21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Variational formulation and structure-preserving discretization of nonlinear electric circuits

Sina Ober-Blöbaum[1] and Henning Lindhorst[2]

***Abstract*— The variational framework for linear electric circuits introduced in [1] is extended to general nonlinear circuits. Based on a constrained Lagrangian formulation that takes the basic circuit laws into account, the equations of motion of a nonlinear electric circuit consisting of inductors, capacitors, resistors and voltage sources are derived. The resulting differential-algebraic system can be reduced by performing the variational principle on a reduced space. It is shown under which conditions the corresponding reduced Lagrangian is regular and thus the equations of motion transform to an ordinary differential equation system. Based on a discrete variational formulation, a variational integrator for the structure-preserving simulation of nonlinear electric circuits is derived and demonstrated by numerical examples.**

I. INTRODUCTION

In this contribution, a variational formulation for nonlinear electric circuits is presented. The corresponding discrete variational formulation provides a structure-preserving simulation method for nonlinear circuits that is called *variational integrator*. Variational integrators are geometric integrators that are symplectic, momentum-preserving and have an excellent long-time energy behavior (see e.g. [2], [3]). This means that energy (for conservative systems) or the energy rate (for systems with external forces) is on average preserved in the discrete solution even for large step sizes, i.e., there is no numerical energy dissipation or generation. For an overview on variational integrators as well as their application to different system classes based on different variational formulations we refer to [2], [4] as well as to the references listed in [1]. Variational integrators have been mainly developed for mechanical systems, however, there exist extensions to the simulation of electric systems as well [1], [5]. In particular, in [1] a unified variational framework for the modeling and simulation of electric circuits is presented. The variational formulation is based on a Lagrangian consisting of magnetic minus electric energy induced by inductors and capacitors present in the circuit, on Lagrangian forces describing the influence of resistors and voltage sources, and on the interconnection constraints which are expressed in the Kirchhoff laws. Since a redundant set of coordinates is used, the Lagrangian is degenerate, that means that the Legendre transform is not invertible. The use of the *Lagrange-d'Alembert-Pontryagin principle* leads to *implicit Euler-Lagrange equations* that involve additional hidden algebraic constraints due to the degeneracy of the Lagrangian [6], [5], [7]. It is shown in [1] that, under certain topology conditions on the circuit, the degeneracy of the Lagrangian system is canceled when the system is formulated on a reduced space. Based on the reduced variational formulation, structure-preserving integrators are derived and demonstrated for linear LCR-curcuits. Whereas in [1] the focus lies on the analysis and simulation of ideal linear circuit elements that consist of inductors, capacitors, resistors, and voltage sources, in this contribution, the variational formulation and simulation is extended to general nonlinear circuits.

In Section II the variational formulations on the *full branch space* and on the *reduced mesh space* are presented and the equations of motion are derived. The main contribution is Theorem 4.1 that provides conditions for a regular (non-degenerated) reduced Lagrangian as an extension of the result in [1] to the nonlinear case. In Section III exemplary variational integrators are derived and demonstrated for the simulation of nonlinear LCR-circuits. A comparison with Runge-Kutta methods shows that a better energy behavior is obtained (cf. also [8]).

II. VARIATIONAL FORMULATION

A. Basic Notation

We introduce an electric circuit as a connected, directed graph G with the following notations: A *branch* and accordingly an *edge* is a connection between two *nodes* in which two or more elements meet. On every branch only one element can be present. A trace of adjacent elements with every element at most included once is called a *path*. A *loop* is a path in G beginning and ending at the same node. If the loop does not contain any other loops, it is also called a *fundamental loop* or *mesh*. Finally, a *planar graph* is a circuit which can be drawn on a plane without crossing branches.

We consider circuits containing n_L inductors, n_C capacitors, n_R resistors, and n_S voltage sources. We assign to each of these components real-valued, time-dependent charges $q(t)$, currents $v(t)$, voltages $u(t)$ and magnetic fluxes $\varphi(t)$ with $t \in [0,T]$. By $q_J(t), v_J(t), u_J(t), \varphi_J(t) \in \mathbb{R}^{n_J} (J \in \{L,C,R,S\})$ we denote the component related quantities. For each device we assign the direction of the voltage and current accordingly to the graph G. Note that we introduce here a set of artificial variables, namely $q_L, q_R, q_S, \varphi_C, \varphi_R$

[1]Sina Ober-Blöbaum is with Faculty of Electrical Engineering, Computer Science and Mathematics, University of Paderborn, 33098 Paderborn, Germany (on leave) and with the Faculty of Science, Technical University Dresden, 01069 Dresden, Germany sinaob@math.uni-paderborn.de

[2]Henning Lindhorst is with the Faculty of Electronic and Computer Science Engineering, Otto-von-Guerike-Universität Magdeburg, 39106 Magdeburg, Germany henning.lindhorst@ovgu.de

ISBN: 978-90-367-6321-9

and φ_S. The characteristic equations for the electric components and differential relations are shown in Table I. To make sure that the resistors are acting strictly passive, we only allow strictly monotonically increasing functions f_R with $f_R(0,t) = 0\ \forall t \in [0,T]$. We assume the inductor, capacitor, and resistor relations to be time-invariant and that the differential relations $\dot{q} = v,\ \dot{\varphi} = u$ are true for all components. The magnetic and electric energy stored in an

TABLE I
CHARACTERISTIC EQUATIONS FOR BASIC CIRCUIT ELEMENTS

device	characteristic	differential
inductor	$\varphi_L = f_L(v_L,t)$	$\frac{d}{dt}\varphi_L = u_L$
capacitor	$u_C = f_C(q_C,t)$	$\frac{d}{dt}q_C = v_C$
resistor	$u_R = f_R(v_R,t)$	-
device	voltage-controlled	time-controlled
voltage source	$u_S = f_S(v_S,t)$	$u_S = f_S(t)$

inductor and a capacitor, respectively, is

$$E_{\text{mag}}(v_L) = \int_0^{v_L} f_L(y)\,dy, \quad E_{\text{el}}(q_C) = \int_0^{q_C} f_C(y)\,dy.$$

Considering the circuit as a directed, connected graph with $n = n_L + n_C + n_R + n_S$ edges and $m+1$ nodes, we represent a component on an edge (branch) by symbols as shown in Fig. 1. For the analysis of the circuit, we introduce local

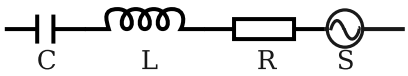

Fig. 1. A typical branch of a circuit with capacitor C, inductor L, resistor R, and voltage source S.

coordinates $q = (q_L, q_C, q_R, q_S)^T$ (resp. for v, φ, u). The two basic circuit laws that we have to take into account are:

1) The Kirchhoff Current Law (KCL) states that the sum of currents leading to and leaving from any node is equal to zero.
2) The Kirchhoff Voltage Law (KVL) states that the sum of voltages along each fundamental loop (or mesh) is equal to zero.

The KCL can be translated to an algebraic constraint $K^T v = 0$ presented by the *Kirchhoff constraint matrix* $K \in \mathbb{R}^{n,m}$ defined by

$$K_{ij} = \begin{cases} -1 & \text{branch } i \text{ connected inward to node } j \\ 1 & \text{branch } i \text{ connected outward to node } j \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The constraint matrix has only m columns as the ground node does not provide an independent constraint. The KVL is presented by the *Fundamental Loop matrix* $K_2 \in \mathbb{R}^{n,l}$, where for a connected, planar graph $l = n - m$ is the number of independent meshes. By defining the Fundamental Loop matrix by

$$K_{2,ij} = \begin{cases} -1 & \text{branch } i \text{ is a backward branch in mesh } j \\ 1 & \text{branch } i \text{ is a forward branch in mesh } j \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

the algebraic constraint resulting from the KVL is written as $K_2^T u = 0$. With coordinates as above we can express these matrices as

$$K = \begin{pmatrix} K_L \\ K_C \\ K_R \\ K_S \end{pmatrix},\ K_2 = \begin{pmatrix} K_{2,L} \\ K_{2,C} \\ K_{2,R} \\ K_{2,S} \end{pmatrix},$$

with the submatrices $K_J \in \mathbb{R}^{n_J,m},\ K_{2,J} \in \mathbb{R}^{n_J,l}, J \in \{L,C,R,S\}$. The voltages satisfying the KVL can also be expressed by the *node voltages* $\hat{u}$ as $K\hat{u} = u$. Since $u \in ker(K_2^T)$ and $u \in im(K)$ it holds $ker(K_2^T) = im(K)$ and $im(K_2) \perp im(K)$, respectively. The last statement is usually refered to as *Tellegen's Theorem*.

B. Geometric Setting

In [1] a geometric approach is introduced for the analysis of an electric circuit. The electric branch charges are assumed to be the configuration manifold called *charge space* with elements $q \in Q \subseteq \mathbb{R}^n$. The associated tangent space $T_qQ \subseteq \mathbb{R}^n$ for a given q consists of the currents v passing through the branches. The tangent bundle TQ, which is the disjoint union of the tangent spaces T_qQ, is therefore called the *current space*. The corresponding cotangent bundle T^*Q with the electromagnetic fluxes $\varphi \in T_q^*Q$ is called the *flux linkage space*. Finally, the branch voltages u are assumed to be covectors in the cotangent space $u \in T_q^*Q$, as branch voltages are the analogy with forces in mechanical systems, see [6].

The KCL and KVL can be represented by the constraint distribution $\Delta_Q \subset TQ$ and its annihilator $\Delta_Q^0(q) \subset T^*Q$, respectively, as described below (cf. [1]). With K^T as local coordinate representation for m independent one-forms w^a, the *constraint KCL space* is given by the submanifold

$$\begin{aligned} \Delta_Q(q) &= \{v \in T_qQ \mid \langle \omega^a, v\rangle = 0, a = 1,\dots,m\} \subset T_qQ \\ &= \{v \in T_qQ \mid K^T v = 0\} \subset T_qQ \end{aligned} \tag{3}$$

which includes all branch currents satisfying the KCL for a given charge.[1] Furthermore, the *constrained KVL space* is given by the annihilator Δ_Q^0 and determined by

$$\begin{aligned} \Delta_Q^0(q) &= \{\omega \in T_q^*Q \mid \langle \omega, v\rangle = 0,\ \forall v \in \Delta_Q(q)\} \subset T_q^*Q \\ &= \{u \in T_q^*Q \mid K_2^T u = 0\} \subset T_q^*Q, \end{aligned} \tag{4}$$

with the natural pairing $\langle\cdot,\cdot\rangle : T_q^*Q \times T_qQ \to \mathbb{R}$ of cotangent and tangent vectors. $\Delta_Q(q)$ and $\Delta_Q^0(q)$ are spaces of dimension l and m, embedded in T_qQ and T_q^*Q, respectively, with local representatives being in $\mathbb{R}^n$. Note that the choices of K and K_2 are in general not unique. The only design criterion for K_2 is the condition $im(K) \perp im(K_2)$. However, due to

[1] Note that since K is constant, $\Delta_Q(q)$ is integrable and thus holonomic.

its physical interpretation we choose the Fundamental Loop matrix as candidate for the matrix K_2.

We already mentioned the possibility to express branch voltages by means of node voltages. This idea can be expanded to express all quantities of the circuit in a nodal or a mesh (fundamental loop) fashion. Therefore, we introduce the *branch space* $\mathfrak{B}$ with quantities (q, v, φ, u), the *node space* $\mathfrak{N}$ with quantities $(\hat{q}, \hat{v}, \hat{\varphi}, \hat{u})$ and the *mesh space* $\mathfrak{M}$ with quantities $(\tilde{q}, \tilde{v}, \tilde{\varphi}, \tilde{u})$. Note that the mesh voltage $\tilde{u}$ and the node current $\hat{v}$ are both zero in accordance with the the KVL and KCL, respectively. For mesh and node space we define the configuration spaces $M \subseteq \mathbb{R}^l$, $N \subseteq \mathbb{R}^m$, the tangent spaces $T_{\tilde{q}}M \subseteq \mathbb{R}^l$, $T_{\hat{q}}N \subseteq \mathbb{R}^m$ and the cotangent space $T^*_{\tilde{q}}M \subseteq \mathbb{R}^l$, $T^*_{\hat{q}}N \subseteq \mathbb{R}^m$, where M is the mesh charge space and N the node charge space. Branch, mesh, and node space are defined to be the Pontryagin bundle which is the direct sum of tangent and cotangent space, that is $\mathfrak{B} = \Delta_Q \oplus \Delta_Q^0$, $\mathfrak{M} = TM \oplus T^*M$, and $\mathfrak{N} = TN \oplus T^*N$. The Fundamental Loop Matrix K_2 and the Kirchhoff Constraint Matrix K define linear relationships between these different spaces as shown in Figure 2 and discussed in [1].

$$\begin{array}{ccccc} T^*_{\hat{q}}N & \xrightarrow{K} & \Delta_Q^0(q) & \xrightarrow{K_2^T} & \{0\} \subset T^*_{\tilde{q}}M \\ \mathfrak{N} & & \mathfrak{B} & & \mathfrak{M} \\ \text{nodes} & & \text{branches} & & \text{meshes} \\ \{0\} \subset T_{\hat{q}}N & \xleftarrow{K^T} & \Delta_Q(q) & \xleftarrow{K_2} & T_{\tilde{q}}M \end{array}$$

Fig. 2. This diagram gives the relations between the different spaces in terms of the Kirchhoff constraint matrix K and the Fundamental Loop matrix K_2.

C. Constrained Variational Formulation

The Lagrangian of an electric circuit is given by the difference between magnetic and electric energy. Thus, for a circuit with n_C capacitors and n_L inductors, the Lagrangian $\mathcal{L} : TQ \to \mathbb{R}$ is defined as

$$\mathcal{L}(q, v) = \mathcal{L}(q_C, v_L) = \sum_{k=1}^{n_L} \int_0^{v_{L_k}} f_{L_k}(y)\, dy - \sum_{j=1}^{n_C} \int_0^{q_{C_j}} f_{C_j}(y)\, dy, \tag{5}$$

with the inductor relations f_{L_k} and capacitor relations f_{C_j}, see Table I. Note that the chosen Lagrangian is independent of the currents across all non-inductor elements. Therefore, the Legendre transform $\mathbb{F}\mathcal{L} : TQ \to T^*Q$, defined by

$$\mathbb{F}\mathcal{L}(q, v) = \left(q, \frac{\partial \mathcal{L}}{\partial v}(q, v)\right) = \left(q, \begin{pmatrix} f_L(v_L) \\ 0_{C,R,S} \end{pmatrix}\right) = (q, \varphi), \tag{6}$$

with the vector

$$f_L(v_L) = (f_{L_j}(v_{L_j}))_{j=1}^{n_L} \tag{7}$$

and $0_{C,R,S} \in \mathbb{R}^{n_C+n_R+n_S}$ is clearly degenerate as no relation between the artificial fluxes $\varphi_{C,R,S}$ and the existing currents $v_{C,R,S}$ is provided.[2] In particular, the Legendre transform states that all artificial fluxes are zero $\varphi_{C,R,S} = 0$. Therefore, we define the *constraint flux linkage subspace* by the Legendre transform as

$$P = \mathbb{F}\mathcal{L}(\Delta_Q) \subset T^*Q,$$

where $\Delta_Q \subset TQ$ is the distribution. The resistors and voltage sources are incorporated in the Lagrangian force $\mathfrak{f}_\mathcal{L} : TQ \to T^*Q$ as

$$\mathfrak{f}_\mathcal{L}(q, v, t) = \begin{pmatrix} 0_L \\ 0_C \\ -f_R(q, v, t) \\ f_S(q, v, t) \end{pmatrix} = \begin{pmatrix} 0_L \\ 0_C \\ -f_R(v_R) \\ f_S(v_S, t) \end{pmatrix}. \tag{8}$$

The resistor relations are negative signed, as they induce a damping force to the circuit, while the voltage sources provide energy to the circuit.

Now the Lagrange-d'Alembert-Pontryagin principle as presented in [5] is used to derive solution curves $q(t), v(t), \varphi(t)$ given as

$$\delta \int_0^T \Big\{ \mathcal{L}(q(t), v(t)) + \varphi(t) \cdot (\dot{q}(t) - v(t)) \Big\}\, dt + \int_0^T \Big\{ \mathfrak{f}_\mathcal{L}(q, v, t) \cdot \delta q \Big\}\, dt = 0. \tag{9}$$

Taking arbitrary variations $\delta v, \delta\varphi$ and constrained variations $\delta q \in \Delta_Q(q)$ with fixed initial and final point $\delta q(0) = \delta q(T) = 0$ gives the *implicit Euler-Lagrange equations* as

$$\frac{\partial \mathcal{L}}{\partial q}(q, v) - \dot{\varphi} + \mathfrak{f}_\mathcal{L}(q, v, t) \in \Delta_Q^0(q) \tag{10a}$$

$$\dot{q} = v, \quad \frac{\partial \mathcal{L}}{\partial v}(q, v) = \varphi \tag{10b}$$

$$v \in \Delta_Q(q). \tag{10c}$$

For the Lagrangian (5), the Lagrangian forces (8), the constrained KCL space (3), and the constrained KVL space (4) we obtain

$$\begin{pmatrix} 0_L \\ -f_C(q_C) \\ 0_R \\ 0_S \end{pmatrix} - \begin{pmatrix} \dot{\varphi}_L \\ \dot{\varphi}_C \\ \dot{\varphi}_R \\ \dot{\varphi}_S \end{pmatrix} + \begin{pmatrix} 0_L \\ 0_C \\ -f_R(v_R) \\ f_S(v_S, t) \end{pmatrix} = K\lambda \tag{11a}$$

$$\dot{q} = v, \quad \begin{pmatrix} f_L(v_L) \\ 0_C \\ 0_R \\ 0_S \end{pmatrix} - \begin{pmatrix} \varphi_L \\ \varphi_C \\ \varphi_R \\ \varphi_S \end{pmatrix} = 0 \tag{11b}$$

$$K^T v = 0, \tag{11c}$$

where $\lambda : [0, T] \to \mathbb{R}^m$ represents the node voltages $\hat{u} \in T^*_{\hat{q}}Q$ (cf. Section II-B). System (11) is a differential-algebraic system with differential variables q and φ and

[2] By (7) we assume that inductors are not magnetically coupled.

algebraic variables v and λ. To eliminate λ and the algebraic constraint (11c), we consider in the following paragraph a reduced formulation that is based on a constrained variational principle on a reduced state space, namely the mesh space.

D. Reduced Variational Formulation

By formulating the variational approach on the mesh space $\mathfrak{M} = TM \oplus T^*M$ (cf. II-B) we can gain multiple advantages contrary to the usage of the branch space. These are:

1) The Lagrange multipliers are eliminated and the number of variables are reduced.
2) The degeneracy of the Lagrangian can be canceled out.
3) The KCL is directly included into the Lagrange-d'Alembert-Pontryagin principle.

The last point is realized by instead of treating the KCL as extra constraint in the form $K^T v = 0$, we express currents that satisfy the KCL with the help of the mesh currents $\tilde{v} \in T_{\tilde{q}}M$ as $v = K_2\tilde{v}$ (cf. Fig. 2). Since K is constant, the KCL are integrable, that is the charges q are constrained to be in the submanifold

$$C = \{q \in Q \mid K^T q = 0\}$$

for consistent initial values $q \in C$. This means that the KCL constraints for the currents v also apply to the charges q up to a constant vector. Thus, we have $T_q C = \Delta_Q(q)$ and the branch charges q can be expressed by the mesh charges $\tilde{q} \in M \subseteq \mathbb{R}^{n-m}$ as $q = K_2\tilde{q}$. Now define the *reduced constrained Lagrangian* $\mathcal{L}^M : TM \to \mathbb{R}$ via pullback as $\mathcal{L}^M := K_2^*\mathcal{L} : TM \to \mathbb{R}$ with

$$\mathcal{L}^M(\tilde{q}, \tilde{v}) = \mathcal{L}(K_2\tilde{q}, K_2\tilde{v}). \tag{12}$$

This reads for the Lagrangian (5) as

$$\mathcal{L}^M(\tilde{q}, \tilde{v}) = \sum_{j=1}^{n_L} \int_0^{(K_2\tilde{v})_{L_j}} f_{L_j}(y)\,dy - \sum_{i=1}^{n_C} \int_0^{(K_2\tilde{q})_{C_i}} f_{C_i}(y)\,dy \tag{13}$$

with the Legendre transform

$$\mathbb{F}\mathcal{L}^M(\tilde{q}, \tilde{v}) = (\tilde{q}, K_{2,L}^T f_L(K_{2,L}\tilde{v})) = (\tilde{q}, \tilde{\varphi}). \tag{14}$$

Define the cotangent bundle T^*M via the Legendre transform as

$$\begin{aligned} T^*M &= \{(\tilde{q}, \tilde{\varphi}) \in \mathbb{R}^{l,l} \mid (\tilde{q}, \tilde{\varphi}) = \mathbb{F}\mathcal{L}^M(\tilde{q}, \tilde{v}), (\tilde{q}, \tilde{v}) \in TM\} \\ &= \{(\tilde{q}, \tilde{\varphi}) \in \mathbb{R}^{l,l} \mid (\tilde{q}, \tilde{\varphi}) = (\tilde{q}, K_2^T\varphi),\ \varphi \in P\}. \end{aligned}$$

Accordingly, we define the reduced Lagrangian force $\mathfrak{f}_{\mathcal{L}}^M \in T^*M$ as

$$\mathfrak{f}_{\mathcal{L}}^M(\tilde{q}, \tilde{v}, t) = K_2^T \mathfrak{f}_{\mathcal{L}}(K_2\tilde{q}, K_2\tilde{v}, t),$$

which reads for the Lagrangian force (8)

$$\mathfrak{f}_{\mathcal{L}}^M(\tilde{q}, \tilde{v}, t) = -K_{2,R}^T f_R(K_{2,R}\tilde{v}) + K_{2,S}^T f_S(K_{2,S}\tilde{v}, t). \tag{15}$$

For the mesh based approach the variations $\delta\tilde{q}$ are not constrained anymore and the *reduced Lagrange-d'Alembert-Pontryagin principle* then reads as

$$\delta \int_0^T \Big\{ \mathcal{L}^M(\tilde{q}, \tilde{v}) + \tilde{\varphi} \cdot \big(\dot{\tilde{q}} - \tilde{v}\big) \Big\} dt + \int_0^T \Big\{ \mathfrak{f}_{\mathcal{L}}^M(\tilde{q}, \tilde{v}, t) \cdot \delta\tilde{q} \Big\} dt = 0. \tag{16}$$

Fixing the initial and final points for $\tilde{q}(0), \tilde{q}(T)$ and taking variations $\delta\tilde{q}, \delta\tilde{v}, \delta\tilde{\varphi}$ gives the implicit Euler-Lagrange equations on the mesh space

$$\frac{\partial \mathcal{L}^M}{\partial \tilde{q}} + \mathfrak{f}_{\mathcal{L}}^M - \dot{\tilde{\varphi}} = 0, \quad \dot{\tilde{q}} = \tilde{v}, \quad \frac{\partial \mathcal{L}^M}{\partial \tilde{v}} - \tilde{\varphi} = 0. \tag{17}$$

Inserting the reduced Lagrangian (13) and the reduced Lagrangian force (15), the implicit Euler-Lagrange equations have the form

$$\begin{aligned} \dot{\tilde{\varphi}} =& - K_{2,C}^T f_C(K_{2,C}\tilde{q}) - K_{2,R}^T f_R(K_{2,R}\tilde{v}) \\ &+ K_{2,S}^T f_S(K_{2,S}\tilde{v}, t) \qquad (18a) \\ \dot{\tilde{q}} =& \tilde{v}, \quad \tilde{\varphi} = K_{2,L}^T f_L(K_{2,L}\tilde{v}). \qquad (18b) \end{aligned}$$

While the Lagrangian $\mathcal{L}$ with the Legendre transform (6) is degenerate if any other components beside inductors are involved in the circuit, the reduction to $\mathcal{L}^M$ can cancel out this degeneracy iff $K_{2,L}^T f_L(K_{2,L}\,\cdot) : T_{\tilde{q}}M \to T_{\tilde{q}}^*M$ is (locally) invertible. The following theorem extends the result in [1] for linear to nonlinear systems.

Theorem 4.1: The reduced Lagrangian $\mathcal{L}^M$ is regular in $\tilde{v} \in T_{\tilde{q}}M$ if

1) the number of capacitors, resistors, and voltage sources equals the number of independent constraints involving the currents through the capacitor, resistor, and source branches, and
2) in $\tilde{v} \in T_{\tilde{q}}M$ it holds

$$rk\left(\nabla f_L(x)|_{x=K_{2,L}\tilde{v}}\right) = n_L.$$

Proof. To show the regularity of $\mathcal{L}^M$, we have to assure the invertibility of the Legendre transform $\mathbb{F}\mathcal{L}^M$ (cf. (14)). The proof is performed in two steps: (i) By using condition 1) on the network topology we show that $rk(K_{2,L}) = l$. (ii) Together with condition 2) we can prove that $rk\left(\frac{\partial}{\partial\tilde{v}} K_{2,L}^T f_L(K_{2,L}\tilde{v})\right) = l$. Then, the implicit function theorem ensures the invertibility of the Legendre transform $\mathbb{F}\mathcal{L}^M$.

Step (i): Let n_L be the number of inductors, $n_X = n_C + n_R + n_S$ the number of non-inductors and m the number of Kirchhoff constraints such that $K_X^T \in \mathbb{R}^{m,n_X}$ and let $l_X \leq l$ be the number of independent constraints involving the currents through the capacitor, resistor, and source branches. By condition 1) we have $n_X = l_X \leq m$ and thus $rk(K_X^T) = n_X$ and equivalently

$$ker(K_X^T) = \{0\}. \tag{19}$$

Furthermore, with $l = n - m$ it follows $n_L \geq l$ for $K_{2,L} \in \mathbb{R}^{n_L,l}$. To determine $rk(K_{2,L})$ we define a matrix L with

$$L = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n,n},\ \Lambda = \mathrm{diag}(\Lambda_i) \in \mathbb{R}^{n_L,n_L},\ \Lambda_i \neq 0\ \forall i,$$

whose kernel is

$$ker(L) = \{v \in T_qQ \mid v_L = 0\}.$$

Expressing $im(K_2)$ by $ker(K^T)$ as

$$im(K_2) = ker(K^T) = \left\{ v \in T_qQ \;\middle|\; (K_L^T \; K_X^T) \begin{pmatrix} v_L \\ v_X \end{pmatrix} = 0 \right\}$$

we obtain

$$im(K_2) \cap ker(L) = \{v \in T_qQ \mid K_X^T v_X = 0\} \overset{(19)}{=} \{0\}$$

and therefore

$$\{0\} = ker(LK_2) = ker \begin{pmatrix} \Lambda K_{2,L} \\ 0 \end{pmatrix} = ker(\Lambda K_{2,L})$$
$$\Leftrightarrow rk(\Lambda K_{2,L}) = l \overset{\Lambda \text{ diag}}{\Longrightarrow} rk(K_{2,L}) = l. \tag{20}$$

Step (ii): Define $\Upsilon = \nabla f_L(x)\Big|_{x=K_{2,L}\tilde{v}}$. Due to the inductor relation (7), Υ is diagonal with the entries $\Upsilon_j = \frac{d}{dx_j} f_{L_j}(x_j)\Big|_{x_j=v_{L_j}}$, $j = 1, \ldots, n_L$. Condition 2) ensures that diagonal elements are not zero. Splitting Υ in $\Upsilon = \sqrt{\Upsilon}^T \sqrt{\Upsilon}$, $\sqrt{\Upsilon} \in \mathbb{C}^{n_L, n_L}$ leads to

$$rk\left(K_{2,L}^T \Upsilon K_{2,L}\right) = rk\left(K_{2,L}^T \sqrt{\Upsilon}^T \sqrt{\Upsilon} K_{2,L}\right)$$
$$= rk\left(\sqrt{\Upsilon} K_{2,L}\right) = rk(K_{2,L}) = l.$$

The last equation holds, since the multiplication with the diagonal matrix $\sqrt{\Upsilon}$ from the left acts as multiplication of the rows of $K_{2,L}$ and therefore does not change the column rank. Finally, the implicit function theorem gives the invertibility of the Legendre transform $\mathbb{F}\mathcal{L}^M$. □

Thus, if the reduced Lagrangian is regular, the DAE system (18) can be written as ordinary differential equation system in Hamiltonian form with the variables $\tilde{q}$ and $\tilde{\varphi}$.

Remark 4.1: The previous proof makes a connection between the Kirchhoff Constraint matrix K and the Fundamental Loop matrix K_2. This is

$$rk(K_X) = n_X,\ n_X \leq m \Longrightarrow rk(K_{2,L}) = l,\ n_L \geq l.$$

Thus, the previous theorem states, that the reduced Lagrangian is regular, if each fundamental loop contains at least one inductor.

Remark 4.2: By following the lines of [1] it can be shown that the implicit Euler-Lagrange equations on the branch space (10) and implicit Euler-Lagrange equations on the mesh space (17) are equivalent that means that a solution of the reduced Lagrangian system can be transformed to a solution of the full Lagrangian system and vice versa.

III. VARIATIONAL INTEGRATORS

A. Discrete Variational Formulation of Electric Circuits

The concept of variational integrators [2] is based on a discretization of the variational principle (9). Due to Remark 4.2 we discretize the reduced variational formulation on the mesh space (16) instead. Consider a discrete time grid $\Delta t = \{t^{(k)} = kh \mid k = 0, \ldots N; Nh = T\} \subset [0, T]$ with constant step size h and accordingly discrete curves $\tilde{q}_d : \Delta t \to M, \tilde{v}_d : \Delta t \to T_{\tilde{q}}M$ and $\tilde{\varphi}_d : \Delta t \to T_{\tilde{q}}^* M$, where we view $\tilde{q}^{(k)} = \tilde{q}_d(hk), \tilde{v}^{(k)} = \tilde{v}_d(hk), \tilde{\varphi}^{(k)} = \tilde{\varphi}_d(hk)$ as approximations of the solution at the given times. For the case that the conditions in Theorem 4.1 are satisfied, we can assure that the degeneration of the Lagrangian system (cf. (6)) is canceled out and obtain a variational integrator suited for simulation. If the system is degenerate, the choice of discretization of the variational principle is important to bypass this intrinsic degeneracy. As a first exemplary discretization we choose a left-sided evaluation of the reduced Lagrangian (13), resp. the Lagrangian force (15), and substitute $\dot{\tilde{q}}(t)$ by the forward difference operator. Therefore, the *discrete Lagrange-d'Alembert-Pontryagin principle* on the mesh space reads

$$\delta\left\{h\sum_{k=0}^{N-1}\left[\mathcal{L}^M(\tilde{q}^{(k)}, \tilde{v}^{(k)}) + \tilde{\varphi}^{(k)}.\left(\frac{\tilde{q}^{(k+1)} - \tilde{q}^{(k)}}{h} - \tilde{v}^{(k)}\right)\right]\right\}$$
$$+h\sum_{k=0}^{N-1} \mathfrak{f}_{\mathcal{L}}^M(\tilde{q}^{(k)}, \tilde{v}^{(k)}, t^{(k)}) \cdot \delta\tilde{q}^{(k)} = 0$$

with fixed initial and final values for the charges $\delta\tilde{q}^{(0)} = \delta\tilde{q}^{(N)} = 0$ and arbitrary variations $\delta\tilde{q}^{(k)}, \delta\tilde{v}^{(k)}, \delta\tilde{\varphi}^{(k)}$. This leads to the *discrete implicit Euler-Lagrange equations on the mesh space*

$$\frac{\partial \mathcal{L}^M}{\partial \tilde{v}}(\tilde{q}^{(0)}, \tilde{v}^{(0)}) = \tilde{\varphi}^{(0)}$$
$$\frac{\partial \mathcal{L}^M}{\partial \tilde{q}}(\tilde{q}^{(k)}, \tilde{v}^{(k)}) - \frac{(\tilde{\varphi}^{(k)} - \tilde{\varphi}^{(k-1)})}{h} = -\mathfrak{f}_{\mathcal{L}}^M(\tilde{q}^{(k)}, \tilde{v}^{(k)}, t^{(k)})$$
$$\frac{\partial \mathcal{L}^M}{\partial \tilde{v}}(\tilde{q}^{(k)}, \tilde{v}^{(k)}) = \tilde{\varphi}^{(k)}$$
$$(\tilde{q}^{(k)} - \tilde{q}^{(k-1)}) = h\tilde{v}^{(k-1)}$$
$$(\tilde{q}^{(N)} - \tilde{q}^{(N-1)}) = h\tilde{v}^{(N-1)}$$

for $k = 1, \ldots, N-1$. This scheme provided by the discrete implicit Euler-Lagrange equations is also denoted as *forward Euler scheme*. In detail, the resulting integrator for the Lagrangian (13) and forces (15) is given as

$$\tilde{q}^{(k)} = \tilde{q}^{(k-1)} + h\tilde{v}^{(k-1)}$$
$$\tilde{\varphi}^{(k)} = K_{2,L}^T f_L(K_{2,L}\tilde{v}^{(k)})$$
$$\frac{\tilde{\varphi}^{(k)} - \tilde{\varphi}^{(k-1)}}{h} = -K_{2,C}^T f_C(K_{2,C}\tilde{q}^{(k)}) - K_{2,R}^T f_R(K_{2,R}\tilde{v}^{(k)})$$
$$+ K_{2,S}^T f_S(K_{2,S}\tilde{v}^{(k)}, t^{(k)})$$

with the explicit update for $\tilde{q}^{(k)}$ and the implicit updates for $\tilde{\varphi}^{(k)}, \tilde{v}^{(k)}$. Similarly (cf. [8]), a backward Euler scheme can be derived. Following the lines of [9] higher order integration schemes can be derived by using an *s-stage Runge-Kutta approximation* to the condition $\dot{\tilde{q}} = \tilde{v}$ with additional internal charges $\tilde{Q}_i^{(k)}$, internal velocities $\tilde{V}_i^{(k)}$ and internal fluxes $\tilde{\Phi}_i^{(k)}$, $i = 1, \ldots, s$ defined on the internal time grid $\Delta\tau = \{\tau_i^{(k)} = t^{(k)} + c_i h \mid i = 1, \ldots, s\}$. The general

discrete approach to (16) then reads

$$\delta \sum_{k=0}^{N-1} \sum_{i=1}^{s} h \Big[\tilde{\Phi}_i^{(k)} \cdot \Big(\frac{\tilde{Q}_i^{(k)} - \tilde{q}^{(k)}}{h} - \sum_{j=1}^{s} a_{ij} \tilde{V}_j^{(k)} \Big) + b_i \mathcal{L}^M(\tilde{Q}_i^{(k)}, \tilde{V}_i^{(k)}) + \tilde{\varphi}^{(k+1)} \cdot \Big(\frac{\tilde{q}^{(k+1)} - \tilde{q}^{(k)}}{h} - \sum_{j=1}^{s} b_j \tilde{V}_j^{(k)} \Big) \Big] + h \sum_{k=0}^{N-1} \sum_{i=1}^{s} b_i \mathfrak{f}_{\mathcal{L}}^M(\tilde{Q}_i^{(k)}, \tilde{V}_i^{(k)}, \tau_i^{(k)}) \cdot \delta \tilde{Q}_i^{(k)} = 0.$$

With $c_i = \sum_j a_{ij}$, the scheme is completely defined by the coefficients b_i, a_{ij}, which can be chosen by the coefficients given by the Butcher tableaus for standard Runge-Kutta methods. As a second example we present an scheme based on the implicit midpoint rule ($s = 1, a = c = 1/2, b = 1$), therefore called *midpoint rule scheme*. The internal time grid simplifies to $\Delta\tau = \{\tau^{(k)} = (k + \frac{1}{2})h \mid k = 0, \dots, N-1\}$ on which the internal stages $\tilde{Q}_d : \Delta\tau \to M$, $\tilde{V}_d : \Delta\tau \to T_{\tilde{q}}M$, $\tilde{\Phi}_d : \Delta\tau \to T^*_{\tilde{q}}M$ are defined. The associated discrete Lagrange-d'Alembert-Pontryagin principle is

$$\delta \Big\{ h \sum_{k=0}^{N-1} \Big[\mathcal{L}^M(\tilde{Q}^{(k)}, \tilde{V}^{(k)}) + \tilde{\Phi}^{(k)} \cdot \Big(\frac{\tilde{Q}^{(k)} - \tilde{q}^{(k)}}{h} - \frac{1}{2} \tilde{V}^{(k)} \Big) + \tilde{\varphi}^{(k+1)} \cdot \Big(\frac{\tilde{q}^{(k+1)} - \tilde{q}^{(k)}}{h} - \tilde{V}^{(k)} \Big) \Big] + \tilde{\varphi}^{(0)} \cdot (\tilde{q}^{(0)} - \tilde{q}_0) \Big\} + h \sum_{k=0}^{N-1} \mathfrak{f}_{\mathcal{L}}^M(\tilde{Q}^{(k)}, \tilde{V}^{(k)}, \tau^{(k)}) \cdot \delta \tilde{Q}^{(k)} = 0$$

where the initial charge condition $\tilde{q}^{(0)} - \tilde{q}_0$ is included in the principle and thus we consider free variations $\delta\tilde{q}^{(0)}$ but zero variations $\delta\tilde{q}^{(N)}$. The resulting integration scheme for the Lagrangian (13) and forces (15) is

$$\begin{aligned} \tilde{\varphi}^{(k+1)} =& \tilde{\varphi}^{(k)} - h K_{2,C}^T f_C\Big(K_{2,C}\Big(\frac{\tilde{q}^{(k)} + \tilde{q}^{(k+1)}}{2}\Big)\Big) \\ & - h K_{2,R} f_R(K_{2,R} \tilde{V}^{(k)}) + h K_{2,S}^T f_S(K_{2,S} \tilde{V}^{(k)}, \tau^{(k)}) \\ \tilde{q}^{(k+1)} =& \tilde{q}^{(k)} + h \tilde{V}^{(k)} \\ \tilde{\varphi}^{(k+1)} =& - \tilde{\varphi}^{(k)} + 2 K_{2,L}^T f_L(K_{2,L} \tilde{V}^{(k)}) \end{aligned}$$

for $k = 0, \dots, N-1$. Note that this scheme determines the discrete charges $\tilde{q}^{(k)}$ and fluxes $\tilde{\varphi}^{(k)}$ at different grid points than the discrete currents $\tilde{V}^{(k)}$. For a more detailed derivation and a discussion on the solvability of these schemes for linear component relations we refer to [1].

B. Numerical Example: LC-Circuit

For a numerical demonstration, we consider the LC-circuit shown in Fig. 3 without taking the resistor into account for now. Therefore, no loss or gain of energy will occur in this academical example. The number of capacitor branches in this circuit is $n_C = 2$ with the characteristic equations

$$u_C = f_C(q_C) = \begin{pmatrix} \operatorname{sgn}(q_{C_1}) a q_{C_1}^2 \\ \frac{1}{b} q_{C_2} \end{pmatrix}$$

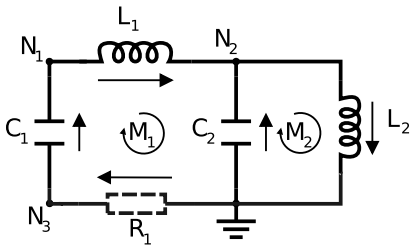

Fig. 3. Graph presentation for the LCR circuit. The resistor (dashed) and therefore node 3 are only used in the second example in Section III-C.

with $a, b \in \mathbb{R}$. With constants $c, d \in \mathbb{R}$ the $n_L = 2$ inductors have the characteristic equations

$$\varphi_L = f_L(v_L) = \begin{pmatrix} c v_{L_1} \\ \tan\left(\frac{v_{L_2}}{d}\right) \end{pmatrix}.$$

Therefore, the circuit consists of both linear and nonlinear components. The Lagrangian is given as

$$\mathcal{L}(q_C, v_L) = \frac{c}{2} v_{L_1}^2 + \frac{d}{2} \log\left(\left(\tan\left(\frac{v_{L_2}}{d}\right)\right)^2 + 1\right) - \operatorname{sgn}(q_{C_1}) \frac{a}{3} q_{C_1}^3 - \frac{1}{2b} q_{C_2}^2.$$

The topology of the circuit with $m + 1 = 3$ nodes (node N_3 is only needed for the LCR-circuit example in Section III-C) provides the Kirchhoff Constraint matrix K and the Fundamental Loop Matrix K_2 as

$$K = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Following the reduction steps described in Section II-D the reduced Lagrangian is given as

$$\mathcal{L}^M(\tilde{q}, \tilde{v}) = \frac{c}{2} \tilde{v}_1^2 + \frac{d}{2} \log\left(\left(\tan\left(\frac{\tilde{v}_2}{d}\right)\right)^2 + 1\right) - \operatorname{sgn}(\tilde{q}_1) \frac{a}{3} \tilde{q}_1^3 - \frac{1}{2b} (\tilde{q}_2 - \tilde{q}_1)^2.$$

Since the number of capacitors ($n_C = 2$) equals the number of independent constraints involving the currents through the capacitors (i.e., $rk(K_C) = n_C = 2$) and since $\nabla f_L(v_L)$ has full rank, the reduced Lagrangian $\mathcal{L}^M$ is regular by Theorem 4.1.

For the numerical computation we choose $a = 0.002$, $b = \frac{1}{30}$, $c = 30$, $d = 1$, and $h = 0.08$ for the initial state $q^{(0)} = (0.4, 0.2, -0.4, 0.2)^T$, $v^{(0)} = (0, 0, 0, 0)^T$, $\varphi^{(0)} = (0, 0, 0, 0)^T$.

We compare three different variational integrators (midpoint rule (VI MPR), backward Euler (VI EBD), forward Euler (VI EBD)) to a forth-order Runge-Kutta method (RK4) and the MATLAB ODE-solver ode45, which is a time adaptive Runge-Kutta solver using the Dormand-Prince method. In Fig. 4 the current through the inductor L_2 is plotted for the different solvers during the time interval $[260, 264]$.

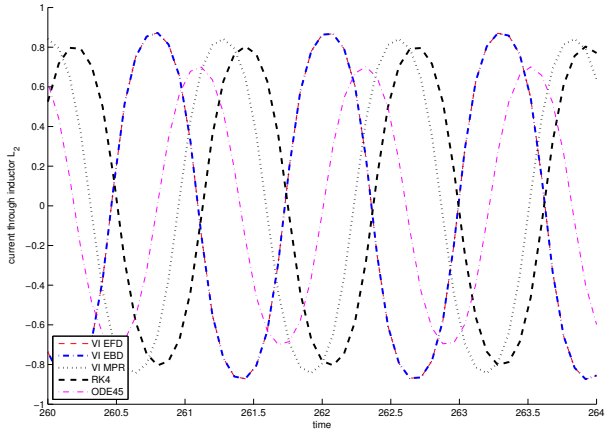

Fig. 4. LC-circuit with no resistors with step size h=0.08. Current through inductor L_2. Comparison of the numerical solution using variational integrators (VI), namely midpoint rule (VI MPR), backward Euler (VI EBD) and forward Euler (VI EFD), a Runge-Kutta method of forth order (RK4) and the MATLAB ODE-solver ode45 (ODE45).

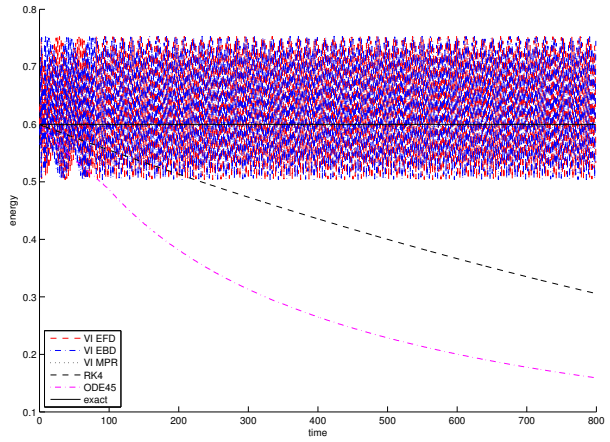

Fig. 5. Long term energy evolution for the LC-circuit example. The solutions of the variational integrators show no energy dissipation compared to the non symplectic Runge-Kutta methods.

Due to the different discretizations, the integrators induce different phase shifts in the discrete solution. In addition, the amplitude of the solutions from the RK4 method and the ode45 solver slowly decreases. This phenomenon can also be observed in the energy behavior of the simulated circuits (see Fig. 5): Although the total energy (magnetic plus electric energy) should be preserved (no resistors or sources are present), the energy for the Runge-Kutta and the ode45 solution is dissipating. For the variational integrators, the system's energy oscillates around its real value, however, no dissipation or artificial growth of the energy occurs due to their symplectic nature.

C. Numerical Example: LCR-Circuit

As second example we consider a LCR-circuit by adding a nonlinear resistor in the previous example as shown in the dashed part of Fig. 3. This induces a damping force to the circuit and therefore the energy is not conserved anymore. We use the same relations for the inductors and the capacitors as for the LC-circuit in the previous subsection. Thus, the (reduced) Lagrangian stays the same. The relation for the resistor is given as

$$f_R(v_R) = ev_R^3.$$

With the new component, the Kirchhoff constraint matrix and the Fundamental loop matrix are given as

$$K = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; K_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that since $rk(K_{C,R}) = n_C + n_R = 3$ the reduced system is still regular by Theorem 4.1. For the numerical example we use a, b, c, d as before and $e = 5000$. The step size is also set to $h = 0.08$. The energy "loss" caused by the resistor is shown in Figure 6. Interestingly the variational integrators keep in the mean a higher energy level at all times. Even so we do not have the exact solution for this problem, the analysis for linear circuits performed in [1], for which the analytical solution is known, allows us to make the assumption that the energy curve provided by the VI MPR is the most accurate one in this example.

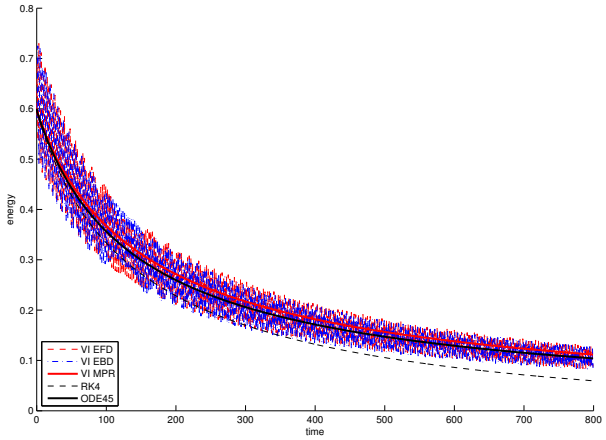

Fig. 6. LCR-circuit with one resistor and the step size h=0.08. Long term energy evolution. Comparison of the numerical solution using variational integrators, namely midpoint rule (VI MPR), backward Euler (VI EBD) and forward Euler (VI EFD), a Runge-Kutta method of forth order (RK4) and the MATLAB ODE-solver ode45 (ODE45).

IV. CONCLUSIONS

In this contribution, the unified framework for the modeling and simulation of linear electric circuits presented in [1] is extended to general nonlinear circuits. A degenerate Lagrangian is defined on the space of branches that consists of electric and magnetic energy, dissipative and external forces that describe the influence of resistors and voltage sources, as well as holonomic constraints given by the KCL of the circuit. The Lagrange-d'Alembert-Pontryagin principle is used to derive the implicit Euler-Lagrange equations. A reduced version on the mesh space is presented for which

under some topology assumptions and regularity conditions on the inductance relations the degeneracy of the Lagrangian is canceled. Based on a variational discretization, variational integrators are derived that ensure a structure-preserving simulation of the nonlinear circuit, in particular a better energy behavior compared to non symplectic Runge-Kutta methods is demonstrated for two examples of LCR-circuits.

With the presented modeling approach, also large-scale nonlinear circuits can be easily formulated in Lagrangian dynamics. In particular, the concept of interconnection for degenerate Lagrangian systems [10] is a useful approach for the modeling of coupled or interconnected circuits. Since the use of geometric integration methods ensures less errors in the qualitative behavior for each sub circuit, it is expected that error accumulation due to numerical errors in each subsystem is less severe compared to non symplectic integration methods.

Of course, for the simulation of circuits involving more complicated devices, a Lagrangian description has to be provided. This is subject of current and future work. Particularly, we are interested in including magnetically coupled inductors in our framework. These are especially important for transformers, and therefore, for power engineering. With this in mind the long-time energy behavior of the derived integrators becomes even more important. Another obvious next step for the variational simulation of electric circuits is the inclusion of transistors, which are used for switching or amplification of electric signals. Transistors naturally introduce a hybrid behavior to the circuit, that means an interaction between continuous dynamics and discrete events [11]. As a first step, we consider a simple diode model that makes it possible to study a basic state-controlled switching feature for the electric circuit. For the variational formulation and simulation of such a hybrid electric systems, we adapt strategies for the construction of nonsmooth variational collision integrators [12]. Due to space limitations, the results are not presented here but will be published in a long version of this paper.

REFERENCES

[1] S. Ober-Blöbaum, M. Tao, M. Cheng, H. Owhadi, and J. E. Marsden, "Variational integrators for electric circuits," *Journal of Computational Physics*, vol. 242, no. C, pp. 498–530, 2013.

[2] J. E. Marsden and M. West, "Discrete mechanics and variational integrators," *Acta Numer.*, vol. 10, pp. 357–514, 2001.

[3] E. Hairer, C. Lubich, and G. Wanner, *Geometric numerical integration*, ser. Springer Series in Computational Mathematics. Berlin: Springer-Verlag, 2002, vol. 31.

[4] A. Lew, J. E. Marsden, M. Ortiz, and M. West, "An overview of variational integrators," in *Finite Element Methods: 1970's and Beyond*, L. P. Franca, T. E. Tezduyar, and A. Masud, Eds. CIMNE, 2004, pp. 98–115.

[5] H. Yoshimura and J. E. Marsden, "Dirac structures in Lagrangian mechanics. II. Variational structures," *J. Geom. Phys.*, vol. 57, no. 1, pp. 209–250, 2006.

[6] ——, "Dirac structures in Lagrangian mechanics. I. Implicit Lagrangian systems," *J. Geom. Phys.*, vol. 57, no. 1, pp. 133–156, 2006.

[7] L. Moreau and A. J. van der Schaft, "Implicit Lagrangian equations and mathematical modeling of physical systems," in *41st IEEE Conference on Decision and Control*, Las Vegas, Nevada, USA, 2002.

[8] H. Lindhorst, "Variational integrators for nonlinear and hybrid electric circuits," Diploma thesis, Paderborn, 2012.

[9] N. Bou-Rabee, "Hamilton-Pontryagin integrators on Lie groups," Ph.D. dissertation, California Institute of Technology, 2007.

[10] H. Jacobs, H. Yoshimura, and J. E. Marsden, "Interconnection of Lagrange-Dirac dynamical systems for electric circuits," *AIP Conference Proceedings*, vol. 1281, no. 1, 2010.

[11] A. van der Schaft and H. Schumacher, *An introduction to hybrid dynamical systems*, ser. Lecture Notes in Control and Information Sciences. London: Springer-Verlag London Ltd., 2000, vol. 251.

[12] R. C. Fetecau, J. E. Marsden, M. Ortiz, and M. West, "Nonsmooth Lagrangian mechanics and variational collision integrators," *SIAM J. Appl. Dyn. Syst.*, vol. 2, no. 3, pp. 381–416, 2003. [Online]. Available: http://dx.doi.org/10.1137/S1111111102406038