S}(R, p, m)$.*

Moreover, stabilizability is a robust property of the plant in this new ν-metric. We first introduce the notion of stability margin for a pair comprising a plant and its controller, which serves as a measure of the performance of the closed loop system comprising P and C. Given $P \in \mathbb{F}(R, p, m)$ and $C \in \mathbb{S}(R, m, p)$, the *stability margin* of the pair (P, C) is defined by $\mu_{P,C} = \begin{cases} \|H(P,C)\|_{S,\infty}^{-1} & \text{if } P \text{ is stabilized by } C, \\ 0 & \text{otherwise.} \end{cases}$

Theorem 0.2. [1] *If $P, P_0 \in \mathbb{S}(R, p, m)$ and $C \in \mathbb{S}(R, m, p)$, then $\mu_{P,C} \geq \mu_{P_0,C} - d_\nu(P, P_0)$.*

The table below gives an overview of the choice of principal objects S, G, ι specific to choices of R as the standard classes of stable transfer functions used in control theory.

R	S	G	ι
$RH^\infty, A(\mathbb{D}), W^+(\mathbb{D})$ $\widehat{L^1[0,\infty)} + \mathbb{C}, \cdots$	$C(\mathbb{T})$	$\mathbb{Z}$	$f \mapsto w(f)$
$\mathcal{A}_+$	$C_0 + AP$	$\mathbb{R}\times\mathbb{Z}$	$f_0 + f_{AP} \mapsto (w_{\text{av}}(f_{AP}) + w(1 + f_{AP}^{-1}\widehat{f_0}))$
H^∞	$\varinjlim C_b(\mathbb{A}_r)$ $\wr\!\!\mid$ $C(\beta\mathbb{A}_0 \setminus \mathbb{A}_0)$	$\mathbb{Z}$	$[(f_r)_r] \mapsto \lim_{r\to 1} w(f_r)$

The case when $R = RH^\infty, A(\mathbb{D}), W^+(\mathbb{D}), \widehat{L^1[0,\infty)}+\mathbb{C}, \cdots$**.** Let $\mathbb{C}_{>0} := \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, and $\overline{\mathbb{D}} := \mathbb{D} \cup \mathbb{T}$. Recall that

$$\begin{aligned} RH^\infty &:= \{f : \mathbb{C}_{>0} \to \mathbb{C} : f \text{ is rational and bounded in } \mathbb{C}_{>0}\}, \\ A(\mathbb{D}) &:= \{f : \overline{\mathbb{D}} \to \mathbb{C} : f \text{ is holomorphic in } \mathbb{D} \text{ and continuous in } \overline{\mathbb{D}}\}, \\ W^+(\mathbb{D}) &:= \{f : f(z) = \sum_{n=0}^{\infty} a_n z^n \ (z \in \overline{\mathbb{D}}) \text{ and } \sum_{n=0}^{\infty} |a_n| < +\infty\}, \\ \widehat{L^1[0,\infty)} + \mathbb{C} &:= \{f : f(s) = \widehat{f_\mathrm{a}}(s) + f_0 \ (s \in \mathbb{C}_{>0}), \text{ where } f_\mathrm{a} \in L^1[0,\infty), \ f_0 \in \mathbb{C}\}. \end{aligned}$$

In the above, $\widehat{f_\mathrm{a}}$ denotes the Laplace transform of $f_\mathrm{a} \in L^1[0,\infty)$. In each of these cases, the values of the function on the boundary of the domain of definition gives rise to a function belonging to the C^*-algebra $S := C(\mathbb{T}) := \{f : \mathbb{T} \to \mathbb{C} : f \text{ is continuous on } \mathbb{T}\}$. We take $G = \mathbb{Z}$, and $\iota : \mathrm{inv}\, C(\mathbb{T}) \to \mathbb{Z}$ to be the winding number w with respect to the origin: $\iota(f) := w(f)$, $f \in \mathrm{inv}\, C(\mathbb{T})$. Then it can be checked that (A1)-(A3) hold [1]. Moreover, the $\|\cdot\|_{S,\infty}$-norm in the definition of the d_ν is the usual supremum $\|\cdot\|_\infty$-norm of functions in $C(\mathbb{T})$.

The case when $R = \mathcal{A}_+$**.** Recall that

$$\mathcal{A}^+ = \left\{ s(\in \mathbb{C}_{>0}) \mapsto \widehat{f_\mathrm{a}}(s) + \sum_{k=0}^{\infty} a_k e^{-st_k} \;\middle|\; \begin{array}{l} f_\mathrm{a} \in L^1[0,\infty), \ (a_k)_{k\geq 0} \in \ell^1, \\ 0 = t_0 < t_1, t_2, t_3, \ldots \end{array} \right\}$$

Let

$$\begin{aligned} C_0 &:= \{f : \mathbb{R} \to \mathbb{C} : f \text{ is continuous on } \mathbb{R} \text{ and } \lim_{x\to\pm\infty} f(x) = 0\} \\ AP &:= \text{closed span in } L^\infty(\mathbb{R}) \text{ of } \{(\mathbb{R} \ni) x \mapsto e^{i\lambda x} : \lambda \in \mathbb{R}\}. \end{aligned}$$

$C_0 + AP$, endowed with pointwise operations, with the supremum norm, and with involution given by pointwise complex conjugation, is a sub-C^*-algebra of $L^\infty(\mathbb{R})$. We will take $S := C_0 + AP$. Moreover, we take $G = \mathbb{R} \times \mathbb{Z}$, and define $\iota : \mathrm{inv}\ (C_0 + AP) \to \mathbb{R} \times \mathbb{Z}$ by

$$\iota(f) = \Big(w_\mathrm{av}(f_{AP}), \ w(1 + f_{AP}^{-1} f_0) \Big), \quad f = f_0 + f_{AP} \in \mathrm{inv}\ (C_0 + AP), \ f_0 \in C_0, \ f_{AP} \in AP.$$

In the above, $w_\mathrm{av} : \mathrm{inv}\ AP \to \mathbb{R}$ denotes the average winding number, defined by

$$w_\mathrm{av}(f_{AP}) := \lim_{x\to+\infty} \frac{\arg(f(x)) - \arg(f(-x))}{2x}, \quad f_{AP} \in \mathrm{inv}\ AP.$$

Again, it can be checked that (A1)-(A3) hold; see [1]. Since $C_0 + AP$ is a sub-C^*-algebra of $L^\infty(\mathbb{R})$, the $\|\cdot\|_{S,\infty}$-norm in the definition of the $d_{c,r}$-metric is the usual $\|\cdot\|_\infty$-norm of functions in $L^\infty(\mathbb{R})$. In [3], it was shown that the topology induced by the ν-metric coincides with the classical gap topology for unstable single-input single-output plants.

The case when $R = H^\infty$**.** The Hardy algebra H^∞ consists of all bounded and holomorphic functions defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, with pointwise operations and the usual supremum norm. We recall the construction of S from [2]. For given $r \in (0,1)$, let $\mathbb{A}_r := \{z \in \mathbb{C} : r < |z| < 1\}$ denote the open annulus and let $C_b(\mathbb{A}_r)$ be the C^*-algebra of all bounded and continuous functions $f : \mathbb{A}_r \to \mathbb{C}$, equipped with pointwise operations and the supremum norm. Moreover, for $0 < r \leq R < 1$ we define the map $\pi_r^R : C_b(\mathbb{A}_r) \to C_b(\mathbb{A}_R)$ by restriction: $\pi_r^R(f) = f|_{\mathbb{A}_R}$, $f \in C_b(\mathbb{A}_r)$. Consider the family $\big(C_b(\mathbb{A}_r), \pi_r^R\big)$ for $0 < r \leq R < 1$. We note that π_r^r is the identity map on $C_b(\mathbb{A}_r)$, and $\pi_r^R \circ \pi_\rho^r = \pi_\rho^R$ for all $0 < \rho \leq r \leq R < 1$. Now consider the $*$-algebra $\prod_{r\in(0,1)} C_b(\mathbb{A}_r)$, and denote by $\mathcal{A}$ its $*$-subalgebra consisting of all elements $f = (f_r) = (f_r)_{r\in(0,1)}$ such that there is an index r_0 with $\pi_r^R(f_r) = f_R$ for all $0 < r_0 \leq r \leq R < 1$. Since every π_r^R is norm decreasing, the net $(\|f_r\|_\infty)$ is convergent and we define $\|f\| := \lim_{r\to 1} \|f_r\|_\infty$. This defines a seminorm on $\mathcal{A}$ that satisfies the C^*-norm identity $\|f^* f\| = \|f\|^2$, where $\cdot^*$ is the involution, that is, complex conjugation. Now, if N is the kernel of $\|\cdot\|$, then the quotient $\mathcal{A}/N$ is a C^*-algebra (and we denote the norm again by $\|\cdot\|$). This algebra is the *direct/inductive limit* of $(C_b(\mathbb{A}_r), \pi_r^R)$ and we denote it by $\varinjlim C_b(\mathbb{A}_r)$. To every element $f \in C_b(\mathbb{A}_{r_0})$, we associate a sequence $f_1 = (f_r)$ in $\mathcal{A}$, where $f_r = \begin{cases} 0 & \text{if } 0 < r < r_0, \\ \pi_{r_0}^r(f) & \text{if } r_0 \leq r < 1. \end{cases}$ We also define a map $\pi_r : C_b(\mathbb{A}_r) \to \varinjlim C_b(\mathbb{A}_r)$ by $\pi_r(f) := [f_1]$, $f \in C_b(\mathbb{A}_r)$, where $[f_1]$ denotes the equivalence class in $\varinjlim C_b(\mathbb{A}_r)$ which contains f_1. The maps π_r are $*$-homomorphisms. Then $\varinjlim C_b(\mathbb{A}_r)$ is a C^*-algebra. The multiplicative identity arises from the constant function $f \equiv 1$ in $C_b(\mathbb{A}_0)$, that is, $\pi_0(f)$. Moreover, we can define an involution in $C_b(\mathbb{A}_r)$ by setting $(f^*)(z) := \overline{f(z)}$, $z \in \mathbb{A}_r$, and this implicitly defines an involution of elements in $\varinjlim C_b(\mathbb{A}_r)$. There is a natural embedding of H^∞ into $\varinjlim C_b(\mathbb{A}_r)$, namely $f \mapsto \pi_0(f) : H^\infty \longrightarrow \varinjlim C_b(\mathbb{A}_r)$. We will take $G = \mathbb{Z}$. For $f \in \text{inv}\ (C_b(\mathbb{A}_\rho))$ and for $0 < \rho < r < 1$ we define the map $f^r : \mathbb{T} \to \mathbb{C}$ by $f^r(\zeta) = f(r\zeta)$, $\zeta \in \mathbb{T}$. If $f \in \text{inv}\ (C_b(\mathbb{A}_\rho))$, then $f^r \in \text{inv}\ (C(\mathbb{T}))$, and so f^r has a winding number $w(f_r)$. We set $w(f) := w(f^r) \in \mathbb{Z}$ with respect to 0, and it can be shown that this is well-defined. Now we define the map $\iota : \text{inv}\ (\varinjlim C_b(\mathbb{A}_r)) \to \mathbb{Z}$. For $[(f_r)] \in \text{inv}\ (\varinjlim C_b(\mathbb{A}_r))$, $\iota(f) := \lim_{r\to 1} w(f_r)$, for $f = [(f_r)] \in \text{inv}\ (\varinjlim C_b(\mathbb{A}_r))$. It can be shown that ι is well-defined and all the properties we demand are satisfied; see [2]. It was also shown there that $\varinjlim C_b(\mathbb{A}_r)$ is isometrically isomorphic to $C(\beta\mathbb{A}_0 \setminus \mathbb{A}_0)$ (where $\beta\mathbb{A}_0$ is the Stone-Čech compactification of $\mathbb{A}_0$, that is, $\beta\mathbb{A}_0$ is the the maximal ideal space of the Banach algebra $C_b(\mathbb{A}_0)$ of all complex-valued bounded continuous functions on $\mathbb{A}_0$), and moreover $\varinjlim C_b(\mathbb{A}_r)$ is a sub-C^*-algebra of $L^\infty(\mathbb{T})$. From here we see that the $\|\cdot\|_{S,\infty}$-norm in the definition of the d_ν-metric is the usual $\|\cdot\|_\infty$-norm of functions in $L^\infty(\mathbb{R})$.

References

[1] J.A. Ball and A.J. Sasane. Extension of the ν-metric. *Complex Analysis and Operator Theory*, 6:65-89, 2012.

[2] M. Frentz and A.J. Sasane. Reformulation of the extension of the ν-metric for H^∞. *Journal of Mathematical Analysis and Applications* , 401:659-671, no. 2, 2013.

[3] A.J. Sasane. The new ν-metric induces the classical gap topology. *Operators and Matrices*, 6:511527, no. 3, 2012.

[4] A.J. Sasane. Extension of the ν-metric for stabilizable plants over H^∞. *Mathematical Control and Related Fields*, 2:29-44, no. 1, 2012.

[5] G. Vinnicombe. Frequency domain uncertainty and the graph topology. *IEEE Transactions on Automatic Control*, no. 9, 38:1371-1383, 1993.

Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom.

E-mail address: sasane@lse.ac.uk