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Two Approaches to Stabilization for Euler-Bernoulli Beam Equation with Boundary Moment Control and Disturbance*

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Abstract— **In this paper, we consider boundary stabilization for a one-dimensional Euler-Bernoulli equation with boundary moment control and disturbance. The active disturbance rejection control (ADRC) and sliding mode control (SMC) approaches are adopted in investigation. By the ADRC approach, an extended state observer with time varying gain is designed to estimate the disturbance. It is shown that the closed-loop system is asymptotically stable after canceling the disturbance in the feedback loop. In the second part, the SMC is applied to reject the disturbance. The well-posedness of the closed-loop system via SMC is proven and the monotonicity of the "reaching condition" is presented without differentiation for the sliding mode function which may not always exist for the weak solution. The numerical experiments are presented to illustrate the convergence and the peaking value reduction caused by the constant high gain in literature. The control energy is compared numerically for two approaches.**

I. INTRODUCTION

In this paper, we are concerned with stabilization for a one-dimensional Euler-Bernoulli beam equation with uncertainty at the input boundary via both the active disturbance rejection control (ADRC) and the sliding mode control (SMC) approaches. The system is governed by the following PDEs:

$$\begin{cases} w_{tt}(x,t)+w_{xxxx}(x,t)=0, 0<x<1, t>0,\\ w(0,t)=w_x(0,t)=0, t\geq 0,\\ w_{xxx}(1,t)=0, t\geq 0,\\ w_{xx}(1,t)=u(t)+d(t), t\geq 0,\\ w(x,0)=w_0(x), w_t(x,0)=w_1(x), 0<x<1,\end{cases} \tag{1}$$

where u is the control input through bending moment, d is the external disturbance at the control end. It is well known that when there is not disturbance, the collocated feedback control $w_{xx}(1,t)=-kw_{xt}(1,t), k>0$ exponentially stabilizes the system (1)([2]). However, this stabilizing control law is not robust to external disturbance. Actually, system (1) under the feedback $u(t)=-kw_{xt}(1,t)$ admits a solution $(w,w_t)=(dx^2/2,0)$ when the disturbance $d(t)=d$ is a constant. Therefore, in the presence of disturbance, the control law must be re-designed.

In [4], an Euler-Bernoulli beam with boundary shear force is considered, which represents a well-posed infinite-dimensional system in the sense of D. Salamon ([1], [9]). However, system (1) is not well-posed ([3]). The objective of this paper is to stabilize system (1) with external disturbance by ADRC with a time varying high gain extended state observer which is different from [6], [7], [4] where the constant high gain is applied and the obvious peaking value problem occurs in the initial time stage.

Another point that we point out is that when the ADRC is applied in [6], [7], [4], the boundedness of the derivative of the disturbance is assumed that is different to SMC where no such limitation. This restriction is also removed in some extent by choosing properly the time varying gain in this paper.

Generally speaking, SMC is a "worst case concern" strategy in dealing with disturbance while ADRC is an estimation/cacellation strategy which reduces significantly the energy costed by control. This is explained numerically in this paper.

We process as follows. In Section 2, we use ADRC approach with time varying high gain extended state observer to reject completely the disturbance by designing an estimator to estimate the disturbance. After canceling the disturbance by the estimated one, we design an collocated like feedback control law. The closed-loop system is shown to be asymptotically stable. Section 3 is devoted to the disturbance rejection by SMC approach. The existence and uniqueness of solution are proved and the monotonicity of the "reaching condition" is presented without differentiating the sliding mode function, for which it does not always exist for the weak solution. In Section 4, we present some numerical simulations for illustration.

II. ACTIVE DISTURBANCE REJECTION CONTROL APPROACH

In this section, we propose a time varying high gain extended state observer to estimate in real time, the disturbance by the ADRC approach. Suppose that the time varying gain function r satisfies

$$\dot{r}(t)>0,\ \lim_{t\to\infty} r(t)=\infty,\ \sup_{t\geq 0}\frac{\dot{r}(t)}{r(t)}\leq \overline{M}, \overline{M}>0. \tag{2}$$

We consider system (1) in the energy Hilbert state space defined by

$$\begin{aligned}&\mathcal{H}=H_e^2(0,1)\times L^2(0,1),\\ &H_e^2(0,1)=\{f\in H^2(0,1)|\ |f(0)=f'(0)=0\},\end{aligned} \tag{3}$$

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with the inner product induced norm given by

$$\|(f,g)^\top\|^2 = \int_0^1 [|f''(x)|^2 + |g(x)|^2]dx, \ \forall\, (f,g)^\top \in \mathcal{H}.$$

Define the operator $\mathcal{A}$ as follows:

$$\begin{cases} \mathcal{A}(f,g)^\top = (g, -f^{(4)})^\top, \ \forall (f,g)^\top \in D(\mathcal{A}) \\ D(\mathcal{A}) = \{(f,g) \in \mathcal{H} \cap (H^4(0,1) \cap H_e^2(0,1)) | \\ \qquad\qquad f''(1) = f'''(1) = 0\} \end{cases} \tag{4}$$

and write system (1) as

$$\begin{aligned} &\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathcal{A}\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathcal{B}[u(t) + d(t)], \\ &\mathcal{B} = (0, -\delta'(x-1))^\top. \end{aligned} \tag{5}$$

By Proposition 2 of [3], $\mathcal{B}$ is not admissible to the semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$. This is different from those in [4] where the systems are always well-posed. To overcome this difficulty, we first introduce a damping on the control boundary by designing

$$u(t) = -k_1 w_{xt}(1,t) + v(t), \ k_1 > 0, t \geq 0,$$

under which, system (1) becomes

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \\ w(0,t) = w_x(0,t) = 0, \\ w_{xxx}(1,t) = 0, \\ w_{xx}(1,t) = -k_1 w_{xt}(1,t) + v(t) + d(t), \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), \end{cases} \tag{6}$$

Define the operator $\mathcal{A}_0$ as follows:

$$\begin{cases} \mathcal{A}_0(f,g)^\top = (g, -f^{(4)})^\top, \ \forall\, (f,g)^\top \in D(\mathcal{A}), \\ D(\mathcal{A}_0) = \{(f,g)^\top \in \mathcal{H} \cap (H^4(0,1) \times H_e^2(0,1)) | \\ \qquad\qquad f'''(1) = 0, f''(1) = -k_1 g'(1)\}. \end{cases} \tag{7}$$

Therefore, system (6) can be written as

$$\begin{aligned} &\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathcal{A}_0\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathcal{B}[u(t) + d(t)], \\ &\mathcal{B} = (0, -\delta'(x-1))^\top. \end{aligned} \tag{8}$$

The following result is straightforward ([10]).

Proposition 2.1: The operator $\mathcal{A}_0$ defined in (7) generates a C_0-semigroup of contractions $e^{\mathcal{A}_0 t}$ on $\mathcal{H}$ and $\mathcal{B}$ is admissible to $e^{\mathcal{A}_0 t}$. Therefore, for any initial value $(w(\cdot,0), \dot{w}(\cdot,0))^\top \in \mathcal{H}$ and control input $v \in L^2_{loc}(0,\infty)$ and $d \in L^2_{loc}(0,\infty)$, (6) admits a unique solution $(w, \dot{w})^\top \in \mathcal{H}$.

By proposition 2.1, the weak solution of (6) is understood in the sense of

$$\begin{aligned} \frac{d}{dt}\left\langle \begin{pmatrix} w \\ w_t \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} w \\ w_t \end{pmatrix}, \mathcal{A}_0^*\begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &+ [u(t) + d(t)]\mathcal{B}^*\begin{pmatrix} f \\ g \end{pmatrix}, \forall\, (f,g)^\top \in D(\mathcal{A}_0^*). \end{aligned} \tag{9}$$

A simple computation shows that

$$\begin{cases} \mathcal{A}_0^*(f,g)^\top = (-g, f^{(4)})^\top, \ \forall\, (f,g)^\top \in D(\mathcal{A}) \\ D(\mathcal{A}_0^*) = \{(f,g)^\top \in \mathcal{H} \cap (H^4(0,1) \times H_e^2(0,1)) | \\ \qquad\qquad f'''(1) = 0, f''(1) = k_1 g'(1)\}, \\ \mathcal{B}^*(f,g)^\top = g'(1). \end{cases} \tag{10}$$

Now, let $(f^t, g^t)^\top = (k_1 x^2 \varepsilon(t)/2, x^2 \varepsilon(t)/2) \in D(\mathcal{A}_0)$, where $\varepsilon(t) \in C([0,\infty), (0,\infty))$ satisfies

$$\lim_{t\to\infty} \varepsilon(t) = 0 \text{ and } \left|\frac{\varepsilon'(t)}{\varepsilon(t)}\right| \text{ is bounded for all } t \geq 0. \tag{11}$$

The existence of such function is obvious. For instance $\varepsilon(t) = \frac{1}{1+t}$. Substitution of $(f^t, g^t)^\top$ into (9) yields

$$\begin{aligned} &\frac{d}{dt}\int_0^1 [k_1 w_{xx}(x,t) + w_t(x,t)x^2/2]\varepsilon(t)dx \\ &= \int_0^1 [k_1 w_{xx}(x,t) + w_t(x,t)x^2]\varepsilon'(t)dx \\ &\quad - \int_0^1 w_{xx}(x,t)\varepsilon(t)dx + (v(t) + d(t))\varepsilon(t). \end{aligned}$$

Define

$$\begin{cases} y(t) = \displaystyle\int_0^1 [k_1 w_{xx}(x,t) + w_t(x,t)x^2/2]\varepsilon(t)dx, \\ y_o(t) = \displaystyle\int_0^1 [k_1 w_{xx}(x,t) + w_t(x,t)x^2]\varepsilon'(t)dx \\ \qquad - \displaystyle\int_0^1 w_{xx}\varepsilon(t)dx. \end{cases} \tag{12}$$

Then

$$\dot{y}(t) = y_o(t) + v(t)\varepsilon(t) + d(t)\varepsilon(t). \tag{13}$$

The system (13) is an ODE with disturbance, which is our starting point to estimate the disturbance d motivated from the ADRC to lumped parameter systems ([5]). To this purpose, we design a time varying high gain extended state observer as follows:

$$\begin{cases} \dot{\hat{y}}(t) = y_o(t) + v(t)\varepsilon(t) + d(t)\varepsilon(t) - r(t)[\hat{y}(t) - y(t)], \\ \dfrac{d}{dt}\left(\hat{d}(t)\varepsilon(t)\right) = -r^2(t)[\hat{y}(t) - y(t)], \end{cases} \tag{14}$$

where r defined by (2) is the time varying gain.

The following Lemma 2.1 is about the convergence of the extended state observer (14). In particular, we can regard $\hat{d}$ as an approximation of d.

Lemma 2.1: Suppose (2) and (11). Assume that the disturbance d is uniformly bounded and satisfies

$$\lim_{t\to\infty} \frac{|\dot{d}(t)|}{r(t)} = 0. \tag{15}$$

Let y and y_0 be defined by (12). Then the solution of (14) satisfies

$$\lim_{t\to\infty} |\hat{y}(t) - y(t)| = \lim_{t\to\infty} |\hat{d}(t) - d(t)| = 0. \tag{16}$$

Proof: Let

$$\tilde{y}(t) = r(t)[\hat{y}(t) - y(t)],\ \tilde{d}(t) = \hat{d}(t) - d(t) \tag{17}$$

be the errors. Then it follows from (13) and (14) that $(\tilde{y}, \tilde{d})$ satisfies

$$\begin{cases} \dot{\tilde{y}}(t) = -r(t)\tilde{y}(t) + r(t)\tilde{d}(t)\varepsilon(t) + \dfrac{\dot{r}(t)}{r(t)}\tilde{y}(t), \\ \dfrac{d}{dt}\left(\tilde{d}(t)\varepsilon(t)\right) = -r(t)\tilde{y}(t) - \dfrac{d}{dt}\left(d(t)\varepsilon(t)\right). \end{cases} \tag{18}$$

We construction a Lyapunov function for system (18):

$$V(y_1, y_2) = (y_1, y_2)P(y_1, y_2)^\top, \tag{19}$$

where the 2×2 positive definite matrix P is the solution of the following Lyapunov equation:

$$F^\top P + PF = -I_{2\times 2}, F = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Notice that, for all $(y_1, y_2) \in \mathbb{R}$, we have

$$\lambda_{\min}(P)\|(y_1, y_2)\|^2 \le V(y_1, y_2) \le \lambda_{\max}(P)\|(y_1, y_2)\|^2, \tag{20}$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the maximal and minimal eigenvalues of P, respectively. Finding the derivative of V along the solution of (18) yields

$$\begin{aligned} \dot{V}(t) \quad &= \left(\dot{\tilde{y}}(t), \frac{d}{dt}\left(\tilde{d}(t)\varepsilon(t)\right)\right) P \left(\tilde{y}(t), \tilde{d}(t)\varepsilon(t)\right)^\top \\ &\quad + \left(\tilde{y}(t), \tilde{d}(t)\varepsilon(t)\right) P \left(\dot{\tilde{y}}(t), \frac{d}{dt}\left(\tilde{d}(t)\varepsilon(t)\right)\right)^\top \\ &= r(t)\left(\tilde{y}(t), \tilde{d}(t)\varepsilon(t)\right)[PF + F^\top P]\left(\tilde{y}(t), \tilde{d}(t)\varepsilon(t)\right)^\top \\ &\quad + \left(\tilde{y}(t), \tilde{d}(t)\varepsilon(t)\right) P \left(\frac{\dot{r}(t)}{r(t)}\tilde{y}(t), -\frac{d}{dt}\left(d(t)\varepsilon(t)\right)\right)^\top \\ &\quad + \left(\frac{\dot{r}(t)}{r(t)}\tilde{y}(t), -\frac{d}{dt}\left(d(t)\varepsilon(t)\right)\right) P \left(\tilde{y}(t), \tilde{d}(t)\varepsilon(t)\right)^\top \\ &\le -r(t)\left\|\left(\tilde{y}(t), \tilde{d}(t)\varepsilon(t)\right)\right\|_{\mathbb{R}^2}^2 + N_1\left\|\left(\tilde{y}(t), \tilde{d}(t)\varepsilon(t)\right)\right\|_{\mathbb{R}^2}^2 \\ &\quad + N_2|\tfrac{d}{dt}\left(d(t)\varepsilon(t)\right)|\left\|\left(\tilde{y}(t), \tilde{d}(t)\varepsilon(t)\right)\right\|_{\mathbb{R}^2}, \end{aligned} \tag{21}$$

where N_1 and N_2 are two positive constants. By (2), one can choose $t_0 > 0$ such that $r(t) > \frac{2\lambda_{\max}(P)}{\lambda_{\min}(P)}N_1$ for all $t \ge t_0$. This together with (20) and (21) gives, for all $t \ge t_0$,

$$\frac{d\sqrt{V(t)}}{dt} \le -\frac{1}{4\lambda_{\max}(P)}r(t)\sqrt{V(t)} + \frac{N_2}{2\lambda_{\min}(P)}\left|\frac{d}{dt}\left(d(t)\varepsilon(t)\right)\right|, \tag{22}$$

which yields further for $t \ge t_0$ that

$$\begin{aligned} \sqrt{V(t)} &\le e^{-\int_{t_0}^t \frac{1}{4\lambda_{\max}(P)}r(\sigma)d\sigma}\sqrt{V(t_0)} \\ &+ \frac{\tilde{N}_2}{2\lambda_{\min}(P)} \frac{\displaystyle\int_{t_0}^t \left|\frac{d}{ds}\left(d(s)\varepsilon(s)\right)\right| e^{\int_{t_0}^s \frac{1}{4\lambda_{\max}(P)}r(\sigma)d\sigma}ds}{e^{\int_{t_0}^t \frac{1}{4\lambda_{\max}(P)}r(\sigma)d\sigma}}. \end{aligned} \tag{23}$$

Passing to the limit as $t \to \infty$ in (23) by using the L'Hospital rule and assumptions (2) and (11) gives $\lim_{t\to\infty}\sqrt{V(t)} = 0$, which implies

$$\lim_{t\to\infty}\tilde{y}(t) = 0 \text{ and hence } \lim_{t\to\infty}|\hat{y}(t) - y(t)| = 0. \tag{24}$$

Using (20) again, we obtain, for $t \ge t_0$, that

$$\begin{aligned} \tilde{d}(t) &\le \frac{\sqrt{V(t_0)}}{\lambda_{\min}^{0.5}(P)\varepsilon(t)e^{\int_{t_0}^t \frac{1}{4\lambda_{\max}(P)}r(\sigma)d\sigma}} \\ &+ \frac{N_2}{2\lambda_{\min}^{1/2}(P)} \frac{\int_{t_0}^t \left|\frac{d}{ds}\left(d(s)\varepsilon(s)\right)\right| e^{\int_{t_0}^s \frac{1}{4\lambda_{\max}(P)}r(\sigma)d\sigma}ds}{\varepsilon(t)e^{\int_{t_0}^t \frac{1}{4\lambda_{\max}(P)}r(\sigma)d\sigma}}. \end{aligned} \tag{25}$$

We claim that $\tilde{d}(t) \to 0$ as $t \to \infty$. To this purpose, it suffices to show the convergence of the second term of the right-hand side of (25) since the first term is less than the second term up to a constant as $t \to \infty$. Using the L'Hospital rule and assumption $\lim_{t\to\infty}|\dot{d}(t)|/r(t) = 0$, we have

$$\begin{aligned} &\lim\nolimits_{t\to\infty} \frac{\displaystyle\int_{t_0}^t |\frac{d}{ds}\left(d(s)\varepsilon(s)\right)| e^{\int_{t_0}^s \frac{1}{4\lambda_{\max}(P)}r(\sigma)d\sigma}ds}{\varepsilon(t)e^{\int_{t_0}^t \frac{1}{4\lambda_{\max}(P)}r(\sigma)d\sigma}} \\ &= \frac{|\frac{d}{dt}\left(d(t)\varepsilon(t)\right)|}{\varepsilon'(t) + \frac{\varepsilon(t)r(t)}{4\lambda_{\max}(P)}} = \frac{|\dot{d}(t) + d(t)\frac{\varepsilon'(t)}{\varepsilon(t)}|}{\frac{\varepsilon'(t)}{\varepsilon(t)} + \frac{r(t)}{4\lambda_{\max}(P)}} = 0. \end{aligned}$$

■

The next objective of the ADRC is to cancel the disturbance in the feedback-loop. Since the collocated feedback control $v(x, t) = -k_2 w_{xt}(1, t), k_2 > 0$ stabilizes the system (6) without the disturbance, a stabilizing control law to (6) is naturally designed as follows:

$$v(x, t) = -k_2 w_{xt}(1, t) - \hat{d}(t), k_2 > 0. \tag{26}$$

It is seen that the second term in the right side of (26) is used to cancel the effect of the disturbance. This is just the estimation/cancellation nature of the ADRC. Under feedback (26), the closed-loop system of (6) becomes

$$\begin{cases} w_{tt}(x, t) + w_{xxxx}(x, t) = 0, 0 < x < 1, t > 0, \\ w(0, t) = w_x(0, t) = 0, t \ge 0, \\ w_{xxx}(1, t) = 0, \\ w_{xx}(1, t) = -kw_{xt}(1, t) - \hat{d}(t) + d(t), k = k_1 + k_2, \\ \dot{\hat{y}}(t) = y_o(t) - k_2 w_{xt}(1, t)\varepsilon(t) - \hat{d}(t)\varepsilon(t) + d(t)\varepsilon(t) \\ \qquad -r(t)[\hat{y}(t) - y(t)], \\ \dfrac{d}{dt}\left(\hat{d}(t)\varepsilon(t)\right) = -r^2(t)[\hat{y}(t) - y(t)], \end{cases} \tag{27}$$

where r is defined by (2) and ε is defined by (11).

Now we are in a position to show the convergence of the closed-loop system (27).

Theorem 2.1: Suppose (2), (11), and that the disturbance d is uniformly bounded. Let y and y_0 be defined by (12).

Then system (27) is asymptotically stable in the sense of

$\lim_{t\to\infty} E(t) = 0$,

$$E(t) = \int_0^1 [w_{xx}^2(x,t) + w_t^2(x,t)]dx + |\hat{y}(t)| + |\hat{d}(t) - d(t)|.$$

Proof: Using the error variables $(\tilde{y}, \tilde{d})$ defined in (17), we can write the equivalent system of (27) as follows:

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, 0 < x < 1, t > 0, \\ w(0,t) = w_x(0,t) = 0, t \geq 0, \\ w_{xxx}(1,t) = 0, \\ w_{xx}(1,t) = -kw_{xt}(1,t) - \tilde{d}(t), k = k_1 + k_2, \\ \dot{\tilde{y}}(t) = -r(t)\tilde{y}(t) + r(t)\tilde{d}(t)\varepsilon(t) + \dfrac{\dot{r}(t)}{r(t)}\tilde{y}(t), \\ \dfrac{d}{dt}\left(\tilde{d}(t)\varepsilon(t)\right) = -r(t)\tilde{y}(t) - \dfrac{d}{dt}\left(d(t)\varepsilon(t)\right). \end{cases} \tag{28}$$

The convergence of the "ODE part" in (28) has been proven in (16). We need only show the convergence of the "w part" of system (28), that is,

$$\lim_{t\to\infty} E(t) = 0,\ E(t) = \int_0^1 [w_{xx}^2(x,t) + w_t^2(x,t)]dx. \tag{29}$$

This is because if (29) is true, then it follows from (24) that

$$\hat{y}(t) = \tfrac{y(t)}{r(t)} + \tilde{y}(t) = \tfrac{\varepsilon(t)}{r(t)} \int_0^1 [k_1 w_{xx}(x,t) + w_t(x,t)x^2/2]dx$$
$$+\tilde{y}(t) \to 0 \text{ as } t \to \infty.$$

Now we prove (29). Similar to (8), we can write the "w part" of (28) in the first order form

$$\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathcal{A}_k \begin{pmatrix} w \\ w_t \end{pmatrix} + \mathcal{B}\tilde{d} \text{ in } D(\mathcal{A}_k), \tag{30}$$

$$\begin{cases} \mathcal{A}_k(f,g)^\top = (g, -f^{(4)})^\top,\ \forall (f,g)^\top \in D(\mathcal{A}_k) \\ D(\mathcal{A}_k) = \{(f,g)^\top \in \mathcal{H} \cap (H^4(0,1) \times H_e^2(0,1)) | \\ \qquad f'''(1) = 0, f''(1) = -kg'(1)\}, \\ \mathcal{B} = (0, -\delta'(x-1))^\top. \end{cases} \tag{31}$$

Owing to (16), for any given $\sigma > 0$, we may suppose that $|\tilde{d}(t)| \leq \sigma$ for all $t > t_0$ for some $t_0 > 0$. The admissibility of $\mathcal{B}$ proved in Proposition 2.1 implies that, $\forall\ t > 0$,

$$\left\| \int_0^t e^{\mathcal{A}_k(t-s)}\mathcal{B}\tilde{d}(s)ds \right\|_{\mathcal{H}}^2 \leq C_t \|\tilde{d}\|_{L^2(0,t)}^2 \leq C_t t^2 \|\tilde{d}\|_{L^\infty(0,\infty)}^2,$$

for some constant C_t that is independent of $\tilde{d}$. On the other hand, under the assumption of the theorem, it is known that $e^{\mathcal{A}_k t}$ is exponentially stable ([2]). By Remark 2.6 of [9], we have

$$\left\| \int_{t_0}^t e^{\mathcal{A}_k(t-s)}\mathcal{B}\tilde{d}(s)ds \right\|_{\mathcal{H}} \leq \left\| \int_0^t e^{\mathcal{A}_k(t-s)}\mathcal{B}(0 \diamond_{t_0} \tilde{d})(s)ds \right\|_{\mathcal{H}}$$
$$\leq L\|\tilde{d}\|_{L^\infty(t_0,\infty)} \leq L\sigma,$$

where L is a constant that is independent of $\tilde{d}$, and

$$(u \diamond_\tau v)(t) = \begin{cases} u(t), 0 \leq t \leq \tau, \\ v(t), t > \tau. \end{cases}$$

Suppose that $\|e^{\mathcal{A}_k t}\| \leq L_0 e^{-wt}$ for some $L_0, w > 0$. Then we have

$$\left\| \begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} \right\|_{\mathcal{H}} \leq L_0 e^{-wt} \left\| \begin{pmatrix} w(\cdot,0) \\ w_t(\cdot,0) \end{pmatrix} \right\|_{\mathcal{H}} \tag{32}$$
$$+L_0 C_{t_0} t_0^2 e^{-\omega(t-t_0)} \|\tilde{d}\|_{L^\infty(0,t_0)} + L\sigma.$$

Passing to the limit as $t \to \infty$ for (32), we finally obtain

$$\lim_{t\to\infty} \left\| \begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} \right\|_{\mathcal{H}} \leq L\sigma. \tag{33}$$

This proves (29). The proof is complete.

■

It is noted that our disturbance is quite general and by (2.1), $\dot{d}$ is allowed to grow exponentially at any growth rate by choosing properly the time gain function r. Certainly, this is not unconditional by the limitation of r. Signals like $d(t) = \sin(e^{e^t})$ are excluded.

III. SLIDING MODE CONTROL APPROACH

In this section, we use the SMC to reject the disturbance by removing the condition that the derivative $\dot{d}$ of disturbance must satisfy (15) in ADRC approach. That is, in this section, the disturbance is assumed to satisfy $|d(t)| \leq M$ for all $t \geq 0$ for some $M > 0$ only.

Consider the following sliding surface

$$S = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H} : f'(1) + 2\int_0^1 m(x)g(x)dx \right.$$
$$\left. -4\int_0^1 m(x)f(x)dx = 0 \right\}.$$

which is a closed-subspace in $\mathcal{H}$, where m satisfies

$$\begin{cases} m''''(x) + 4m(x) = 0 \\ m(0) = m'(0) = m'''(1) = 0, \\ m''(1) = 1, m'(1)(\approx 0.8485) > 0. \end{cases} \tag{34}$$

The existence of the solution of (34) is guaranteed by a simple computation (or by Maple Soft).

The corresponding sliding mode function is designed then as

$$S(t) = w_x(1,t) + 2\int_0^1 m(x)w_t(x,t)dx - 4\int_0^1 m(x)w(x,t)dx. \tag{35}$$

On the sliding surface $S(t) = 0$, the system (1) becomes

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, 0 < x < 1, t > 0, \\ w(0,t) = w_x(0,t) = 0, \\ w_{xxx}(1,t) = 0, \\ w_x(1,t) = -2\displaystyle\int_0^1 m(x)w_t(x,t)dx + 4\int_0^1 m(x)w(x,t)dx, \end{cases} \tag{36}$$

It is a trivial exercise to show that on the sliding surface S, system (36) associates with a C_0-semigroup of contractions solution, which is displayed by the dissipativity of the following function $E_0(t)$:

$$E_0(t)=\frac{1}{2}\int_0^1 w_t^2(x,t)+w_{xx}^2(x,t)dx.$$

Actually, differentiating E_0 along the solution of system (36) gives

$$\dot{E}_0(t)=w_{xx}(1,t)w_{xt}(1,t)=-2m'(1)w_{xx}^2(1,t)\le 0,$$

where the fact $w_{xt}(1,t)=-2m'(1)w_{xx}(1,t)$ in the sliding surface was used. Moreover, it is shown, by the Riesz base method in [2], that system (36) is exponentially stable in S owing to $w_{xt}(1,t)=-2m'(1)w_{xx}(1,t)$ in the sliding surface.

Next we seek the "reaching condition" by designing the sliding mode feedback. Differentiating the sliding surface function formally gives

$$\begin{aligned}
&\dot{S}(t)\\
&=w_{xt}(1,t)+2\int_0^1 m(x)w_{tt}(x,t)dx-4\int_0^1 m(x)w_t(x,t)dx\\
&=w_{xt}(1,t)+2m'(1)w_{xx}(1,t)-2S(t)\\
&=w_{xt}(1,t)+2m'(1)(u(t)+d(t))-2S(t).
\end{aligned}$$

Design the feedback control law:

$$u(t)=\frac{1}{2m'(1)}(-w_{xt}(1,t)+2S(t))-(M+\eta)\text{sign}(S(t)). \tag{37}$$

Then

$$S(t)\dot{S}(t)\le -2m'(1)\eta|S(t)|, \tag{38}$$

which is just the "reaching condition" owing to $-2m'(1)<0$. However, we do not know if $\dot{S}$ always exists which is remarkably different to the lumped parameter systems where the derivative is always the classical derivative.

Under state feedback (37), the closed-loop system of (1) reads

$$\begin{cases}
w_{tt}(x,t)+w_{xxxx}(x,t)=0, 0<x<1, t>0,\\
w(0,t)=w_x(0,t)=0,\\
w_{xxx}(1,t)=0,\\
w_{xx}(1,t)=\dfrac{1}{2m'(1)}(-w_{xt}(1,t)+2S(t))\\
\qquad\qquad -(M+\eta)sign(S(t))+d(t)\\
\qquad\quad =\dfrac{1}{2m'(1)}(-w_{xt}(1,t)+2S(t))+\tilde{d}(t),\\
w(x,0)=w_0(x), w_t(x,0)=w_1(x),
\end{cases} \tag{39}$$

where

$$\tilde{d}(t)=-(M+\eta)\text{sign}(S(t))+d(t).$$

Define the operator $\mathbb{A}$ as follows:

$$\begin{cases}
\mathbb{A}(f,g)^\top=(g,-f^{(4)})^\top,\ \forall (f,g)^\top\in D(\mathbb{A})\\
D(\mathbb{A})=\big\{(f,g)^\top\in\mathcal{H}\cap(H^4(0,1)\times H_e^2(0,1))|f'''(1)=0,\\
\quad f''(1)=\dfrac{1}{2m'(1)}\left(-g'(1)+2f'(1)+4\displaystyle\int_0^1 m(x)g(x)dx\right.\\
\quad\left.-8\displaystyle\int_0^1 m(x)f(x)dx\right)\Big\}.
\end{cases} \tag{40}$$

Then system (39) can be written as

$$\frac{d}{dt}\begin{pmatrix} w\\ w_t\end{pmatrix}=\mathbb{A}\begin{pmatrix} w\\ w_t\end{pmatrix}+\mathbb{B}\tilde{d}(t),\ \mathbb{B}=(0,-\delta'(x-1))^\top. \tag{41}$$

Proposition 3.1: The operator $\mathbb{A}$ defined in (40) generates a C_0-semigroup on $\mathcal{H}$ and $\mathbb{B}$ defined in (41) is admissible to the semigroup $e^{\mathbb{A}t}$ generated by $\mathbb{A}$. Therefore, for any initial value $(w(\cdot,0),\dot{w}(\cdot,0))^\top\in\mathcal{H}$ and the disturbance $\tilde{d}\in L^2_{loc}(0,\infty)$, (39) admits a unique solution $(w,\dot{w})^\top\in\mathcal{H}$.

Proof: We first show the first assertion. Actually, for any $(f,g)^\top\in D(\mathbb{A})$, we have

$$\begin{aligned}
&\text{Re}\langle\mathbb{A}(f,g)^\top,(f,g)^\top\rangle=\text{Re}f''(1)\overline{g'(1)}\\
&=-\frac{|g'(1)|^2}{2m'(1)}+\frac{1}{2m'(1)}\text{Re}\left(2f'(1)+4\int_0^1 m(x)g(x)dx\right.\\
&\quad\left.-8\int_0^1 m(x)f(x)dx\right)\overline{g'(1)}\\
&\le\frac{1}{2m'(1)}\left|f'(1)+2\int_0^1 m(x)g(x)dx-4\int_0^1 m(x)f(x)dx\right|^2\\
&\le\frac{3}{2m'(1)}\left(\int_0^1|f''(x)|^2dx+4\int_0^1 m^2(x)dx\int_0^1|g(x)|^2dx\right.\\
&\quad\left.+16\int_0^1 m^2(x)dx\int_0^1|f(x)|^2dx\right)\\
&\le L_1\left(\int_0^1|f''(x)|^2dx+\int_0^1|g(x)|^2dx\right)=L_1\|(f,g)^\top\|^2,
\end{aligned} \tag{42}$$

where L_1 is a positive constant and the fact $\max\{|f'(x)|^2,|f(x)|^2\}\le\int_0^1|f''(x)|^2dx$ was used.

A direct computation shows that $\mathbb{A}^*$, the adjoint of $\mathbb{A}$, is given by

$$\begin{cases}
\mathbb{A}^*\begin{pmatrix} f(x)\\ g(x)\end{pmatrix}=\begin{pmatrix} -g(x)+\frac{g'(1)}{m'(1)}m(x)\\ f^{(4)}(x)+2\frac{g'(1)}{m'(1)}m(x)\end{pmatrix},\\
D(\mathbb{A}^*)=\left\{\begin{pmatrix} f\\ g\end{pmatrix}\in\mathcal{H}\cap(H^4(0,1)\times H_e^2(0,1))|\right.\\
\qquad\qquad\left. f'''(1)=0, f''(1)=\dfrac{g'(1)}{2m'(1)}\right\}.
\end{cases} \tag{43}$$

Then for any $(f(x), g(x))^\top \in D(\mathbb{A}^*)$,

$$
\begin{aligned}
&\mathrm{Re}\langle \mathbb{A}^*(f,g)^\top, (f,g)^\top\rangle \\
&= -\frac{|g'(1)|^2}{2m'(1)} + \frac{1}{2m'(1)}\mathrm{Re}\left(2f'(1) + 4\int_0^1 m(x)g(x)dx\right. \\
&\qquad \left. -8\int_0^1 m(x)f(x)dx\right)\overline{g'(1)} \\
&\le \frac{1}{2m'(1)}\left|f'(1) + 2\int_0^1 m(x)g(x)dx - 4\int_0^1 m(x)f(x)dx\right|^2 \\
&\le L_1\|(f,g)\top\|^2.
\end{aligned}
\tag{44}
$$

It follows from (42) and (44) that, for any $M_1 \ge L_1$, $\mathbb{A} - M_1$ and $\mathbb{A}^* - M_1$ are dissipative in $\mathcal{H}$. By Lemma 3.2 below $\mathbb{A}$ is a closed operator. This together with Corollary 4.4 of [8, p.15] shows that $\mathbb{A} - M_1$ generates a C_0-semigroup of contractions on $\mathcal{H}$. Therefore, $\mathbb{A}$ generates a C_0-semigroup on $\mathcal{H}$.

Now show the second assertion. For any $(\varphi,\psi)^\top \in \mathcal{H}$, find the $(f,g)^\top \in D(\mathbb{A}^*)$ such that

$$(\mathbb{A}^* - 2M_2^2)(f,g)^\top = (\varphi,\psi)^\top,$$

where $M_2 > 0$ is a constant. Then from the definition of $\mathbb{A}^*$ and the boundary conditions in (43), we can obtain

$$g(x) = -2M_2^2 f(x) + \frac{g'(1)}{m'(1)}m(x) - \varphi(x),$$

and

$$
\begin{aligned}
&f(x) = C_5\sin(M_2x)\sinh(M_2x) + C_6(\sin(M_2x)\cosh(M_2x) \\
&-\cos(M_2x)\sinh(M_2x)) + \frac{1}{4M_2^3}\int_0^x(\sin(M_2(x-y)) \\
&\cdot\cosh(M_2(x-y)) - \cos(M_2(x-y))\sinh(M_2(x-y)))h(y)dy,
\end{aligned}
$$

where

$$h(x) = \psi + (2M_2^2-2)g'(1)m'(1)^{-1}m(x) - 2M_2^2\varphi(x).$$

For simplicity, we can simply choose $2M_2^2 > M_1$ such that $\cos(M_2) = 1$ and $\sin(M_2) = 0$. Taking the boundary condition in (43) and $f'(1) = -\frac{\varphi'(1)}{2M_2^2}$ into account, we can obtain

$$C_5 = \frac{\Delta_1}{\Delta},\ C_6 = \frac{\Delta_2}{\Delta},\ g'(1) = \frac{\Delta_3}{\Delta},$$

where

$$\Delta_1 = \begin{vmatrix} -b_1 & 0 & a_1 \\ -b_2 & M_2^2(\sinh(M_2)+\cosh(M_2)) & a_2 \\ -b_3 & 2M_2^3(\sinh(M_2)+\cosh(M_2)) & a_3 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} M_2\sinh(M_2) & -b_1 & a_1 \\ 2M_2^2\cosh(M_2) & -b_2 & a_2 \\ 2M_2^3\sinh(M_2) & -b_3 & a_3 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} M_2\sinh(M_2) & 0 & -b_1 \\ 2M_2^2\cosh(M_2) & M_2^2(\sinh(M_2)+\cosh(M_2)) & -b_2 \\ 2M_2^3\sinh(M_2) & 2M_2^3(\sinh(M_2)+\cosh(M_2)) & -b_3 \end{vmatrix},$$

and

$$\Delta = \begin{vmatrix} M_2\sinh(M_2) & 0 & a_1 \\ 2M_2^2\cosh(M_2) & M_2^2(\sinh(M_2)+\cosh(M_2)) & a_2 \\ 2M_2^3\sinh(M_2) & 2M_2^3(\sinh(M_2)+\cosh(M_2)) & a_3 \end{vmatrix},$$

where $a_1, a_2, a_3,\ b_1, b_2, b_3$ are constants as follows:

$$
\begin{aligned}
a_1 &= \frac{(2M_2^2-2)}{4M_2^2m'(1)}\int_0^1[\sin M_2(1-y)\cosh M_2(1-y) \\
&\quad -\cos M_2(1-y)\sinh M_2(1-y)]m(y)dy, \\
a_2 &= \frac{(2M_2^2-2)}{4M_2m'(1)}\int_0^1[\cos M_2(1-y)\cosh M_2(1-y) \\
&\quad +\cos M_2(1-y)\sinh M_2(1-y) \\
&\quad +\sin M_2(1-y)\sinh M_2(1-y) \\
&\quad +\sin M_2(1-y)\cosh M_2(1-y)]m(y)dy - \frac{1}{2m'(1)}, \\
a_3 &= \frac{(2M_2^2-2)}{2m'(1)}\int_0^1[\cos M_2(1-y)\sinh M_2(1-y) \\
&\quad +\cos M_2(1-y)\cosh M_2(1-y)]m(y)dy,
\end{aligned}
$$

and

$$
\begin{aligned}
b_1 &= \frac{1}{4M_2^2}\int_0^1[\sin M_2(1-y)\cosh M_2(1-y)- \\
&\cos M_2(1-y)\sinh M_2(1-y)](\psi(y) - 2M_2^2\varphi(y))dy + \frac{\varphi'(1)}{2M_2^2}, \\
b_2 &= \frac{1}{4M_2}\int_0^1[\cos M_2(1-y)\cosh M_2(1-y) \\
&\quad +\cos M_2(1-y)\sinh M_2(1-y) \\
&\quad +\sin M_2(1-y)\sinh M_2(1-y) \\
&\quad +\sin M_2(1-y)\cosh M_2(1-y)](\psi(y) - 2M_2^2\varphi(y))dy, \\
b_3 &= \frac{1}{2}\int_0^1[\cos M_2(1-y)\sinh M_2(1-y) \\
&\quad +\cos M_2(1-y)\cosh M_2(1-y)](\psi(y) - 2M_2^2\varphi(y))dy.
\end{aligned}
$$

Therefore, $(\mathbb{A}^* - 2M_2^2)^{-1}$ exists and is bounded, and $\mathbb{B}^*(\mathbb{A}^* - 2M_2^2)^{-1}(\varphi,\psi)^\top = g'(1)$ which is bounded from $\mathcal{H}$ to $\mathbb{C}$.

Furthermore, we consider the dual system:

$$\frac{d}{dt}\begin{pmatrix} p(\cdot,t) \\ q(\cdot,t) \end{pmatrix} = \mathbb{A}^*\begin{pmatrix} p(\cdot,t) \\ q(\cdot,t) \end{pmatrix}$$

That is,

$$\begin{cases} p_t(x,t) = -q(x,t) + \frac{q_x(1,t)}{m'(1)} m(x), \\ q_t(x,t) = p_{xxxx}(x,t) + 2\dfrac{q_x(1,t)}{m'(1)} m(x), \\ p(0,t) = p_x(0,t) = q(0,t) = q_x(0,t) = 0, \\ p_{xxx}(1,t) = 0, p_{xx}(1,t) = \dfrac{q_x(1,t)}{2m'(1)} \end{cases} \tag{45}$$

Define

$$F(t) = \frac{1}{2}\int_0^1 [p_{xx}^2(x,t) + q^2(x,t)].$$

Since $\mathbb{A}$ generates a C_0-semigroup on $\mathcal{H}$, and so does $\mathbb{A}^*$, there exist constants ω, M_ω such that $F(t) \leq M_\omega e^{\omega t} F(0)$ for all $t \geq 0$. Finding the derivative of F along the solution of (45) gives

$$\begin{aligned} F'(t) \quad &= -\tfrac{q_x^2(1,t)}{2m'(1)} \\ &+ \frac{q_x(1,t)}{m'(1)} \int_0^1 [k''(x) p_{xx}(x,t) + 2m(x) q(x,t)] dx, \end{aligned}$$

which implies

$$\begin{aligned} & q_x^2(1,t) \\ & \leq -2m'(1)F'(t) + \frac{1}{2} q_x^2(1,t) \\ & \quad + 2\left(\int_0^1 [m''(x) p_{xx}(x,t) + 2m(x) q(x,t)] dx\right)^2 \\ & \leq -2m'(1)F'(t) + \frac{1}{2} q_x^2(1,t) + 4\int_0^1 m''^2(x) dx \int_0^1 p_{xx}^2(x,t) dx \\ & \quad + 16\int_0^1 m^2(x) dx \int_0^1 q^2(x,t) dx. \end{aligned}$$

Therefore, for any given $T > 0$, we obtain

$$\begin{aligned} \int_0^1 q_x^2(1,t) dt & \leq -4m'(1)(F(T) - F(0)) \\ & \quad + 8\int_0^1 m''^2(x) dx \int_0^T \int_0^1 p_{xx}^2(x,t) dx dt \\ & \quad + 32\int_0^1 m^2(x) dx \int_0^T \int_0^1 q^2(x,t) dx dt \\ & \leq C_T F(0) \end{aligned}$$

where C_T is independent of $F(0)$. This fact together with the boundedness of $\mathbb{B}^*(\mathbb{A}^* - 2M_2^2)^{-1}$ shows the admissibility of $\mathbb{B}$([9]). ■

We are now in a position to show the main result of this section.

Theorem 3.1: Suppose that the disturbance d is bounded measurable and $S(t)$ is defined by (35). Then for any $(w(\cdot,0), w_t(\cdot,0))^\top \in \mathcal{H}$, $S(0) \neq 0$, there exists a $t_0 \geq 0$ such that (39) admits a unique solution $(w, w_t)^\top \in C(0,t_0;\mathcal{H})$ and $S(t) = 0$ for all $t \geq t_0$. Moreover, $S(t)$ is continuous, monotone in $[0,t_0]$. On the sliding surface $S(t) = 0$, the system (1) becomes (36) which is exponentially stable.

Proof: We need only prove that $S(t)$ is continuous, monotone in $[0,t_0]$. Suppose without lost of generality that $S(0) > 0$ since the proof for $S(0) < 0$ is similar.

Since by Proposition 3.1, $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$, the solution to system (39) can be written as

$$\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} = e^{\mathbb{A}t}\begin{pmatrix} w(\cdot,0) \\ w_t(\cdot,0) \end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)} \mathbb{B}\tilde{d}(s) ds, \tag{46}$$

which means, for any $(f,g)^\top \in D(\mathbb{A}^*)$, that

$$\begin{aligned} & \frac{d}{dt}\int_0^1 w_{xx}(x,t) f''(x) + w_t(x,t) g(x) dx \\ & = \int_0^1 w_{xx}(x,t)\left(-g(x) + \frac{g'(1)}{m'(1)} m(x)\right) \\ & \qquad + w_t(x,t)\left(f^{(4)} + 2\frac{g'(1)}{m'(1)} m(x)) dx + g'(1)\right)\tilde{d}(t) \end{aligned} \tag{47}$$

Substitute $(f,g)^\top = (m(x), 2m(x))\top \in D(\mathbb{A}^*)$, where m is defined in (34), into (47) to obtain

$$\begin{aligned} & \frac{d}{dt}\left(w_x(1,t) + 2\int_0^1 m(x) w_t(x,t) dx\right. \\ & \qquad \left. -4\int_0^1 m(x) w(x,t) dx\right) = 2m'(1)\tilde{d}(t), \end{aligned}$$

which is just

$$\dot{S}(t) = 2m'(1)\tilde{d}(t), \text{ for } S(t) \neq 0.$$

This shows that S is continuous in the interval where $S \neq 0$. Moreover, (38) holds true. Therefore, there exists a $t_0 > 0$ such that $S(t)$ is monotone in $[0,t_0]$ and $S(t) = 0$ for all $t > t_0$. In particular, if $S(0) = 0$, then $t_0 = 0$. This completes the proof. ■

IV. NUMERICAL SIMULATION

In this section, we present some numerical simulations for illustration. The purpose are twofold. The first is to verify the theoretical results of both ADRC and SMC and the second is to look at the peaking value reduction by the time varying gain approach in ADRC. We apply the finite difference method to compute the displacement for system (1). For numerical computations, here we take parameter $k = 2$, $d = \sin t$. We also take the grid size $N = 20$ for x, the time step $dt = 0.0001$, and the initial values as follows:

$$w(x,0) = -3x + 1.5; \; w_t(x,0) = 2x - 1. \tag{48}$$

$\varepsilon(t)$ defined in (11) is taken as

$$\varepsilon(t) = \frac{1}{1+t}, \tag{49}$$

and the time varying gain function r is taken as

$$r(t) = \begin{cases} e^{5t}, \; t \leq \dfrac{\log(30)}{5}, \\ 30, \; t > \dfrac{\log(30)}{5}. \end{cases} \tag{50}$$

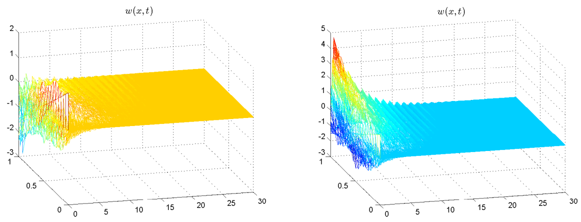
It is seen that r grows slowly from the small value in the beginning to its maximum value $r = 30$ which is used as the constant gain in our numerical simulations.

Figures 1(a) and 1(b) display the displacement w under the controller by ADRC with the time varying gain and constant gain, respectively. It is clearly seen the convergence of w. In addition, from these figures, we see that the peaking value from Figure 1(b) is dramatically reduced by the time varying gain in Figure 1(a).

Figure 2 plots the tracking errors for the disturbance where Figure 2(a) is with the time varying gain (50) and Figure 2(b) is with the constant gain $r = 30$. It is clearly seen from these figures that the peaking value from Figure 2(b) is dramatically reduced by the time varying gain in Figure 2(a). This is the most advantage of the application of the time varying gain compared with the constant gain in existing literature [6], [7], [4]. This is also a remarkable property of the ADRC in dealing with the disturbance. We actually do not need much high gain for the convergence due to the nature of estimation/cacellation in ADRC although it is difficult to prove this fact theoretically.

Figure 3 shows the behavior of the feedback control law u with time under both time varying gain and constant gain. It is seen that the control is much less by time varying than constant gain. This is another advantage of the time varying gain. The price is that the convergence is slightly slowly which is seen from figure 2.

Figure 4 displays the displacement w by SMC and the control u with time. It is clearly seen the convergence for w. The chattering phenomena is also observed from Figure 3(b). Compared with Figure 2, we see that ADRC takes advantage of much less energy over SMC.



(a) The displacement w with the time varying gain (b) The displacement w with the constant gain

Fig. 1. The displacement w with both time varying gain and constant gain.

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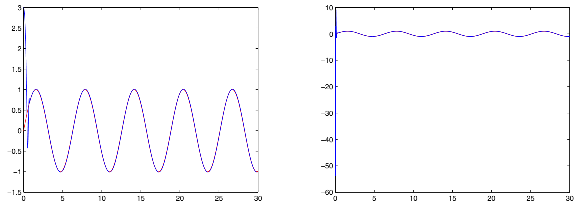
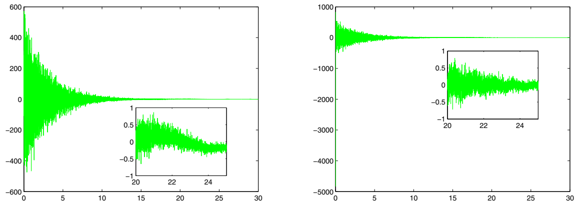


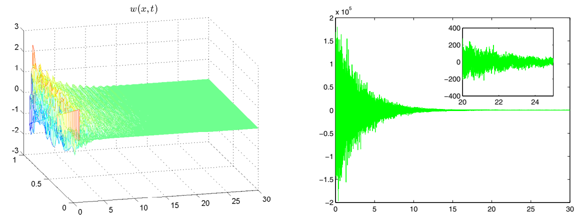
(a) d (red), $\hat{d}$ (blue) (b) d (red), $\hat{d}$ (blue)

Fig. 2. (a): Tracking of disturbance with $d(t) = \sin(t)$ by time varying high gain; (b): Tracking of disturbance with $d(t) = \sin(t)$ by constant high gain.



(a) The control law u by time varying gain (b) The control law u by constant high gain

Fig. 3. The feedback control law u with both time varying gain and constant gain by ADRC.



(a) The displacement w by SMC (b) The control law u by SMC

Fig. 4. (a): The displacement w by SMC; (b): The control law u by SMC.

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