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A Douglas-Shapiro-Shields factorization approach to the Leech equation

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***Abstract*— In a recent series of papers concrete procedures were derived to compute a stable rational matrix solution to the Leech equation with rational matrix data. In one of these papers the procedure involved a specific Douglas-Shapiro-Shields (DSS) factorization. In the present paper it is shown that one can take a class of DSS factorizations as the starting point of the procedure, leading to solutions which may have a lower McMillan degree as the one obtained in the original procedure.**

I. Introduction

Let G and K be stable rational complex-valued matrix functions of sizes $m \times p$ and $m \times q$, respectively. Here stable means that G and K have no poles in the closed unit disc $\overline{\mathbb{D}}$, where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the open unit disc. In particular, G and K are rational matrix H^∞ functions. We denote this as $G \in \mathfrak{R}H^\infty_{m\times p}$ and $K \in \mathfrak{R}H^\infty_{m\times q}$, where $\mathfrak{R}$ stands for rational. A $p \times q$ matrix-valued H^∞ function X is called a *contractive analytic solution to* $GX = K$ if

$$G(z)X(z) = K(z) \quad (z \in \mathbb{D}) \quad \text{and} \quad \|X\|_\infty \le 1. \tag{1}$$

Here $\|X\|_\infty = \sup_{z\in\mathbb{D}} \|X(z)\|$ denotes the supremum norm of X on $\mathbb{D}$. A result by R.W. Leech [18], cf., [10, Section VIII.6] and [19, p. 107], yields that a contractive analytic solution to $GX = K$ exists if and only if the function

$$L(z,w) = \frac{G(z)G(w)^* - K(z)K(w)^*}{1 - z\overline{w}} \quad (z, w \in \mathbb{D}) \tag{2}$$

is a positive kernel on $\mathbb{D}\times\mathbb{D}$, that is, for any finite sequence $z_1, \ldots, z_n \in \mathbb{D}$ the block operator matrix $[L(z_i, z_j)]_{i,j=1,\ldots,n}$ defines a positive operator on the Hilbert space direct sum of n copies of $\mathbb{C}^m$. As one might expect, but which is not immediately obvious, if a contractive analytic solution to $GX = K$ exists, then there also exists a rational matrix solution, i.e., an $X \in \mathfrak{R}H^\infty_{p\times q}$ satisfying (1). This was proved in [21] via a reduction to the case of matrix polynomials and in [17] without using this reduction step. Both papers present a method to construct a solution. The paper [13] provides a state space procedure that parallels the construction of [17]. Independently of the constructions of [21], [17] and [13], in [14] it is shown that the maximum entropy solution in the suboptimal case is rational, and a state space representation for this solution is presented.

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The special case of Leech's theorem with $q = m$ and K identically equal to the $m \times m$ identity matrix I_m ($K \equiv I_m$) is part of the corona theorem, which is due to Carlson [6], for $m = 1$, and Fuhrmann [15] for arbitrary m. This special case appears more frequently in the engineering literature; see [23], [22] and the references therein for an engineering perspective and related applications in signal processing. The paper [20] provides a procedure for the construction of a solution for the case of polynomial data and $m = 1$, which can be seen as a forerunner for [21].

The approaches of [21] and [17] rely on the so-called "lurking isometry" approach from [2], which was used in [3] to derive solutions to the Leech equation (1) for functions on the polydisc. The first step in this approach is to make a factorization $L(z,w) = \Lambda(z)\Lambda(w)^*$ of the positive kernel (2) with Λ a function on $\mathbb{D}$ whose values are bounded linear operators mapping some Hilbert space $\mathcal{H}$ into $\mathbb{C}^m$. Such a factorization is referred to as a *Kolmogorov decomposition* in the literature, cf., Sections 1.3 and 1.5 in [7]. If $\dim \mathcal{H} < \infty$, we refer to the factorization $L = \Lambda\Lambda^*$ as a *finite Kolmogorov decomposition*, and in this case the lurking isometry approach yields a rational contractive analytic solution to $GX = K$, following Procedure 1.2 below with F given by $F(z) = 0$, $z \in \mathbb{D}$. However, it rarely happens that one can obtain a finite Kolmogorov decomposition of L. The following result, proved as Theorem 3.2 in [17], makes this precise.

Theorem 1.1: Let $G \in \mathfrak{R}H^\infty_{m\times p}$ and $K \in \mathfrak{R}H^\infty_{m\times q}$ such that L in (2) is a positive kernel. Then there exists a finite Kolmogorov decomposition $L = \Lambda\Lambda^*$ of L if and only if $G(e^{it})G(e^{it})^* = K(e^{it})K(e^{it})^*$ for any $t \in [0, 2\pi]$.

A corollary of this result is that in the corona case ($K \equiv I_m$) finite Kolmogorov decompositions only occurs if G is a co-isometric constant [17, Corollary 3.4].

The idea behind the approach of [17], [13] is the following: Find a function $F \in \mathfrak{R}H^\infty_{m\times r}$, for some positive integer r, such that

$$\widetilde{L}(z,w) = \frac{G(z)G(w)^* - K(z)K(w)^* - F(z)F(w)^*}{1 - z\overline{w}} \tag{3}$$

defines a positive kernel on $\mathbb{D}\times\mathbb{D}$ that admits a finite Kolmogorov decomposition. Once such a function F is found, a rational contractive analytic solution to $GX = K$ is obtained by the following procedure. We prove the claims in the procedure in Section III.

Procedure 1.2: Assume $F \in \mathfrak{R}H^\infty_{m\times r}$ is such that $\widetilde{L}$ in (3) is a positive kernel that admits a finite Kolmogorov decomposition $\widetilde{L}(z,w) = \widetilde{\Lambda}(z)\widetilde{\Lambda}(w)^*$, $z, w \in \mathbb{D}$, say $\widetilde{\Lambda}(z) \in \mathbb{C}^{m\times k}$ for each $z \in \mathbb{D}$.

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(i) The factorization $\widetilde{L}(z,w) = \widetilde{\Lambda}(z)\widetilde{\Lambda}(w)^*$ implies there exists a partial isometry $M \in \mathbb{C}^{(k+p)\times(k+q+p)}$ such that for every $z \in \mathbb{D}$

$$\begin{bmatrix} z\widetilde{\Lambda}(z) & G(z) \end{bmatrix} M = \begin{bmatrix} \widetilde{\Lambda}(z) & K(z) & F(z) \end{bmatrix},$$

with

$$\operatorname{Ker} M^\perp = \overline{\operatorname{span}} \left(\bigcup_{z\in\mathbb{D}} \operatorname{Im} \begin{bmatrix} \widetilde{\Lambda}(z)^* \\ K(z)^* \\ F(z)^* \end{bmatrix} \right),$$
$$\operatorname{Im} M = \overline{\operatorname{span}} \left(\bigcup_{z\in\mathbb{D}} \operatorname{Im} \begin{bmatrix} \bar{z}\widetilde{\Lambda}(z)^* \\ G(z)^* \end{bmatrix} \right).$$

(ii) Decompose M as

$$M = \begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2 \end{bmatrix} : \begin{bmatrix} \mathbb{C}^k \\ \mathbb{C}^q \\ \mathbb{C}^r \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^k \\ \mathbb{C}^p \end{bmatrix}.$$

Then $X(z) = D_1 + zC(I - zA)^{-1}B_1$, $z \in \mathbb{D}$, defines a rational contractive analytic solution X to $GX = K$. In addition we obtain that $Y(z) = D_2 + zC(I - zA)^{-1}B_2$, $z \in \mathbb{D}$, defines a rational contractive analytic solution Y to $GY = F$.

The main result of [17], [13] provide a construction of a function $F \in \Re H^\infty_{m\times r}$ such that $\widetilde{L}$ in (3) is a positive kernel that admits a finite Kolmogorov decomposition. We shall briefly sketch the procedure from [17] in Procedure 1.3 below. The focus of the current paper is on extending the procedure of [17] in such a way that we obtain a class of functions $F \in \Re H^\infty_{m\times r}$ to choose from. Making a 'better' choice for F can lead to a finite Kolmogorov decomposition $\widetilde{L} = \widetilde{\Lambda}\widetilde{\Lambda}^*$ of $\widetilde{L}$ such that $\widetilde{\Lambda}$ acts on a space of lower dimension, and consequently, the rational solution X determined in Step (viii) may have a lower McMillan degree.

In order to explain the construction of [17], we require a bit more notation. In addition to $\Re H^\infty_{k\times l}$, we use $\Re L^\infty_{k\times l}$ to indicate the rational matrix L^∞ functions of size $k \times l$. With $H^\infty_{k\times l}$ and $L^\infty_{k\times l}$ we denote the spaces of $k\times l$ matrix H^∞ and L^∞ functions, respectively. Given a $k \times l$ matrix function V, the symbol V^* stands for the $l \times k$ matrix function defined by $V^*(z) = V(1/\bar{z})^*$ for each $z \in \mathbb{C}$ such that $V(1/\bar{z})$ is defined. Note that $V \in L^\infty_{k\times l}$ implies $V^* \in L^\infty_{l\times k}$, but $V \in H^\infty_{k\times l}$ only implies $V^* \in H^\infty_{l\times k}$ if V is constant. The function V^* is a rational matrix function if and only if V is a rational matrix function, in contrast with the function $z \mapsto V(z)^*$. We write L^2_k for the Hilbert space of vector-valued square integrable functions on the unit circle $\mathbb{T}$ of size k and H^2_k for the Hardy space of analytic vector-valued functions on $\mathbb{D}$, of size k, which extend a.e. to a function in L^2_k on $\mathbb{T}$. The orthogonal complement of H^2_k in L^2_k is denoted by K^2_k and we write P_+ and P_- for the orthogonal projections of L^2_k on H^2_k and K^2_k, respectively. The McMillan degree of a rational matrix function V is denoted by $\delta(V)$. For a function $V \in L^\infty_{k\times l}$ we define the Hankel operator $H_V : K^2_l \to H^2_k$ and Toeplitz operator $T_V : H^2_l \to H^2_k$ by

$$H_V g = P_+ V g \quad (g \in K^2_l) \quad \text{and} \quad T_V f = P_+ V f \quad (f \in H^2_l).$$

Procedure 1.3: Let $G \in \Re H^\infty_{m\times p}$ and $K \in \Re H^\infty_{m\times q}$ be such that L in (2) is a positive kernel.

(i) Define $R = GG^* - KK^* \in \Re L^\infty_{m\times m}$. The fact that L in (2) is a positive kernel implies $R(e^{it}) \geq 0$ for every $t \in [0, 2\pi]$.

(ii) Let Φ be an outer spectral factor of R, i.e., $\Phi \in \Re H^\infty_{r\times m}$ for some $r \leq m$,

$$R = \Phi^*\Phi \quad \text{and} \quad \overline{\Phi H^2_m} = H^2_r.$$

(iii) Define

$$\mathcal{M}_\Phi = \{ f \in H^2_r \mid T^*_\Phi f \in (\operatorname{Im} H_G + \operatorname{Im} H_K)\}. \quad (4)$$

Then $\mathcal{M}_\Phi$ is a finite dimensional backward shift-invariant subspace of H^2_r.

(iv) Let $\Theta \in H^\infty_{r\times r}$ be inner such that $\mathcal{M}_\Phi^\perp = \Theta H^2_r$, which exists by the Beurling-Lax theorem.

(v) Define $F = \Phi^*\Theta$. Then $F \in \Re H^\infty_{m\times r}$ and $\widetilde{L}$ in (3) defined a positive kernel on $\mathbb{D}\times\mathbb{D}$ which admits a finite Kolmogorov decomposition $\widetilde{L}(z,w) = \widetilde{\Lambda}(z)\widetilde{\Lambda}(w)^*$, $z, w \in \mathbb{D}$, say $\Lambda(z) \in \mathbb{C}^{m\times k}$. This is possible with $k \leq \delta([G\ K])$, where $\delta([G\ K])$ is the McMillan degree of the rational matrix function $[G(z)\ K(z)]$, see Part (i) of [13, Theorem 1.1].

Note that the outer spectral factor Φ is uniquely determined by G and K, up to multiplication with a unitary $r \times r$ matrix on the left. Likewise, the inner function Θ is uniquely determined by $\mathcal{M}_\Phi$, up to multiplication with a unitary $r \times r$ matrix on the right, the function F is uniquely determined by Θ and Φ. Hence, apart from orthonormal transformations of the basis of $\mathbb{C}^r$ and $\mathbb{C}^k$, the freedom in this procedure is only in the choice of $\mathcal{M}_\Phi$.

One can further derive the following relations between the McMillan degrees of the function constructed in this procedure [17], [13]:

$$\delta(\Phi) \leq \delta(F) = \delta(\Theta) = \dim \mathcal{M}_\Phi \leq \delta[G\ K] \geq \delta(X),$$
$$\text{and } \delta(X) \leq \dim \mathcal{M}_\Phi \text{ if } H_G H_G^* - H_K H_K^* \geq 0.$$

In the corona case ($K \equiv I_m$) $H_K = 0$, so that we obtain $\delta(X) \leq \dim \mathcal{M}_\Phi$.

Given a state space realization of the rational matrix function $[G\ K] \in H^\infty_{m\times(p+q)}$, it is possible to compute state space realizations for all functions appearing in this procedure, expressed in terms of the matrices appearing in the realization of $[G\ K]$, see [13].

Note that Θ being a square inner function implies Θ is two-sided inner, which yields

$$R = FF^* \quad \text{and} \quad \Phi = \Theta F^*. \quad (5)$$

The first identity shows that $GG^* - KK^* = FF^*$, and thus $GG^* - KK^* - FF^* = 0$. Using Theorem 1.1, this implies that the positive kernel $\widetilde{L}$ in (3) indeed admits a finite Kolmogorov decomposition $\widetilde{L} = \widetilde{\Lambda}\widetilde{\Lambda}^*$. In fact, the first identity in (5) turns out to be one of two conditions on F that are necessary and sufficient for $\widetilde{L}$ in (3) to be a positive kernel with a finite Kolmogorov decomposition.

Proposition 1.4: Let $F \in \Re H^{\infty}_{m\times s}$ for some positive integer s. Then $\widetilde{L}$ in (3) is a positive kernel which admits a finite Kolmogorov decomposition if and only if

$$FF^* = R \quad \text{and} \quad H_F H_F^* \geq H_G H_G^* - H_K H_K^*, \qquad (6)$$

where $R = GG^* - KK^*$.

A proof of Proposition 1.4 is given in Section III. If $F \in \Re H^{\infty}_{m\times s}$ satisfies (6), then necessarily $s \geq r$, where r is as in item (ii) of Procedure 1.3. In the sequel we take s equal to this r.

In general it is not so clear how to compute the functions $F \in \Re H^{\infty}_{m\times r}$ that satisfy (6). Therefore we restrict to a subclass of functions F that meet (6) and which still contains the function F determined by Procedure 1.3. Example 4.1 below shows that this subclass can be strict.

Assume $G \in \Re H^{\infty}_{m\times p}$ and $K \in \Re H^{\infty}_{m\times q}$ are such that the Leech equation (1) admits a solution. Then, by item (i) of Procedure 1.3, the $m \times m$ rational matrix function $R = GG^* - KK^*$ is non-negative on the unit circle. In what follows Φ is the outer spectral factor of R, as in item (ii) of Procedure 1.3.

Definition 1.5: A finite dimensional subspace $\mathcal{M}$ of H^2_r will be called Φ-*admissible* or simply *admissible* if $\mathcal{M}$ has the following properties:

(C1) $\mathcal{M}$ is invariant under the backward shift on H^2_r;
(C2) $\operatorname{Im} H_\Phi \subset \mathcal{M}$;
(C3) $(T_\Phi^*)^{-1}[\operatorname{Im}(H_G H_G^* - H_K H_K^*)] \subset \mathcal{M}$.

The left hand side of the inclusion in (C3) indicates the inverse image of $\operatorname{Im}(H_G H_G^* - H_K H_K^*)$ under T_Φ^*.

Note that the space $\mathcal{M}_\Phi$ defined in item (iii) of Procedure 1.3 is Φ-admissible. Many other admissible subspaces exist. Indeed, given an admissible subspace $\mathcal{M}$, condition (C1) tells us that there exists a unique rational two-sided inner function $\Theta \in \Re H^{\infty}_{r\times r}$ such that $\mathcal{M} = (\Theta H^2_r)^\perp$. If Ξ is another two-sided inner function in $\Re H^2_{r\times r}$, then $\mathcal{M} \oplus \operatorname{Im} T_\Theta H_\Xi$ is admissible as well.

Condition (C2) is equivalent to the requirement that the function $F = \Phi^*\Theta$ is in $\Re H^{\infty}_{m\times r}$. The latter follows from the fact that $\mathcal{M} = \operatorname{Ker} T_{\Theta^*}$ and $T_{\Theta^*} H_\Phi = H_{\Theta^*\Phi} = H_{F^*}$, see (7) below. In the sequel we shall refer to F as the *function determined by the* Φ*-admissible subspace* $\mathcal{M}$. Condition (C3) allows us to prove the following result.

Theorem 1.6: Let $\mathcal{M}$ be a Φ-admissible subspace, and let F be the function determined by $\mathcal{M}$. Then (6) holds, and hence $\widetilde{L}$ in (3) is a positive kernel which admits a finite Kolmogorov decomposition $\widetilde{L} = \widetilde{\Lambda}\widetilde{\Lambda}^*$ on $\mathbb{D} \times \mathbb{D}$, with $\widetilde{\Lambda}(z) \in \mathbb{C}^{m\times k}$. Furthermore, $k \leq \delta([G\ K])$, and if $H_G H_G^* - H_K H_K^* \geq 0$, which is the case in the corona problem, then $k \leq \dim \mathcal{M}$.

The inequality $k \leq \dim \mathcal{M}$ in the above theorem implies that the degree of the rational contractive analytic solution constructed using this admissible subspace $\mathcal{M}$ is bounded by the dimension of $\mathcal{M}$. Hence it is of interest to take $\mathcal{M}$ such that its dimension is as small as possible. We shall prove Theorem 1.6 in Section III.

DSS factorizations. Let F be the function determined by a Φ-admissible subspace $\mathcal{M}$. The fact that Θ is two-sided inner allows us to rewrite the identity $F = \Phi^*\Theta$ as $\Phi = \Theta F^*$. Since $F \in \Re H^{\infty}_{m\times r}$, this implies that $\Phi = \Theta F^*$ is a so-called Douglas-Shapiro-Shields (DSS) factorization of Φ; see Section II for the definition and basic properties of DSS factorizations. More information on DSS factorizations can be found in Sections 4.6 and 4.7 of [12], including an efficient way to compute DSS factorizations.

Conversely, let $\Phi = \Theta F^*$ be a DSS factorization of Φ. Put $\mathcal{M} = (\Theta H^2_r)^\perp$. Then (C1) and (C2) are satisfied. However, it may happen that condition (C3) is not satisfied. In particular, for the canonical DSS factorization $\Phi = \Theta_0 F_0^*$ of Φ, the corresponding subspace $\mathcal{M}_0 = \operatorname{Im} H_\Phi$ may not be Φ-admissible. In Section IV we present two examples where this phenomenon occurs, one in which the canonical DSS factorization does lead to a positive kernel $\widetilde{L}$, and one where this in general does not happen. There does exist a minimal admissible subspace $\mathcal{M}_{min}$, both with respect to inclusion and dimension. The two examples show that the minimal admissible subspace $\mathcal{M}_{min}$ need not correspond to the canonical DSS factorization. However, we prove in Proposition 4.4 that in the case of the square corona problem, the minimal admissible subspace $\mathcal{M}_{min}$ always equals $\operatorname{Im} H_\Phi$, and hence corresponds to the canonical DSS factorization of Φ.

Besides the current introduction, this paper consists of 3 sections. Section II contains preliminary results on DSS factorizations. We prove Proposition 1.4, the claims of Procedure 1.2 and Theorem 1.6 in Section III. In the final section we look at the class of DSS factorizations that meet the constraints of Theorem 1.6 and provide a few examples.

We conclude this introduction with some further notation and terminology, and some elementary preliminaries that will be used throughout the paper. Let $U \in H^{\infty}_{n\times p}$, $V \in H^{\infty}_{m\times p}$ and $W \in H^{\infty}_{m\times q}$. Then the following useful identities apply (cf., [5, Proposition 2.14]):

$$\begin{aligned} &T_{V^*W} = T_V^* T_W, \quad T_{UV^*} = T_U T_V^* + H_U H_V^*, \\ &H_{V^*W} = T_V^* H_W, \quad H_{UV^*} = H_U T_{V^t}^*. \end{aligned} \qquad (7)$$

Here for any matrix function Y, the function Y^t is defined by $Y^t(z) = Y(1/z)$ for any $z \in \mathbb{D}$ for which $Y(1/z)$ exists. Furthermore, we have

$$T_{U^*} = T_U^*, \qquad H_{U^*} = H_U^*,$$
$$\|T_U\| = \|U\|_\infty \quad \text{and} \quad \delta(U) = \operatorname{rank}(H_U),$$

with $\operatorname{rank}(H_U) < \infty$ if and only if U is rational. The function U is inner if and only if T_U is an isometry; the function V is outer if and only if $\operatorname{Im} T_V$ is dense in H^2_m.

II. DSS FACTORIZATIONS

In this section we review some facts about Douglas-Shapiro-Shields (DSS) factorizations. The concept of DSS factorization originates from the paper [9]. The case of matrix-valued functions was studied by Fuhrmann in [16]. In the present paper we follow the treatment in Sections 4.7

and 4.8 of [12]. We shall restrict to the case of rational matrix functions.

A *Douglas-Shapiro-Shields (DSS) factorization* of a function $V \in \Re L^\infty_{r\times m}$ is a factorization of the form

$$V(e^{it}) = \Theta(e^{it})F^*(e^{it}) \quad (t \in [0, 2\pi))\ a.e., \tag{8}$$

with $F \in \Re H^\infty_{m\times r}$ and $\Theta \in \Re H^\infty_{r\times r}$ a two-sided inner function, i.e., $\Theta\Theta^* = \Theta^*\Theta$ is identically equal to I_r, the $r \times r$ identity matrix. A DSS factorization $V = \Theta F^*$ of V is called *canonical* in case the only common right inner factor between Θ and F is a unitary constant $r \times r$ matrix. Hence a canonical DSS factorization of V is unique up to multiplication with a unitary $r\times r$ matrix. With some abuse of terminology, we will refer to the (rather than a) canonical DSS factorization of V, and indicate the functions in the canonical DSS factorization by Θ_0 and F_0.

Theorem 2.1: Let $V \in \Re L^\infty_{r\times m}$ and $\Theta \in \Re H^\infty_{r\times r}$ two-sided inner. Set $\mathcal{M} = (\Theta H^2_r)^\perp \subset H^2_r$ and $F = V^*\Theta$. Then $V = \Theta F^*$ is a DSS factorization of V if and only if $\mathrm{Im}\, H_V \subset \mathcal{M}$. Moreover, $V = \Theta F^*$ is the canonical DSS factorization of V if and only if $\mathcal{M} = \mathrm{Im}\, H_V$.

Proof: Since Θ is two-sided inner, we have $V = \Theta F^*$. Hence it remains to show that $\mathrm{Im}\, H_V \subset \mathcal{M}$ is equivalent to $F \in \Re H^\infty_{m\times r}$. By the third identity in (7) we have

$$H_{F^*} = H_{\Theta^* V} = T^*_\Theta H_V.$$

Now, $F \in \Re H^\infty_{m\times r}$ if and only if $H_{F^*} = 0$, or equivalently $T^*_\Theta H_V = 0$. The latter identity is satisfied precisely when $\mathrm{Im}\, H_V \subset \mathrm{Ker}\, T^*_\Theta = \mathcal{M}$. Due to the lattice properties of (two-sided) inner functions, see Theorem 2.2 below, a DSS factorization $V = \Theta F^*$ of V is canonical if and only if $\mathcal{M} := (\Theta H^2_r)^\perp$ is the smallest subspace in H^2_r of this form (ranging over all DSS factorizations of V). Since $\mathrm{Im}\, H_V$ is backward shift invariant, it is obvious that this corresponds to the case $\mathcal{M} = \mathrm{Im}\, H_V$. ■

One can also take a finite dimensional subspace $\mathcal{M} \subset H^2_r$ as a starting point. Besides $\mathrm{Im}\, H_V \subset \mathcal{M}$ one then also needs the assumption that $\mathcal{M}$ is S^*_r-invariant, which by the Beurling-Lax theorem determines Θ uniquely up to a unitary transformation of $\mathbb{C}^r$. In the context of the outer spectral factor Φ in Procedure 1.3, these are exactly conditions (C1) and (C2).

The lattice properties of the set of two-sided rational inner functions in $H^2_{r\times r}$ yield the following structure in the set of DSS factorizations, cf., Subsection 3.1.1 in [12].

Theorem 2.2: Let $V \in \Re L^\infty_{r\times m}$. For any DSS factorization $V = \Theta F^*$ of V there exists a two-sided inner function $\Xi \in \Re H^\infty_{r\times r}$ such that

$$\Theta = \Theta_0 \Xi \quad \text{and} \quad F = F_0 \Xi,$$

and Ξ is uniquely determined by Θ and F up to multiplication with a constant unitary $r \times r$ matrix on the right. Moreover, if $V = \Theta_j F^*_j$, $j = 1, 2$, are two DSS factorizations of V, then there exists a DSS factorization $V = \Theta F^*$ of V and two two-sided inner functions Ξ_1 and Ξ_2 in $\Re H^\infty_{r\times r}$ such that

$$\Theta_j = \Theta \Xi_j \quad \text{and} \quad F_j = F\Xi_j, \quad j = 1, 2,$$
$$(\Theta H^2_r)^\perp = (\Theta_1 H^2_r)^\perp \cap (\Theta_2 H^2_r)^\perp,$$

and the latter condition determines Θ and F unique up to multiplication with a constant unitary $r \times r$ matrix on the right.

We conclude this section with some properties of the function F in case V is an outer function. Recall that $V \in H^\infty_{m\times m}$ is called *invertible outer* if $V(z)$ is invertible for each $z \in \mathbb{D}$ and $z \mapsto V(z)^{-1}$ is in $H^\infty_{m\times m}$.

Lemma 2.3: Let $V = \Theta F^*$ be a DSS factorization of an outer function $V \in \Re H^\infty_{r\times m}$. Set $\mathcal{M} = (\Theta H^2_r)^\perp$. Then

$$\mathrm{Ker}\, T_F = \{0\} \quad \text{and} \quad \mathcal{M} = (T^*_V)^{-1}[\mathrm{Im}\, H_F]. \tag{9}$$

Moreover, we have

$$\delta(V) \le \delta(\Theta) = \delta(F) = \dim \mathcal{M},$$

and $\delta(V) = \dim \mathcal{M}$ holds if and only if $V = \Theta F^*$ is the canonical DSS factorization of V. Furthermore, if V is invertible outer, then F is invertible in $L^\infty_{m\times m}$ and all Fourier coefficients of $z \mapsto F(e^{it})^{-1}$ with positive index are equal to 0.

Proof: Most of the results in this lemma are classical; proofs are added for the sake of completeness.

We have $F = V^*\Theta$. Since both V and Θ are matrix H^∞ functions, we obtain from (7) that

$$T_F = T^*_V T_\Theta \quad \text{and} \quad H_F = T^*_V H_\Theta.$$

Since Θ is two-sided inner we have $\mathrm{Ker}\, T_\Theta = \{0\}$ and $\mathrm{Im}\, H_\Theta = (\mathrm{Im}\, T_\Theta)^\perp = \mathcal{M}$. The fact that $\mathrm{Ker}\, T^*_V = \{0\}$, because V is outer, then yields (9). Again using $\mathrm{Im}\, H_\Theta = \mathcal{M}$ and $H_F = T^*_V H_\Theta$, together with $\mathrm{Ker}\, T^*_V = \{0\}$, gives

$$\delta(\Theta) = \mathrm{rank}\, H_\Theta = \dim \mathcal{M} = \mathrm{rank}\, H_F = \delta(F).$$

Since $\mathrm{Im}\, H_V \subset \mathcal{M}$ and $V \in \Re H^\infty_{r\times m}$, we obtain

$$\delta(V) = \mathrm{rank}\,(H_V) = \dim(\mathrm{Im}\, H_V) \le \dim \mathcal{M}.$$

This also shows that $\delta(V) = \dim \mathcal{M}$ holds if and only if $\mathrm{Im}\, H_V = \mathcal{M}$, that is, if and only if $V = \Theta F^*$ is the canonical DSS factorization of V.

Now assume V is invertible outer, and thus $r = m$. Write V^{-1} for the function $z \mapsto V(z)^{-1}$ in $H^\infty_{m\times m}$. Then both V and Θ are invertible in $L^\infty_{m\times m}$ and thus $F = V^*\Theta$ is invertible in $L^\infty_{m\times m}$, with inverse $F^{-1} = \Theta^{-1}(V^*)^{-1} = \Theta^*(V^{-1})^*$. Since the Fourier coefficients with positive index of Θ^* and $(V^{-1})^*$ are all 0, the same is true for F^{-1}. ■

III. Proofs of Proposition 1.4, Procedure 1.2 and Theorem 1.6

In this section we prove Proposition 1.4, the claims of Procedure 1.2 and Theorem 1.6. We first rephrase the positivity of the kernels L and $\widetilde{L}$ in terms of Toeplitz operators and show how a (finite) Kolmogorov decomposition can be obtained from this. Positivity of the kernel functions L and $\widetilde{L}$ is equivalent to positivity of the operators $T_G T_G^* - T_K T_K^*$ and $T_G T_G^* - T_K T_K^* - T_F T_F^*$, respectively, cf., page 107 in [19]. Assume $T_G T_G^* - T_K T_K^* - T_F T_F^* \geq 0$ and set $k = \mathrm{rank}\,(T_G T_G^* - T_K T_K^* - T_F T_F^*)$, with possibly $k = \infty$. Then we can factor

$$T_G T_G^* - T_K T_K^* - T_F T_F^* = \widetilde{\Delta}\widetilde{\Delta}^* \text{ with } \widetilde{\Delta} : \mathbb{C}^k \to H_m^2.$$

(Here $\mathbb{C}^k$ is to be interpreted as a separable Hilbert space in case $k = \infty$.) The Kolomogorov decomposition $\widetilde{L} = \widetilde{\Lambda}\widetilde{\Lambda}^*$ of $\widetilde{L}$ can then be achieved with a $\widetilde{\Lambda}$ having values in $\mathbb{C}^{m\times k}$, namely with $\widetilde{\Lambda}(z)x = (\widetilde{\Delta}x)(z)$, $z \in \mathbb{D}$, $x \in \mathbb{C}^k$. In a similar way one can obtain a Kolmogorov decomposition of L from a factorization of $T_G T_G^* - T_K T_K^*$.

Proof of Proposition 1.4: As before we set $R = GG^* - KK^*$. Using the second identity of (7) we obtain

$$T_R = T_G T_G^* - T_K T_K^* + H_G H_G^* - H_K H_K^*.$$

Assume (6) holds. Then $R = FF^*$ yields

$$T_R = T_F T_F^* + H_F H_F^*.$$

This shows that

$$T_G T_G^* - T_K T_K^* - T_F T_F^* = H_F H_F^* + H_K H_K^* - H_G H_G^*. \quad (10)$$

Hence $H_F H_F^* \geq H_G H_G^* - H_K H_K^*$ implies the positivity of $T_G T_G^* - T_K T_K^* - T_F T_F^*$ and hence the positivity of the kernel $\widetilde{L}$ in (3). Note that $\widetilde{L}$ in (3) is of the form of L in (2) when K is replaced by $\widetilde{K} = [K\ F]$. Since $FF^* = R = GG^* = KK^*$, we have $GG^* = \widetilde{K}\widetilde{K}^*$. Applying Theorem 1.1 with K replaced by $\widetilde{K}$ then shows $\widetilde{L}$ admits a finite Kolmogorov decomposition.

Conversely, assume the kernel $\widetilde{L}$ in (3) admits a finite Kolmogorov decomposition. Then $\widetilde{L}$ is a positive kernel and we obtain from Theorem 1.1, again with K replaced by $\widetilde{K} = [K\ F]$, that $GG^* = \widetilde{K}\widetilde{K}^* = KK^* + FF^*$. Hence $FF^* = GG^* - KK^* = R$. In particular, identity (10) holds. Now, positivity of the kernel $\widetilde{L}$ implies positivity of $T_G T_G^* - T_K T_K^* - T_F T_F^*$ and thus, by (10), we have $H_F H_F^* \geq H_G H_G^* - H_K H_K^*$. ■

Proof of claims in Procedure 1.2: Assume $\widetilde{L}$ in (3) is a positive kernel which admits a finite Kolmogorov decomposition $\widetilde{L} = \widetilde{\Lambda}\widetilde{\Lambda}^*$. This yields

$$\begin{aligned} G(z)G(w)^* - K(z)K(w)^* - F(z)F(w)^* &= \\ &= (1 - z\bar{w})\widetilde{\Lambda}(z)\widetilde{\Lambda}(w)^* \quad (z, w \in \mathbb{D}), \end{aligned}$$

in other words for any $z, w \in \mathbb{D}$

$$\begin{aligned} &\begin{bmatrix} z\widetilde{\Lambda}(z) & G(z) \end{bmatrix} \begin{bmatrix} \bar{w}\widetilde{\Lambda}(w)^* \\ G(w)^* \end{bmatrix} = \\ &\qquad = \begin{bmatrix} \widetilde{\Lambda}(z) & K(z) & F(z) \end{bmatrix} \begin{bmatrix} \widetilde{\Lambda}(w)^* \\ K(w)^* \\ F(w)^* \end{bmatrix}. \end{aligned}$$

In particular, for any $z_1, \ldots, z_n \in \mathbb{D}$, where n is an arbitrary positive integer, and any vector $x \in \mathbb{C}^m$ this gives

$$\begin{aligned} &\left\| \sum_{j=1}^n \begin{bmatrix} \bar{z}_j\widetilde{\Lambda}(z_j)^* \\ G(z_j)^* \end{bmatrix} x \right\|^2 = \\ &\quad = \left\langle \sum_{j=1}^n \begin{bmatrix} \bar{z}_j\widetilde{\Lambda}(z_j)^* \\ G(z_j)^* \end{bmatrix} x, \sum_{j=1}^n \begin{bmatrix} \bar{z}_j\widetilde{\Lambda}(z_j)^* \\ G(z_j)^* \end{bmatrix} x \right\rangle = \\ &\quad = \left\langle \sum_{j=1}^n \begin{bmatrix} \widetilde{\Lambda}(z_j)^* \\ K(z_j)^* \\ F(z_j)^* \end{bmatrix} x, \begin{bmatrix} \widetilde{\Lambda}(z_j)^* \\ K(z_j)^* \\ F(z_j)^* \end{bmatrix} x \right\rangle = \\ &\quad = \left\| \sum_{j=1}^n \begin{bmatrix} \widetilde{\Lambda}(z_j)^* \\ K(z_j)^* \\ F(z_j)^* \end{bmatrix} x \right\|^2. \end{aligned}$$

As a result, we can define a partial isometry $M \in \mathbb{C}^{(k+p)\times(k+q+r)}$, with $\mathrm{Ker}\, M$ and $\mathrm{Im}\, M$ as in Step (i) of Procedure 1.2, such that

$$\begin{bmatrix} z\widetilde{\Lambda}(z) & G(z) \end{bmatrix} M = \begin{bmatrix} \widetilde{\Lambda}(z) & K(z) & F(z) \end{bmatrix}.$$

Note that this identity after extension by linearity and continuity, together with the specification of $\mathrm{Ker}\, M$ and $\mathrm{Im}\, M$, determines M uniquely.

Decompose M as in Step (ii). Then for and $z \in \mathbb{D}$

$$\begin{aligned} z\widetilde{\Lambda}(z)A + G(z)C &= \widetilde{\Lambda}(z), \\ z\widetilde{\Lambda}(z)B_1 + G(z)D_1 &= K(z), \\ z\widetilde{\Lambda}(z)B_2 + G(z)D_2 &= F(z). \end{aligned}$$

Since M is a partial isometry, it is a contraction, which implies the matrix A is a contraction. Hence $I - zA$ is invertible. We can then solve for $\widetilde{\Lambda}(z)$ in the first identity, leading to

$$\widetilde{\Lambda}(z) = G(z)C(I - zA)^{-1}.$$

Inserting this into the second and third identities gives

$$\begin{aligned} G(z)(C(I - zA)^{-1}B_1 + D_1) &= K(z), \\ G(z)(C(I - zA)^{-1}B_2 + D_2) &= F(z). \end{aligned}$$

Hence $GX = K$ and $GY = F$, with X and Y as defined in Step (ii). The basic theory of contractive realizations implies that $\|[X\,Y]\|_\infty \leq 1$, in particular $\|X\|_\infty \leq 1$ and $\|Y\|_\infty \leq 1$. Hence X and Y are contractive analytic solutions to $GX = K$ and $GY = F$, respectively. ■

Proof of Theorem 1.6: Let $\mathcal{M}$ be a Φ-admissible subspace of H_r^2, let F be the function defined by $\mathcal{M}$ and $\Theta \in \mathfrak{R}H_{r\times r}^\infty$ the two-sided inner function such that $\mathcal{M} = (\Theta H_r^2)^\perp$. The fact that Θ is inner implies

$$R = \Phi^*\Phi = F\Theta^*\Theta F^* = FF^*.$$

Hence, by Proposition 1.4, the kernel $\widetilde{\Lambda}$ is positive and admits a finite Kolmogorov decomposition if and only if $H_F H_F^* \geq H_G H_G^* - H_K H_K^*$. Since Θ is two-sided inner, the orthogonal projection $P_\mathcal{M}$ on $\mathcal{M}$ is given by $P_\mathcal{M} = H_\Theta H_\Theta^*$. Moreover, $F = \Phi^*\Theta$ gives $H_F = T_\Phi^* H_\Theta$, so that

$$H_F H_F^* = T_\Phi^* H_\Theta H_\Theta^* T_\Phi = T_\Phi^* P_\mathcal{M} T_\Phi.$$

Hence we need to show that

$$T_\Phi^* P_\mathcal{M} T_\Phi \geq H_G H_G^* - H_K H_K^*.$$

Note that

$$\begin{aligned} T_\Phi^* T_\Phi = T_R &= T_G T_G^* - T_K T_K^* + H_G H_G^* - H_K H_K^* \\ &\geq H_G H_G^* - H_K H_K^*, \end{aligned}$$

since L being a positive kernel implies $T_G T_G^* - T_K T_K^* \geq 0$. Now set $\mathcal{N} = \operatorname{Im}(H_G H_G^* - H_K H_K^*)$. Then condition (C3) implies that T_Φ^* admits a block decomposition of the form

$$T_\Phi^* = \begin{bmatrix} A & 0 \\ * & * \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{N} \\ \mathcal{N}^\perp \end{bmatrix}$$

with $A = P_\mathcal{N} T_\Phi^* P_\mathcal{M}$, $P_\mathcal{N}$ and $P_\mathcal{M}$ the projections on $\mathcal{N}$ and $\mathcal{M}$, respectively, and the $*$'s indicating operators that are irrelevant for the remainder of the proof. Then

$$\begin{aligned} H_G H_G^* - H_K H_K^* \leq T_\Phi^* T_\Phi &= \begin{bmatrix} A & 0 \\ * & * \end{bmatrix} \begin{bmatrix} A^* & * \\ 0 & * \end{bmatrix} \\ &= \begin{bmatrix} AA^* & * \\ * & * \end{bmatrix}. \end{aligned}$$

Since $H_G H_G^* - H_K H_K^*$ acts on $\mathcal{N}$, it follows that

$$H_G H_G^* - H_K H_K^* \leq \begin{bmatrix} AA^* & 0 \\ 0 & 0 \end{bmatrix} = T_\Phi^* P_\mathcal{M} T_\Phi.$$

Hence we can conclude that $\widetilde{L}$ is a positive kernel that admits a finite Kolmogorov decomposition.

Since

$$H_F H_F^* + H_K H_K^* - H_G H_G^* = T_G T_G^* - T_K T_K^* - T_F T_F^* \geq 0,$$

and F, K and G are rational matrix H^∞ functions, we obtain that $T_G T_G^* - T_K T_K^* - T_F T_F^*$ has finite rank, say $\operatorname{rank}(T_G T_G^* - T_K T_K^* - T_F T_F^*) = k$, and we can factor

$$T_G T_G^* - T_K T_K^* - T_F T_F^* = \Delta\Delta^* \text{ with } \Delta : \mathbb{C}^k \to H_m^2.$$

As observed at the beginning of this section, a Kolomogorov decomposition $\widetilde{L} = \widetilde{\Lambda}\widetilde{\Lambda}^*$ of $\widetilde{L}$ can be achieved with a $\widetilde{\Lambda}$ having values in $\mathbb{C}^{m\times k}$, via $\widetilde{\Lambda}(z)x = (\Delta x)(z)$, $z \in \mathbb{D}$, $x \in \mathbb{C}^k$. It remains to prove the bounds on k. Note that $H_F H_F^* = T_\Phi^* P_\mathcal{M} T_\Phi$ together with the definition of $\mathcal{M}$ shows that $H_F H_F^* + H_K H_K^* - H_G H_G^*$ admits a block decomposition of the form

$$H_F H_F^* + H_K H_K^* - H_G H_G^* = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{N} \\ \mathcal{N}^\perp \end{bmatrix},$$

with $\mathcal{N} = \operatorname{Im}(H_G H_G^* - H_K H_K^*)$. This shows that $k = \operatorname{rank}(H_F H_F^* + H_K H_K^* - H_G H_G^*) \leq \dim \mathcal{N}$. Now

$$\begin{aligned} \dim \mathcal{N} &= \operatorname{rank}(H_G H_G^* - H_K H_K^*) \\ &\leq \operatorname{rank}(H_G H_G^* + H_K H_K^*) = \operatorname{rank}(H_{[G\ K]} H_{[G\ K]}^*) \\ &= \delta([G\ K]). \end{aligned}$$

Assume in addition that $H_G H_G^* - H_K H_K^* \geq 0$. Then

$$\begin{aligned} \operatorname{rank}(H_F H_F^* + H_K H_K^* - H_G H_G^*) &\leq \operatorname{rank}(H_F H_F^*) \\ &= \delta(F) = \dim \mathcal{M}. \end{aligned}$$

This completes the proof. ■

IV. Admissible subspaces and DSS factorization

Assume $G \in \Re H_{m\times p}^\infty$ and $K \in \Re H_{m\times q}^\infty$ are such that the Leech equation (1) admits a solution. Let Φ be the outer spectral factor of R as in item (ii) of Procedure 1.3. We define $\mathcal{F}_{adm}$ to be the set of all functions $F \in \Re H_{m\times r}^\infty$ such that F is determined by a Φ-admissible subspace (see Definition 1.5 for the definition of the latter notion). Any $F \in \mathcal{F}_{adm}$ comes from a DSS factorization of Φ. Indeed, if F is determined by an admissible subspace $\mathcal{M}$, then $\mathcal{M} = (\Theta H_r^2)^\perp$ where Θ is a 2-sided inner function and $\Phi = \Theta F^*$.

The converse is not true, that is, if $\Phi = \Theta F^*$ is a DSS factorization of Φ, then is does not follow that F is determined by an admissible subspace. The only candidate for the admissible subspace is the space $\mathcal{M}_0 = (\Theta H_r^2)^\perp$. This space satisfies conditions (C1) and (C2) in Definition 1.5 but not necessarily condition (C3). The following example, which is a continuation of the example given in Section 6 of [13], shows that this indeed happens. In this example the DSS factorization is actually a canonical one.

Example 4.1: Take

$$G(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad K(z) = \frac{z}{2}.$$

In Section 6 of [13] it was noted that for any H^∞ function τ with $\|\tau\|_\infty \leq 1$ the function

$$X(z) = \frac{z}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tau(z) \quad (z \in \mathbb{D}) \quad (11)$$

is a contractive analytic solutions to $GX = K$, and it was shown that Procedure 1.3 provides the solution with $\tau \equiv 0$, i.e., $X(z) = \frac{z}{2\sqrt{2}} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. The corresponding kernel $\widetilde{L}$ in (3) is given by

$$\widetilde{L}(z,w) = \frac{1 - \frac{z\bar{w}}{4} - \frac{3z\bar{w}}{4}}{1 - z\bar{w}} = 1.$$

Note that

$$G(z)G^*(z) - K(z)K^*(z) = 1 - \frac{z}{2} \cdot \frac{1}{2z} = 3/4.$$

Hence $R(z) = 3/4$ and $\Phi(z) = \sqrt{3}/2$. Since Φ is a non-zero constant function, we obtain that

$$\mathcal{M}_\Phi = (T_\Phi^*)^{-1}[\operatorname{Im}(H_G + H_K)] = \operatorname{Im} H_K = \mathbb{C},$$

with $\mathbb{C}$ interpreted as the constant functions in H_1^2. Hence $\Theta(z) = z$, and the function F determined by the admissible subspace $\mathcal{M}_\Phi$ is given by $F(z) = z\sqrt{3}/2$.

In this case among all possible admissible subspaces the space $\mathcal{M}_\Phi$ is the minimal one. To see this it suffices to show $\mathcal{M}_\Phi \subset \mathcal{M}$ for any admissible subspace $\mathcal{M}$. But this inclusion follows from condition (C3) and the fact that

$$\begin{aligned} (T_\Phi^*)^{-1}[\operatorname{Im}(H_G H_G^* - H_K H_K^*)] &= \operatorname{Im}(H_G H_G^* - H_K H_K^*) \\ &= \operatorname{Im}(H_K H_K^*) = \mathbb{C} \\ &= \mathcal{M}_\Phi. \end{aligned}$$

Here again we use that Φ is a non-zero constant function.

Next let us consider DSS factorizations of Φ. Since $\Phi(z) = \sqrt{3}/2$, the canonical DSS factorizations of Φ is given by $\Phi = \Theta_0 F_0^*$, where for Θ_0 and F_0 one can take $\Theta_0(z) = 1$ and $F_0(z) = \Phi(z) = \sqrt{3}/2$ for each z. It follows that $\mathcal{M}_0 = (\Theta_0 H^2)^\perp$ is the zero space, and thus $\mathcal{M}_\Phi$ is not a subset of $\mathcal{M}_0$. We conclude that in this case $\mathcal{M}_0$ is not an admissible subspace.

On the other hand, note that $F = F_0$ does lead to a positive kernel $\widetilde{L}$, as in (3), which admits a finite Kolmogorov decomposition. In fact for $F = F_0$ we have

$$\widetilde{L}(z,w) = \frac{1 - \frac{z\bar{w}}{4} - \frac{3}{4}}{1 - z\bar{w}} = \frac{1}{4}.$$

Hence we can take $\widetilde{\Lambda}(z) = 1/2$ for each z. Thus in this example the canonical DSS factorization of Φ leads to a rational solution but not via Procedure 1.3. Using Procedure 1.2 one computes that this solution is given by

$$X(z) = \frac{z}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad (12)$$

Remark 4.2: Note that the solution given by (12) is just the solution in (11) with τ is identically equal to zero. Thus it may happen that two different functions F lead to the same solution X. In other words two different positive kernels of type (3) may lead to the same solution X. We intend to return to this phenomenon in a later publication.

We give another example which shows that for any $K \in \Re H^\infty_{m\times q}$ one can construct a $G \in \Re H^\infty_{m\times p}$ for some p such that the canonical DSS factorization of the outer function Φ does not provide a function F leading to a positive kernel $\widetilde{L}$.

Example 4.3: Take $K \in \Re H^\infty_{m\times q}$ arbitrarily and $G = \begin{bmatrix} K & \widetilde{G} \end{bmatrix} \in \Re H^\infty_{m\times q}$, $q = p + q_1$, with $\widetilde{G} \in \Re H^\infty_{m\times q_1}$ such that $\widetilde{G}\widetilde{G}^*$ is identically equal to a positive $m \times m$ matrix Υ. Then $\widetilde{G} = C\Xi$ with C a $m \times q_1$ matrix with $CC^* = \Upsilon$ and $\Xi \in \Re H^\infty_{q_1\times q_1}$ two-sided inner. We have $T_G T_G^* - T_K T_K^* = T_{\widetilde{G}} T_{\widetilde{G}}^* \geq 0$, hence there exists contractive analytic solution X to $GX = K$. In fact, we can take $X(z) = \begin{bmatrix} I & 0 \end{bmatrix}$, $z \in \mathbb{D}$; with $\delta(X) = 0$ obviously a minimal McMillan degree solution. Next observe that $R = GG^* - KK^* = \widetilde{G}\widetilde{G}^*$ is the constant function with value Υ. Then the outer factor Φ of R is a constant function as well with value $\Phi_0 : \mathbb{C}^m \to \mathbb{C}^r$ such that $\Phi_0^*\Phi_0 = \Upsilon$ and $r = \operatorname{rank}\Upsilon$. In particular, we have $H_\Phi = 0$, so that any backward shift invariant subspace $\mathcal{M}$ of $\ell^2_+(\mathbb{C}^r)$ satisfies conditions (C1) and (C2).

Furthermore, $H_G H_G^* = H_K H_K^* + H_{\widetilde{G}} H_{\widetilde{G}}^*$. Hence $H_G H_G^* - H_K H_K^* = H_{\widetilde{G}} H_{\widetilde{G}}^*$. Put $\mathcal{M} = (T_\Phi^*)^{-1}\left[\operatorname{Im} H_{\widetilde{G}}\right]$. Then $\mathcal{M}$ is finite dimensional and invariant under the backward shift. It follows that $\mathcal{M} = (T_\Phi^*)^{-1}\left[\operatorname{Im} H_{\widetilde{G}}\right]$ satisfies (C1)–(C3), and it is the smallest subspace of H_r^2 with this property. Note that $\dim \operatorname{Im} H_{\widetilde{G}}$ can be much smaller that $\dim(\operatorname{Im} H_G + \operatorname{Im} H_K) = \dim(\operatorname{Im} H_{\widetilde{G}} + \operatorname{Im} H_K)$. Hence this choice of $\mathcal{M}$ may lead to a solution of smaller McMillan degree than the one obtained from Procedure 1.3. We claim that $\dim \mathcal{M} = \delta(\widetilde{G})$. To see that this is the case note that $T_\Phi = T_{\Phi_0}$, the Toeplitz operator defined by the constant function with value Φ_0. Since $\widetilde{G}(z) = C\Xi(z)$, $z \in \mathbb{D}$, we have $H_{\widetilde{G}} = T_C H_\Xi$, with T_C the Toeplitz operator defined by the constant function with value C. Now $\operatorname{Im}\Phi_0^* = \operatorname{Im}\Upsilon = \operatorname{Im} C$ implies that $\operatorname{Im} T_{\Phi_0}^* = \operatorname{Im} T_{\Phi_0^*} = \operatorname{Im} T_C$, and thus $\operatorname{Im} H_{\widetilde{G}} \subset \operatorname{Im} T_\Phi^*$. Since $\mathcal{M} = (T_\Phi^*)^{-1}[\operatorname{Im} H_{\widetilde{G}}]$, this shows that $\dim \mathcal{M} = \dim(\operatorname{Im} H_{\widetilde{G}}) = \delta(\widetilde{G})$, as claimed. In particular, $\dim \mathcal{M} = \delta(\widetilde{G}) > \delta(\Phi)$, unless $\widetilde{G}$ is constant.

Now let us consider the canonical DSS factorization $\Phi = \Theta_0 F_0^*$. As observed in Example 4.1, the fact that $\mathcal{M}_0 = \operatorname{Im} H_\Phi = \{0\}$ does not satisfy (C3), does not imply one cannot take $F = F_0$ in (3). However, doing so gives $\Theta_0 = I_r$ and thus $F_0 = \Phi^*\Theta = \Phi_0^*$. Hence

$$T_{F_0} T_{F_0}^* = T_\Upsilon = T_{\widetilde{G}\widetilde{G}^*} = T_{\widetilde{G}} T_{\widetilde{G}}^* + H_{\widetilde{G}} H_{\widetilde{G}}^*.$$

Therefore, we obtain

$$\begin{aligned} T_G T_G^* - T_K T_K^* - T_{F_0} T_{F_0}^* &= T_{\widetilde{G}} T_{\widetilde{G}}^* - T_F T_F^* \\ &= -H_{\widetilde{G}} H_{\widetilde{G}}^*. \end{aligned}$$

This shows that $T_G T_G^* - T_K T_K^* - T_{F_0} T_{F_0}^*$ is positive if and only if $H_{\widetilde{G}} H_{\widetilde{G}}^* = 0$, i.e., $\widetilde{G}$ constant. Hence, in general, the canonical DSS factorization does not lead to a positive kernel.

In connection with the two preceding examples we mention two problems:

(a) Let $F \in \Re H^\infty_{m\times r}$ and assume that the two conditions in (6) are satisfied. Does it follow that F comes from a DSS factorization of Φ? We expect the answer to be negative.
(b) Assume that $F_0 \in \Re H^\infty_{m\times r}$ comes from a canonical DSS factorization of Φ. Under which additional conditions does F satisfy the two conditions in (6).

We conclude this paper with a proposition which shows that in the special case of the corona problem where G is square, Procedure 1.3 does lead to the canonical DSS factorization.

Proposition 4.4: Let $G \in \Re H^\infty_{m\times m}$ and take $K \equiv I_m$. Assume L in (2) is a positive kernel. Let Φ be the outer spectral factor of $R = GG^* - I$. Define Θ and F as in Procedure 1.3. Then $\delta(\Phi) = \delta(G)$ and $\Phi = \Theta F^*$ is the canonical DSS factorization of Φ.

Proof: The fact that L is a positive kernel implies that $G(z)G(z)^* \geq I_m$ for all $z \in \mathbb{D}$. Since G is square, we thus have $\det G(z) \neq 0$ on $\mathbb{D}$. In fact, by the continuity of G, we have $G(z)G(z)^* \geq \frac{1}{2} I_m$ on some open neighborhood of $\overline{\mathbb{D}}$, which implies that the inequality $\det G(z) \neq 0$ extends to this open neighborhood as well. Furthermore, the domain of analyticity G also includes an open neighborhood of $\overline{\mathbb{D}}$. Hence there exists an open neighborhood $\mathcal{D}$ of $\overline{\mathbb{D}}$ on which G is analytic and $\det G$ bounded away from 0. In particular, G has no poles and no zeros on $\mathcal{D}$. See [4, Proposition 8.1], and the paragraph following its proof, and the second paragraph of Section 8.2 in [4].

Set $\mathcal{D}^* = \{z \in \mathbb{C} \cup \{\infty\} \mid 1/\bar{z} \in \mathcal{D}\}$. Then $\mathbb{C} \ominus \mathbb{D} \subset \mathcal{D}^*$ and G^* has no poles and no roots on $\mathcal{D}^*$. Similarly, G^T and G^{*T} have no poles and no zeros on $\mathcal{D}$ and $\mathcal{D}^*$, respectively, where G^T and G^{*T} are defined by $G^T(z) = G(z)^T$ and

$G^{*T} = G^*(z)^T$, $z \in \mathbb{D}$, T indicating the transpose. By Theorem 9.1 in [4], we then obtain that the factorization GG^* is minimal, i.e., the rational function $S = GG^*$ has $\delta(S) = \delta(G) + \delta(G^*) = 2\,\delta(G)$. However, we have $S = R + I$, and thus $\delta(S) = \delta(R)$. This shows that $2\delta(G) = \delta(R)$.

Next observe that $H_K = 0$ implies

$$\mathcal{M}_\Phi = (T_\Phi^*)^{-1}[\operatorname{Im} H_G].$$

Moreover, we have

$$T_\Phi^* T_\Phi = T_R = T_{GG^*-I} = T_G T_G^* - I + H_G H_G^* \geq H_G H_G^*.$$

This implies $\operatorname{Im} H_G \subset \operatorname{Im} T_\Phi^*$, and consequently, we have $\dim \mathcal{M}_\Phi = \dim(\operatorname{Im} H_G) = \delta(G)$. Hence $2 \dim \mathcal{M}_\Phi = \delta(R)$. Now $R = \Phi^*\Phi$ implies $\delta(R) \leq 2\delta(\Phi)$. Hence we obtain $\dim \mathcal{M}_\Phi \leq \delta(\Phi)$. However, we also have $\delta(\Phi) \leq \dim \mathcal{M}_\Phi$. Hence $\delta(\Phi) = \dim \mathcal{M}_\Phi$. Since $\operatorname{Im} H_\Phi \subset \mathcal{M}_\Phi$ and $\dim(\operatorname{Im} H_\Phi) = \delta(\Phi)$, we obtain that $\mathcal{M}_\Phi = \operatorname{Im} H_\Phi$, and thus that $\Phi = \Theta F^*$ is the canonical DSS factorization of Φ. ■

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