21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Control Design Problems for Multidimensional Behaviours

Martin Scheicher[1,2,*], Ingrid Blumthaler[1,3], Mauro Bisiacco[1,4], Maria Elena Valcher[1,5]

Abstract— **In this paper we study a generalized tracking and disturbance rejection problem for multidimensional linear behaviours. Given a multidimensional plant, our first goal is to design a compensator to be connected to the plant through regular partial interconnection, in such a way that the overall controlled system is autonomous and stable, when no exogenous signal acts on the system. On the other hand, when exogenous signals affect the controlled system evolution, we want to impose that a suitable linear combination of the overall system trajectories is "negligibile" in a sense we will clarify within the paper. This problem set-up formalizes a number of classical control problems, first of all tracking of some (reference) signal together with rejection of another (disturbance) signal. The adopted approach is extremely general and it is based on the idea of describing all behaviour trajectories as the sum of a "transient signal" and a "steady state" component, a decomposition that relies on Gabriel's localization theory. Necessary and sufficient conditions for the problem solvability are provided, and the compensators that satisfy the control goal are characterized in terms of an internal model condition.**

I. Introduction

Stabilization and regulation problems within the behavioral approach have a long history. Stimulated by two milestone contributions [16], [17], dealing with the control of one-dimensional behaviours, a long stream of research on this topics flourished, dealing either with one-dimensional (1D) behaviours (e.g. [1], [3], [4], [9]) or with the wider class of multidimensional (nD) behaviours [5], [6], [8], [10], [14].

In the behavioural framework, controlling a plant amounts to restricting the set of all its trajectories to a proper subset, whose elements display desired properties (typically, but not solely, some form of convergence). So, stabilization, either by partial or by full interconnection, consists in designing a second system, the controller, that, once connected with the original plant (either by means of all the plant variables or by a proper subset of them), makes it possible to achieve this goal. One of the main features of the behavioural approach lies in its capability of treating the system dynamics without imposing any input/output partition on the system variables. This is particularly appropriate when the plant under investigation has to become part of a larger, interconnected, system, since it is the specific interconnection structure that determines what are the inputs and what are the outputs. A simple but rather paradigmatic example is represented by passive circuits, for which the choice of assuming the voltage or the current as input signal is strictly related to the way they are connected with the external generators. Consistently with this perspective, in most of the aforementioned references stabilization and regulation problems have been posed and solved without assuming any input/output partition of the system variables.

The tracking and disturbance rejection problem for 1D behaviours was first addressed in [3], where necessary and sufficient conditions for the problem solvability, under the assumption that the exogenous system generating both the reference signal and the disturbance is autonomous, have been provided. Interestingly enough, the solvability conditions involve the well-known internal model principle, first pointed out in the behavioural context for observers in [15]. An algorithm to explicitly construct controllers that achieve these goals was also proposed in [3].

In [2], the set-up introduced in [3] was generalized, to deal with more general stabilization goals (design of T-stabilizing compensators) and by introducing a target function that can formalize a number of classical control problems, first of all the tracking of a reference signal, meanwhile rejecting a disturbance acting on the system.

The aim of this paper is to extend the results derived in [2] to the multidimensional case, to deal with stabilization and regulation problems for nD behaviours. The main results derived in this mansucript represent neat, but highly non-trivial generalizations of the results provided in section 4 of [2]. Indeed, the mathematical set-up required to extend the analysis to the nD case deeply relies on advanced algebraic concepts, like Serre subcategories [12] and localization according to Gabriel [13]. Gabriel localization was first applied to system theoretic questions in [8], and later refined. In order to make the proofs accessible also to non-specialists, we have tried to briefly recall the main definitions and results about these topics in a preliminary section. We refer the reader to [11] for a more thorough description of the theory.

In detail, the paper is organized as follows: Section II recalls the basic concepts about behaviours, interconnection of behaviours, and negligibility of modules and signals, as well as some technical results about Gabriel localization and its use in defining the steady-state and the transient part of the behaviour trajectories. Section III addresses stabilization by partial interconnection, by assuming that no exogenous signal acts on the overall controlled system, while section IV tackles the same problem in the presence of the exogenous signals, which will later represent the reference signal and the disturbance. Finally, in section V, the general tracking and

[1] Dipartimento di Ingegneria dell'Informazione, Università di Padova, via Gradenigo 6/B, 35131 Padova, Italia
[2] martin.scheicher@uibk.ac.at
[3] ingrid.blumthaler@uibk.ac.at
[4] bisiacco@dei.unipd.it
[5] meme@dei.unipd.it
* Martin Scheicher has been supported by the Austrian FWF via project J3411-N25.

ISBN: 978-90-367-6321-9

disturbance rejection problem for nD behaviours is posed and solved.

II. Preliminaries

In this section we introduce the framework adopted in the paper to investigate the control design problems.

We consider an affine integral domain A, called the *ring of operators*, and a *signal space* $\mathscr{F}$, which is a large injective cogenerator over A [7, p. 31]. The scalar product of $f \in A$ with $w \in \mathscr{F}$ is denoted by $f \circ w$. Given an A-module (in the following simply "a module") $U = A^{1\times k}R \subseteq A^{1\times l}$ generated by the rows of a matrix $R \in A^{k\times l}$, we define the *behaviour* associated with U as the set[6]

$$\begin{aligned}\mathscr{B} = U^{\perp} &:= \{w \in \mathscr{F}^l :\ U \circ w = 0,\ \text{i.e., } \forall \eta \in U,\ \eta \circ w = 0\}\\ &= \{w \in \mathscr{F}^l :\ R \circ w = 0\}\end{aligned}$$

Conversely, given a behaviour $\mathscr{B}$, the *module of equations* associated with $\mathscr{B}$ is

$$\mathscr{B}^{\perp} := \{\eta \in A^{1\times l} :\ \eta \circ \mathscr{B} = 0,\ \text{i.e., } \forall w \in \mathscr{B},\ \eta \circ w = 0\}.$$

From the definition of behaviour, it follows immediately that $\mathscr{B}^{\perp\perp} = \mathscr{B}$, but since $\mathscr{F}$ is a cogenerator the identity $\mathscr{B}^{\perp} = U^{\perp\perp} = U$ holds, too [7, Cor. 47, Cor. 48, p. 29].

Because of the injectivity of the signal module $\mathscr{F}^l$, the image of a behaviour $\mathscr{B} \subseteq \mathscr{F}^l$ under a map $P\circ$ given by a matrix $P \in A^{v\times l}$, i.e. $P \circ \mathscr{B} = \{P \circ w \in \mathscr{F}^v :\ w \in \mathscr{B}\}$, is again a behaviour. Let $(X,\ Y) \in A^{k_1\times(l+v)}$ be a universal (or minimal) left annihilator of the block matrix $\binom{R}{P}$, i.e., the rows of $(X,\ Y)$ generate the syzygy module of $\binom{R}{P}$,

$$\left\{(\eta,\ \eta_1) \in A^{1\times(k+v)} :\ (\eta,\ \eta_1)\binom{R}{P} = 0\right\} = A^{1\times k_1}(X,\ Y).$$

With this, the image of $\mathscr{B}$ under $P\circ$ is [7, Thm. 34, p. 24]

$$P \circ \mathscr{B} = (A^{1\times k_1}Y)^{\perp} = \{\widetilde{w} \in \mathscr{F}^v :\ Y \circ \widetilde{w} = 0\}.$$

Consider the special case when $\mathscr{B} = \{\binom{w_1}{w_2} \in \mathscr{F}^{l_1+l_2} :\ (R_1,\ R_2)\circ\binom{w_1}{w_2} = 0\}$ and $P = (0,\ \mathrm{id}_{l_2})$ is the matrix associated with the projection $\mathrm{proj}_{w_2} :\ \binom{w_1}{w_2} \longmapsto w_2$. The projection of the behaviour $\mathscr{B}$ on the variable w_2 is

$$\begin{aligned}\mathrm{proj}_{w_2}(\mathscr{B}) = P \circ \mathscr{B} &= \{w_2 \in \mathscr{F}^{l_2} :\ \exists w_1 \in \mathscr{F}^{l_1} :\ \tbinom{w_1}{w_2} \in \mathscr{B}\}\\ &= \{w_2 \in \mathscr{F}^{l_2} :\ (XR_2)\circ w_2 = 0\},\end{aligned}$$

where, in this case, X is a universal left annihilator of R_1.

In this set-up, the variables w_2 are *free* for the behaviour $\mathscr{B}$ if $\mathrm{proj}_{w_2}(\mathscr{B}) = \mathscr{F}^{l_2}$. This property is equivalent to the rank condition $\mathrm{rank}(R_1) = \mathrm{rank}(R_1,\ R_2)$ and implies that there exists a matrix $H \in K^{l_1\times l_2}$ with entries in the quotient field $K := \mathrm{quot}(A)$, such that $R_2 = R_1H$. A behaviour $\mathscr{B} = (A^{1\times k}R)^{\perp} \subseteq \mathscr{F}^l$ with no free variables is called *autonomous* and satisfies $\mathrm{rank}(R) = l$.

[6]The notation $\perp$ was first used in [7, p. 21] in this context. The notion of orthogonal complement of a submodule $U \subseteq A^{1\times l}$ and of a behaviour $\mathscr{B} \subseteq \mathscr{F}^l$ is induced by the bilinear form

$$A^{1\times l} \times \mathscr{F}^l \longrightarrow A :\ (\eta,\ w) \longmapsto \eta \circ w,$$

i.e., $\eta \perp w$ if and only if $\eta \circ w = 0$.

We denote the subbehaviour of $\mathscr{B}$ consisting of the trajectories of $\mathscr{B}$ whose components w_2 are identically zero by

$$\begin{aligned}\mathscr{N}_{w_2}(\mathscr{B}) &:= \{w_1 \in \mathscr{F}^{l_1} :\ (R_1,\ R_2)\circ\tbinom{w_1}{0} = R_1 \circ w_1 = 0\}\\ &= (A^{1\times k}R_1)^{\perp}.\end{aligned}$$

A behaviour

$$\mathscr{B} = \left\{\tbinom{y}{u} \in \mathscr{F}^{p+m} : P \circ y = Q \circ u\right\}, \quad (P, -Q) \in A^{k\times(p+m)}, \tag{1}$$

is an *input/output (IO) behaviour* with input u and output y if u is maximally free in $\mathscr{B}$, i.e., if u is free and $\mathscr{B}^0 := \mathscr{N}_u(\mathscr{B}) = \{y \in \mathscr{F}^p :\ P \circ y = 0\}$ is autonomous. This is equivalent to the rank condition $p = \mathrm{rank}(P) = \mathrm{rank}(P,\ -Q)$ and implies the existence of a unique transfer matrix $H \in K^{p\times m}$ satisfying $Q = PH$.

The *(full) interconnection* of two behaviours $\mathscr{B}_i = U_i^{\perp} \subseteq \mathscr{F}^l$, $U_i = A^{1\times k_i}R_i$, $i = 1, 2$, is their intersection

$$\begin{aligned}\mathscr{B}_1 \cap \mathscr{B}_2 &= \{w \in \mathscr{F}^l :\ U_1 \circ w = 0 \text{ and } U_2 \circ w = 0\}\\ &= \{w \in \mathscr{F}^l :\ (U_1 + U_2)\circ w = 0\} = (U_1 + U_2)^{\perp}\\ &= \left(A^{1\times(k_1+k_2)}\begin{pmatrix}R_1\\ R_2\end{pmatrix}\right)^{\perp}.\end{aligned}$$

Such interconnection is *regular* if the sum $U_1 + U_2$ is a direct sum, i.e., if the intersection $U_1 \cap U_2$ is zero, or, equivalently,

$$\mathrm{rank}\begin{pmatrix}R_1\\ R_2\end{pmatrix} = \mathrm{rank}(R_1) + \mathrm{rank}(R_2).$$

Given two behaviours

$$\begin{aligned}\mathscr{B} &= \left\{\tbinom{w_1}{w_2} \in \mathscr{F}^{l_1+l_2} :\ (R_1,\ R_2)\circ\tbinom{w_1}{w_2} = 0\right\} \text{ and}\\ \widetilde{\mathscr{B}} &= \left\{\tbinom{w_2}{w_3} \in \mathscr{F}^{l_2+l_3} :\ (\widetilde{R_2},\ \widetilde{R_3})\circ\tbinom{w_2}{w_3} = 0\right\}\end{aligned}$$

with $(R_1,\ R_2) \in A^{k_1\times(l_1+l_2)}$ and $(\widetilde{R_2},\ \widetilde{R_3}) \in A^{k_2\times(l_2+l_3)}$,

their *partial interconnection via* w_2 is defined as

$$\begin{aligned}\mathscr{B} \wedge_{w_2} \widetilde{\mathscr{B}} &= \left\{\begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix} \mathscr{F}^{l_1+l_2+l_3} :\ \tbinom{w_1}{w_2} \in \mathscr{B},\ \tbinom{w_2}{w_3} \in \widetilde{\mathscr{B}}\right\}\\ &= \left\{\begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix} \mathscr{F}^{l_1+l_2+l_3} :\ \begin{pmatrix}R_1 & R_2 & 0\\ 0 & \widetilde{R_2} & \widetilde{R_3}\end{pmatrix}\circ\begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix} = 0\right\}.\end{aligned}$$

The partial interconnection is called *regular* if the sum $A^{1\times k_1}(R_1,\ R_2,\ 0) + A^{1\times k_2}(0,\ \widetilde{R_2},\ \widetilde{R_3})$ is direct.

We define negligibility of modules and signals via a Serre subcategory of modules $\emptyset \neq \mathfrak{C} \subsetneq \mathrm{Mod}_A$, where Mod_A denotes the category of all A-modules. A *Serre subcategory* is a full subcategory closed under isomorphism, subobjects, factor objects, extensions and direct sums [12, Chap. I]. From $\mathfrak{C} \neq \mathrm{Mod}_A$ it follows that $\mathfrak{C}$ consists of torsion modules only. Also, we define the subcategory $\mathrm{Mod}_{A,\mathfrak{C}} \subseteq \mathrm{Mod}_A$ of $\mathfrak{C}$-*closed modules* consisting of those A-modules M such that for all ideals $\mathfrak{a} \subseteq A$ with $A/\mathfrak{a} \in \mathfrak{C}$ the map

$$M \longrightarrow \mathrm{Hom}_A(\mathfrak{a}, M) :\ x \longmapsto (a \mapsto ax)$$

is an isomorphism. An A-module, in particular a behaviour, is $\mathfrak{C}$-*negligible* if it belongs to $\mathfrak{C}$, and a signal $w \in \mathscr{F}$ is $\mathfrak{C}$-*negligible* if $A \circ w \in \mathfrak{C}$. As a consequence, a behaviour is $\mathfrak{C}$-negligible if and only if all its trajectories are $\mathfrak{C}$-negligible.

Given any A-module M, its largest $\mathfrak{C}$-negligible submodule is denoted by

$$\mathrm{Ra}_{\mathfrak{C}}(M) = \sup\{N \subseteq M:\ N \in \mathfrak{C}\} \in \mathfrak{C}$$

and is called the $\mathfrak{C}$-*radical* of M.

The Serre subcategory $\mathfrak{C}$ induces a direct sum decomposition of the signal space

$$\mathscr{F} = \mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}) \oplus \mathscr{F}_2.$$

The radical is unique, however, in general, its direct complement is not, i.e., there are many possible choices of $\mathscr{F}_2 \cong \mathscr{F}/\mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})$ and none of them is preferrable over the others. The signals $w \in \mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})$ are $\mathfrak{C}$-negligible and often called *transient signals*, while the signals in $\mathscr{F}_2$ are called *steady states*. The latter ones form a system of representatives of the equivalence classes of signals up to $\mathfrak{C}$-negligible ones. The direct sum decomposition carries over to vectors of signals

$$\mathscr{F}^l = \mathrm{Ra}_{\mathfrak{C}}(\mathscr{F})^l \oplus \mathscr{F}_2^l = \mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}^l) \oplus \mathscr{F}_2^l$$

and to behaviours

$$\mathscr{B} = (\mathscr{B} \cap \mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}^l)) \oplus (\mathscr{B} \cap \mathscr{F}_2^l) = \mathrm{Ra}_{\mathfrak{C}}(\mathscr{B}) \oplus (\mathscr{B} \cap \mathscr{F}_2^l). \tag{2}$$

From this decomposition it follows immediately that $\mathscr{B}$ is $\mathfrak{C}$-negligible if and only if $\mathscr{B} \cap \mathscr{F}_2^l = \{0\}$. Furthermore, since $\mathfrak{C}$ consists only of torsion modules, a $\mathfrak{C}$-negligible behaviour cannot have free variables and therefore it is always autonomous.

It should be remarked that, in general, neither $\mathrm{Ra}_{\mathfrak{C}}(\mathscr{B})$ nor $\mathscr{B} \cap \mathscr{F}_2^l$ are behaviours in $\mathscr{F}^l$. This implies, in particular, that the A-submodule

$$\mathscr{B} \cap \mathscr{F}_2^l = \{w \in \mathscr{F}^l:\ R \circ w = 0\} \cap \mathscr{F}_2^l = \{w \in \mathscr{F}_2^l:\ R \circ w = 0\}$$

of $\mathscr{F}^l$, in general, cannot be written in the form

$$\mathscr{B} \cap \mathscr{F}_2^l = \{w \in \mathscr{F}^l:\ \widetilde{R} \circ w = 0\} = \left(A^{1 \times \tilde{k}} \tilde{R}\right)^{\perp}$$

for any matrix $\widetilde{R} \in A^{\tilde{k} \times l}$. However, it is still possible to interpret this set as a behaviour. To explain what we mean, some preliminary definitions and results are required.

Every Serre subcategory induces a *Gabriel localization functor*

$$\mathscr{Q} = \mathscr{Q}_{\mathfrak{C}}:\ \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{A,\mathfrak{C}}$$

into the subcategory $\mathrm{Mod}_{A,\mathfrak{C}} \subseteq \mathrm{Mod}_A$ of $\mathfrak{C}$-closed modules. The Gabriel localization functor $\mathscr{Q}$ is the left-adjoint of the injection functor $\mathrm{Mod}_{A,\mathfrak{C}} \subseteq \mathrm{Mod}_A$.

The Gabriel localization $\mathscr{Q}(M)$ of an A-module M is a $\mathscr{Q}(A)$-module. Furthermore, the functor $\mathscr{Q}$ is such that, for every $M,\ M_1,\ M_2 \in \mathrm{Mod}_A$,

$$M \in \mathrm{Mod}_{A,\mathfrak{C}} \Longleftrightarrow M = \mathscr{Q}(M), \tag{3a}$$
$$\mathscr{Q}(\mathscr{Q}(M)) = \mathscr{Q}(M), \tag{3b}$$
$$M \in \mathfrak{C} \Longleftrightarrow \mathscr{Q}(M) = \{0\}, \tag{3c}$$
$$M_1 \subseteq M_2 \Longrightarrow \mathscr{Q}(M_1) \subseteq \mathscr{Q}(M_2), \tag{3d}$$
$$\mathrm{Ra}_{\mathfrak{C}}(M) = \{0\} \Longrightarrow M \subseteq \mathscr{Q}(M). \tag{3e}$$

In general, however, M cannot be embedded in $\mathscr{Q}(M)$. Also, Gabriel localization preserves direct sums and intersections, i.e.,

$$\mathscr{Q}(M_1 \oplus M_2) = \mathscr{Q}(M_1) \oplus \mathscr{Q}(M_2) \text{ for } M_1, M_2 \in \mathrm{Mod}_A \tag{3f}$$
$$\mathscr{Q}(M_1 \cap M_2) = \mathscr{Q}(M_1) \cap \mathscr{Q}(M_2) \text{ for } M_1, M_2 \subseteq N \in \mathrm{Mod}_A, \tag{3g}$$

but not arbitrary sums. Indeed, only the inclusion

$$\mathscr{Q}(M_1 + M_2) \supseteq \mathscr{Q}(M_1) + \mathscr{Q}(M_2) \text{ for } M_1, M_2 \subseteq N \in \mathrm{Mod}_A \tag{3h}$$

holds, in general.

Gabriel localization generalizes the usual localization with respect to a multiplicatively closed and saturated set $T \subseteq A$, i.e.,

$$(-)_T:\ M \longmapsto M_T = \left\{\frac{m}{t}:\ m \in M,\ t \in T\right\},\ M \in \mathrm{Mod}_A\,.$$

The localized modules M_T are A_T-modules via $\frac{a}{t_1}\frac{m}{t_2} = \frac{am}{t_1 t_2}$ for $a \in A$, $t_1, \in T$ and $\frac{m}{t_2} \in M_T$. Given such a set $T \subseteq A$, the appropriate Serre subcategory $\mathfrak{C}(T)$ such that $\mathscr{Q}_{\mathfrak{C}(T)}(M) = M_T$ holds for every $M \in \mathrm{Mod}_A$ is $\mathfrak{C}(T) = \{M \in \mathrm{Mod}_A:\ M_T = 0\}$. It is easily seen that the modules A/At for $t \in T$ lie in $\mathfrak{C}(T)$, in fact the identity $T = \{t \in A:\ A/At \in \mathfrak{C}(T)\}$ holds, i.e., T can be retrieved from $\mathfrak{C}(T)$.

For an arbitrary Serre subcategory $\mathfrak{C}$ (which is not necessarily of the form $\mathfrak{C}(T)$), this motivates the multiplicatively closed set $T(\mathfrak{C}) = \{t \in A:\ A/At \in \mathfrak{C}\}$ and the associated localization

$$(-)_{T(\mathfrak{C})}:\ M \longmapsto M_{T(\mathfrak{C})} = \left\{\frac{m}{t}:\ m \in M,\ t \in T(\mathfrak{C})\right\},$$

for $M \in \mathrm{Mod}_A$. In general, however, the localizations $(-)_{T(\mathfrak{C})}$ and $\mathscr{Q}_{\mathfrak{C}}(-)$ are not equal and the localization theory according to Gabriel is a proper extension of the usual one.

Assumption 1. *In the rest of the article we will make the steady assumption that the two localizations of the ring A are equal, i.e.,* $\mathscr{Q}_{\mathfrak{C}}(A) = A_{T(\mathfrak{C})}$ *holds. This assumption holds, for instance, whenever the ring A is a unique factorization domain, in particular if it is a polynomial ring* $A = F[s_1, \ldots, s_n]$ *over some field F. Furthermore, since the Serre subcategory* $\mathfrak{C}$ *will be fixed, we will write* $T := T(\mathfrak{C})$.

The Gabriel localization $\mathscr{Q}(U) \subseteq \mathscr{Q}(A^{1\times l}) = A_T^{1\times l}$ of a module of equations $U = A^{1\times k}R$ is an A_T-submodule of $A_T^{1\times l}$, and thus it is finitely generated by a matrix $R' \in A_T^{k'\times l}$, i.e., we have

$$\mathscr{Q}(U) = A_T^{1\times k'} R' \subseteq A_T^{1\times l} = \mathscr{Q}(A^{1\times l}). \tag{4}$$

The matrix R' can be computed as shown in [11, Alg. 3.9]. The chain of inclusions

$$\begin{aligned} U \subseteq U_T = A_T^{1\times k} R \subseteq \mathscr{Q}(U) = A_T^{1\times k'} R' = \mathscr{Q}(U_T) \\ \subseteq KU = K^{1\times k} R = K\mathscr{Q}(U) = K^{1\times k'} R' \subseteq K^{1\times l} \end{aligned} \tag{5}$$

does always hold, but in general the two sets U_T and $\mathscr{Q}(U)$ are not equal. As a consequence of property (3b) and of (5), one has the set of equivalences

$$\mathscr{Q}(U_1) \subseteq \mathscr{Q}(U_2) \Longleftrightarrow U_1 \subseteq \mathscr{Q}(U_2) \Longleftrightarrow (U_1)_T \subseteq \mathscr{Q}(U_2) \tag{6}$$

for $U_1,\ U_2 \subseteq A^{1\times l}$.

A homomorphism

$$\cdot P:\ U \longrightarrow V:\ \eta \longmapsto \eta P,$$

$U \subseteq A^{1\times l_1}$, $V \subseteq A^{1\times l_2}$, given by a matrix $P \in A^{l_1\times l_2}$, is mapped by the functor $\mathscr{Q}$ to the homomorphism

$$\begin{aligned}\mathscr{Q}(\cdot P) : \mathscr{Q}(U) = A_T^{1\times k'}R' &\longrightarrow \mathscr{Q}(V) \subseteq A_T^{1\times l_2},\\ \eta &\longmapsto \mathscr{Q}(\cdot P)(\eta) = \eta P,\end{aligned} \tag{7}$$

i.e., the map $\mathscr{Q}(\cdot P)$ is again the multiplication of row vectors by P. As a consequence, $\mathscr{Q}(U)P \subseteq \mathscr{Q}(UP) = \mathscr{Q}(\mathrm{im}(\cdot P)) \subseteq \mathscr{Q}(V)$. The same holds for A_T-modules $U \subseteq A_T^{1\times l_1}$ and $V \subseteq A_T^{1\times l_2}$.

The set $\mathscr{F}_2$ is isomorphic to $\mathscr{Q}(\mathscr{F})$ and thus it is an $A_T = \mathscr{Q}(A)$-module with the scalar multiplication

$$\frac{a}{t} \circ w = y_2 \quad \text{for } a \in A,\ t \in T, \text{ and } w \in \mathscr{F}_2,$$

where $y = y_1 + y_2 \in \mathrm{Ra}_{\mathfrak{C}}(\mathscr{F}) \oplus \mathscr{F}_2$ is a solution of $t \circ y = a \circ w$. Such a solution y always exists since $\mathscr{F}$ is an injective A-module and hence divisible over A.

The fact that $\mathscr{F}$ is a large injective cogenerator in Mod_A implies that the module $\mathscr{F}_2$ is an injective cogenerator in the category $\mathrm{Mod}_{A,\mathfrak{C}}$ and this, in turn, induces the following duality theory between steady state behaviours in $\mathscr{F}_2^l$ and $\mathfrak{C}$-closed submodules of equations of $A_T^{1\times l}$. The A_T-scalar multiplication on $\mathscr{F}_2$ carries over to multiple components. It gives rise to the bilinear form

$$A_T^{1\times l} \times \mathscr{F}_2^l \longrightarrow A_T:\ (\eta,\ w) \longmapsto \eta \circ w,$$

and to a corresponding notion of orthogonality:

$$\widetilde{U}^{\perp_2} = \{w \in \mathscr{F}_2^l:\ \widetilde{U} \circ w = 0\}$$

for A_T-submodules $\widetilde{U} \subseteq A_T^{1\times l}$ and

$$\widetilde{\mathscr{B}}^{\perp_2} = \{\eta \in A_T^{1\times l}:\ \eta \circ \widetilde{\mathscr{B}} = 0\}$$

for A_T-submodules $\widetilde{\mathscr{B}} \subseteq \mathscr{F}_2^l$. Also, $\widetilde{U}^{\perp_2\perp_2} = \widetilde{U}$ for $\widetilde{U} \in \mathrm{Mod}_{A,\mathfrak{C}}$ and $\widetilde{\mathscr{B}}^{\perp_2\perp_2} = \widetilde{\mathscr{B}}$ for a behaviour $\widetilde{\mathscr{B}} \in \mathrm{Mod}_{A,\mathfrak{C}}$. We denote this concept of orthogonality by $\perp_2$ to distinguish it from the earlier one of submodules of $A^{1\times l}$ and of $\mathscr{F}^l$.

The steady state behaviour $\mathscr{B} \cap \mathscr{F}_2^l$ is $\mathfrak{C}$-closed. It is therefore an A_T-module and

$$\begin{aligned}(\mathscr{B} \cap \mathscr{F}_2^l)^{\perp_2} &= \{\eta \in A_T^{1\times l} : \eta \circ (\mathscr{B} \cap \mathscr{F}_2^l) = 0\}\\ &= \mathscr{Q}(A^{1\times k}R) = A_T^{1\times k'}R'\end{aligned} \tag{8}$$

holds, i.e., the module of equations of $\mathscr{B} \cap \mathscr{F}_2^l$ is finitely generated as an A_T-module by the rows of R'. This means that $\mathscr{B} \cap \mathscr{F}_2^l$, although in general it is not an A-behaviour in $\mathscr{F}^l$, is an A_T-behaviour in $\mathscr{F}_2^l$. As a consequence, given two behaviours $\mathscr{B} = (A^{1\times k}R)^\perp$ and $\widetilde{\mathscr{B}} = (A^{1\times \widetilde{k}}\widetilde{R})^\perp$, one has the following set of equivalent conditions:

$$\begin{aligned}&\mathscr{B} \cap \mathscr{F}_2^l \subseteq \widetilde{\mathscr{B}} \cap \mathscr{F}_2^l\\ \Longleftrightarrow\ & \{w \in \mathscr{F}_2^l:\ R \circ w = 0\} \subseteq \{w \in \mathscr{F}_2^l:\ \widetilde{R} \circ w = 0\}\\ \Longleftrightarrow\ & \mathscr{Q}(A^{1\times k}R) \supseteq \mathscr{Q}(A^{1\times \widetilde{k}}\widetilde{R}) \Longleftrightarrow \exists X \in A_T^{\widetilde{k}'\times k'}:\ \widetilde{R}' = XR'\\ \overset{(6)}{\Longleftrightarrow}\ & \mathscr{Q}(A^{1\times k}R) \supseteq A^{1\times \widetilde{k}}\widetilde{R} \Longleftrightarrow \exists X \in A_T^{\widetilde{k}'\times k'}:\ \widetilde{R} = XR',\end{aligned} \tag{9}$$

where $\mathscr{Q}(A^{1\times k}R) = A_T^{1\times k'}R'$ and $\mathscr{Q}(A^{1\times \widetilde{k}}\widetilde{R}) = A_T^{1\times \widetilde{k}'}\widetilde{R}'$.

Furthermore, if $P \in A^{l_1\times l}$ then the image of $\mathscr{B} \cap \mathscr{F}_2^l$ under $P\circ$ is

$$P \circ (\mathscr{B} \cap \mathscr{F}_2^l) = (P \circ \mathscr{B}) \cap \mathscr{F}_2^{l_1}.$$

In other words,

$$\begin{aligned}\{w_1 \in \mathscr{F}^{l_1}:\ \exists w \in \mathscr{B} \cap \mathscr{F}_2^l : P \circ w = w_1\}\\ = \{w_1 \in \mathscr{F}_2^{l_1} : \exists w \in \mathscr{B}:\ P \circ w = w_1\}.\end{aligned}$$

Since the concept of free variables of a behaviour (in particular, of an IO behaviour) depends only on the rank of the matrices involved in the behaviour representation, a variable is free in $\mathscr{B}$ if and only if the same variable is free in $\mathscr{B} \cap \mathscr{F}_2^l$.

An IO behaviour $\mathscr{B} = (A^{1\times k}(P,\ -Q))^\perp$ is *$\mathfrak{C}$-stable* if its autonomous part $\mathscr{B}^0$ is $\mathfrak{C}$-negligible, i.e., equivalently, $\mathscr{B}^0 \in \mathfrak{C}$ or $\mathscr{B}^0 \cap \mathscr{F}_2^p = \{0\}$ or $\mathscr{Q}(A^{1\times k}P) = A_T^{1\times p}$ [8, Thm. and Def. 4.2]. If this is the case then the entries of the transfer matrix lie in A_T and

$$\mathscr{B} \cap \mathscr{F}_2^{p+m} = \left\{\binom{y}{u} \in \mathscr{F}_2^{p+m}:\ y = H \circ u\right\}, \tag{10}$$

i.e., $H \in A_T^{p\times m}$ can be seen as an operator that maps every steady state input to the corresponding steady state output [11, Cor. 3.8].

Lemma 2. *Let $\mathscr{B} = (A^{1\times k}(P,\ -Q))^\perp$ be a $\mathfrak{C}$-stable IO behaviour. The module of equations of its steady state behaviour $\mathscr{B} \cap \mathscr{F}_2^{p+m}$ is*

$$(\mathscr{B} \cap \mathscr{F}_2^{p+m})^{\perp_2} = A_T^{1\times p}(\mathrm{id}_p,\ -H). \tag{11}$$

Proof. Since

$$\begin{aligned}(\mathscr{B} \cap \mathscr{F}_2^{p+m})^{\perp_2} &\overset{(8)}{=} \mathscr{Q}(A^{1\times k}(P,\ -Q)) = \mathscr{Q}(A_T^{1\times k}(P,\ -Q))\\ &= \mathscr{Q}(A_T^{1\times k}P(\mathrm{id}_p,\ -H)),\end{aligned}$$

we need to show that

$$\mathscr{Q}(A_T^{1\times k}P(\mathrm{id}_p,\ -H)) = A_T^{1\times p}(\mathrm{id}_p,\ -H).$$

Consider the isomorphism

$$A_T^{1\times k}P \longrightarrow A_T^{1\times k}P(\mathrm{id}_p,\ -H),\ \eta_1 \longmapsto \eta_1(\mathrm{id}_p,\ -H)$$

with inverse

$$A_T^{1\times k}P(\mathrm{id}_p,\ -H) \longrightarrow A_T^{1\times k}P,\ (\eta_1,\ \eta_2) \longmapsto \eta_1.$$

Applying the functor $\mathscr{Q}$ leads to the isomorphism

$$\begin{aligned}\psi:\ \mathscr{Q}(A_T^{1\times k}P) &\longrightarrow \mathscr{Q}(A_T^{1\times k}P(\mathrm{id}_p,\ -H)),\\ \eta_1 &\longmapsto \eta_1(\mathrm{id}_p,\ -H)\end{aligned}$$

with inverse

$$\begin{aligned}\psi^{-1}:\ \mathscr{Q}(A_T^{1\times k}P(\mathrm{id}_p,\ -H)) &\longrightarrow \mathscr{Q}(A_T^{1\times k}P),\\ (\eta_1,\ \eta_2) &\longmapsto \eta_1.\end{aligned}$$

Notice that ψ and ψ^{-1} are given by the same matrices as the original maps. Since $\mathscr{B}$ is a $\mathfrak{C}$-stable IO behaviour, we have $\mathscr{Q}(A^{1\times k}P) = \mathscr{Q}(A_T^{1\times k}P) = A_T^{1\times p}$ and thus

$$\begin{aligned}\mathscr{Q}(A_T^{1\times k}P(\mathrm{id}_p,\ -H)) &= \psi\left(\psi^{-1}\left(\mathscr{Q}(A_T^{1\times k}P(\mathrm{id}_p,\ -H))\right)\right)\\ &= \psi(A_T^{1\times p}) = A_T^{1\times p}(\mathrm{id}_p,\ -H).\end{aligned}$$

□

III. Stabilization by Partial Interconnection

As a first step, we formally introduce the stabilization problem.

Definition 3. Given a plant

$$\mathscr{P}=\left\{\left(\begin{smallmatrix} w \\ c \end{smallmatrix}\right)\in\mathscr{F}^{l_w+l_c}:\ (R_w,\ R_c)\circ\left(\begin{smallmatrix} w \\ c \end{smallmatrix}\right)=0\right\}, \tag{12}$$

with $(R_w,\ R_c)\in A^{k_p\times(l_w+l_c)}$, we say that the compensator

$$\mathscr{C}=\{c\in\mathscr{F}^{l_c}:\ C_c\circ c=0\}, \tag{13}$$

with $C_c\in A^{k_c\times l_c}$, $\mathfrak{C}$*-stabilizes the plant* $\mathscr{P}$ if the partial interconnection

$$\mathscr{P}\wedge_c\mathscr{C}=\left\{\left(\begin{smallmatrix} w \\ c \end{smallmatrix}\right)\in\mathscr{F}^{l_w+l_c}:\ \left(\begin{smallmatrix} R_w & R_c \\ 0 & C_c \end{smallmatrix}\right)\circ\left(\begin{smallmatrix} w \\ c \end{smallmatrix}\right)=0\right\}$$

is *(i)* regular, i.e., $A^{1\times k_p}(R_w,\ R_c)\cap A^{1\times k_c}(0,\ C_c)=\{0\}$, and *(ii)* $\mathfrak{C}$-negligible, i.e., $\mathscr{P}\wedge_c\mathscr{C}\in\mathfrak{C}$.

The diagram illustrating the connection of a plant and a compensator is given in Figure 1. We refer to w as the to be controlled variable, and to c as the control variable.

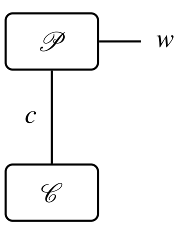

Fig. 1. The interconnection diagram for stabilization by partial interconnection.

Our goal is to find necessary and sufficient conditions for the existence of a $\mathfrak{C}$-stabilizing compensator for a given plant. We say that a plant is $\mathfrak{C}$*-stabilizable* if it admits a (partial regular) $\mathfrak{C}$-stabilizing compensator.

Lemma 4 (Characterization of $\mathfrak{C}$-stabilizing compensators)**.** *Let* $R'=(R'_w,\ R'_c)\in A_T^{k'_p\times(l_w+l_c)}$ *and* $C'_c\in A_T^{k'_c\times l_c}$ *be matrices such that*

$$\mathscr{Q}(A^{1\times k_p}(R_w,\ R_c))=A_T^{1\times k'_p}R' \text{ and } \mathscr{Q}(A^{1\times k_c}C_c)=A_T^{1\times k'_c}C'_c.$$

The behaviour $\mathscr{C}$ *described by* (13) *is a* $\mathfrak{C}$*-stabilizing compensator for the plant* $\mathscr{P}$ *given in* (12) *if and only if the sum of* $A_T^{1\times k'_p}R'$ *and* $A_T^{1\times k'_c}(0,\ C'_c)$ *is a direct one and it coincides with* $A_T^{1\times(l_w+l_c)}$*, namely* $A_T^{1\times k'_p}R'\oplus A_T^{1\times k'_c}(0,\ C'_c)=A_T^{1\times(l_w+l_c)}$.

Proof. By properties (3g) and (5) and the fact that $\mathscr{Q}(\{0\})=\{0\}$ (which follows from $\{0\}\in\mathfrak{C}\neq\emptyset$ and (3c)) we have

$$\begin{aligned} &A^{1\times k_p}(R_w,\ R_c)\cap A^{1\times k_c}(0,\ C_c)=\{0\}\\ &\quad\Longleftrightarrow \mathscr{Q}(A^{1\times k_p}(R_w,\ R_c))\cap\mathscr{Q}(A^{1\times k_c}(0,\ C_c))=\{0\}. \end{aligned}$$

Hence, the interconnection $\mathscr{P}\wedge_c\mathscr{C}$ is regular if and only if the sum of the Gabriel localizations of the modules of equations of $\mathscr{P}$ and $\mathscr{C}$ is direct.

Moreover, due to the direct sum decomposition (2) of a behaviour, we have that $\mathscr{P}\wedge_c\mathscr{C}=\mathrm{Ra}_{\mathfrak{C}}(\mathscr{P}\wedge_c\mathscr{C})\in\mathfrak{C}$ if and only if

$$\begin{aligned}(\mathscr{P}\wedge_c\mathscr{C})\cap\mathscr{F}_2^{l_w+l_c}&=\left\{\left(\begin{smallmatrix} w \\ c \end{smallmatrix}\right)\in\mathscr{F}_2^{l_w+l_c}:\ \left(\begin{smallmatrix} R_w & R_c \\ 0 & C_c \end{smallmatrix}\right)\circ\left(\begin{smallmatrix} w \\ c \end{smallmatrix}\right)=0\right\}\\ &=\{0\}.\end{aligned}$$

This is equivalent to

$$\begin{aligned} A_T^{1\times(l_w+l_c)} &\overset{\text{ass. on } A}{=} \mathscr{Q}(A^{1\times(l_w+l_c)})=\mathscr{Q}\left(A^{1\times(k_p+k_c)}\left(\begin{smallmatrix} R_w & R_c \\ 0 & C_c \end{smallmatrix}\right)\right)\\ &=\mathscr{Q}\left(A^{1\times k_p}(R_w,\ R_c)\oplus A^{1\times k_c}(0,\ C_c)\right)\\ &\overset{(3f)}{=}\mathscr{Q}\left(A^{1\times k_p}(R_w,\ R_c)\right)\oplus\mathscr{Q}\left(A^{1\times k_c}(0,\ C_c)\right)\\ &=A_T^{1\times k'_p}R'\oplus A_T^{1\times k'_c}(0,\ C'_c). \end{aligned}$$

□

This characterization allows us to derive a necessary and sufficient condition for the existence of a $\mathfrak{C}$-stabilizing compensator.

Theorem 5 (Existence of $\mathfrak{C}$-stabilizing compensators)**.** *Given a plant* $\mathscr{P}$*, described as in* (12)*, let* $R'=(R'_w,\ R'_c)\in A_T^{k'_p\times(l_w+l_c)}$ *be a matrix such that* $\mathscr{Q}(A^{1\times k_p}(R_w,\ R_c))=A_T^{1\times k'_p}R'$*. The following statements are equivalent:*

1. *There exists a* $\mathfrak{C}$*-stabilizing compensator* $\mathscr{C}$ *for* $\mathscr{P}$*.*
2. *The module* $A_T^{1\times k'_p}R'$ *has a direct complement in* $A_T^{1\times(l_w+l_c)}$ *and* $A_T^{1\times k'_p}R'_w=A_T^{1\times l_w}$*, i.e.,* R'_w *has a left inverse over* A_T*.*

Proof. $1\Longrightarrow 2$. Let $\mathscr{C}=(A^{1\times k_c}C_c)^{\perp}$ be a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$. Lemma 4 implies that

$$A_T^{1\times k'_p}(R'_w,\ R'_c)\oplus A_T^{1\times k'_c}(0,\ C'_c)=A_T^{1\times(l_w+l_c)},$$

where $C'_c\in A_T^{k'_c\times l_c}$ is a matrix such that $\mathscr{Q}(A^{1\times k_c}C_c)=A_T^{1\times k'_c}C'_c$. This means that $A_T^{1\times k'_p}(R'_w,\ R'_c)=A_T^{1\times k'_p}R'$ is a direct summand of $A_T^{1\times(l_w+l_c)}$. Furthermore, by focusing only on the first components in the previous identity, we get $A_T^{1\times k'_p}R'_w\oplus A_T^{1\times k'_c}0=A_T^{1\times l_w}$, i.e., $A_T^{1\times k'_p}R'_w=A_T^{1\times l_w}$.

$2\Longrightarrow 1$. Let $\widetilde{C}=(\widetilde{C}_w,\ \widetilde{C}_c)\in A_T^{k_c\times(l_w+l_c)}$ be such that $A_T^{1\times k_c}\widetilde{C}$ is a direct complement of $A_T^{1\times k'_p}R'$ in $A_T^{1\times(l_w+l_c)}$ and let $X\in A_T^{l_w\times k'_p}$ be a left inverse of R'_w. From $\widetilde{C}$ we can obtain the matrix

$$\begin{aligned} C=(C_w,\ C_c)&:=\widetilde{C}-\widetilde{C}_wXR'\\ &=(\widetilde{C}_w-\widetilde{C}_wXR'_w,\ \widetilde{C}_c-\widetilde{C}_wXR'_c)=(0,\ C_c) \end{aligned}$$

whose first block is zero. We want to show that also the row space of C is a direct complement of $A_T^{1\times k'_p}R'$. The identity

$$\begin{aligned} A_T^{1\times(l_w+l_c)}&=A_T^{1\times k'_p}R'+A_T^{1\times k_c}\widetilde{C}=A_T^{1\times(k'_p+k_c)}\left(\begin{smallmatrix} R' \\ \widetilde{C} \end{smallmatrix}\right)\\ &=A_T^{1\times(k'_p+k_c)}\underbrace{\begin{pmatrix} \mathrm{id}_{k'_p} & 0 \\ -\widetilde{C}_wX & \mathrm{id}_{k_c} \end{pmatrix}}_{=:Y}\left(\begin{smallmatrix} R' \\ \widetilde{C} \end{smallmatrix}\right)\\ &=A_T^{1\times(k'_p+k_c)}\left(\begin{smallmatrix} R' \\ C \end{smallmatrix}\right)=A_T^{1\times k'_p}R'+A_T^{1\times k_c}C \end{aligned}$$

holds, since Y invertible over A_T. We infer that $A_T^{1\times k_c}C$ is a complement of $A_T^{1\times k'_p}R'$ and, in particular, that rank $\left(\begin{smallmatrix} R' \\ C \end{smallmatrix}\right) =$ rank $\left(\begin{smallmatrix} R' \\ \widetilde{C} \end{smallmatrix}\right)$. One easily verifies that the equivalences

$$\begin{aligned} & A_T^{1\times k'_p}R' \cap A_T^{1\times k_c}\widetilde{C} = \{0\} \\ \Longleftrightarrow\ & K^{1\times k'_p}R' \cap K^{1\times k_c}\widetilde{C} = \{0\} \\ \Longleftrightarrow\ & \operatorname{rank}\begin{pmatrix} R' \\ \widetilde{C} \end{pmatrix} = \operatorname{rank}(R') + \operatorname{rank}(\widetilde{C}) \end{aligned}$$

hold. From $C = \widetilde{C}\left(\mathrm{id}_{l_w+l_c} - \left(\begin{smallmatrix} \mathrm{id}_{l_w} \\ 0 \end{smallmatrix}\right) XR'\right)$ we infer that $\operatorname{rank}(C) \leqslant \operatorname{rank}(\widetilde{C})$. These facts imply that the relations

$$\begin{aligned} \operatorname{rank}\begin{pmatrix} R' \\ \widetilde{C} \end{pmatrix} &= \operatorname{rank}\left(\begin{smallmatrix} R' \\ C \end{smallmatrix}\right) \leqslant \operatorname{rank}(R') + \operatorname{rank}(C) \\ &\leqslant \operatorname{rank}(R') + \operatorname{rank}(\widetilde{C}) = \operatorname{rank}\begin{pmatrix} R' \\ \widetilde{C} \end{pmatrix} \end{aligned}$$

are all equalities and thus

$$A_T^{1\times k'_p}R' \oplus A_T^{1\times k_c}C = A_T^{1\times(l_w+l_c)}.$$

We apply the functor $\mathscr{Q}$ to this equation, use the identities $\mathscr{Q}(A_T^{1\times(l_w+l_c)}) = A_T^{1\times(l_w+l_c)}$ and

$$\mathscr{Q}(A_T^{1\times k'_p}R') = \mathscr{Q}(\mathscr{Q}(A^{1\times k_p}R)) \overset{(3b)}{=} A_T^{1\times k'_p}R'$$

as well as the property (3f) and obtain

$$A_T^{1\times k'_p}R' \oplus \mathscr{Q}(A_T^{1\times k_c}C) = A_T^{1\times(l_w+l_c)}.$$

Let $t \in T$ be a common denominator of the entries of the matrix C, i.e., $tC \in A^{k_c\times(l_w+l_c)}$. This implies $A_T^{1\times k_c}C = A_T^{1\times k_c}tC$ and $\mathscr{Q}(A_T^{1\times k_c}C) = \mathscr{Q}(A_T^{1\times k_c}tC)$. Consequently, the behaviour $(A^{1\times k_c}tC)^{\perp}$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$ by Lemma 4. □

Remark 6. The second condition of item 2. in the previous theorem is equivalent to the existence of a $\mathfrak{C}$-observer of w from c for $\mathscr{P}$ [11, Thm. 4.4]. The existence of a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$ by (regular) partial interconnection via c is thus equivalent to the existence of a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$ by (regular) full interconnection and the existence of a $\mathfrak{C}$-observer of w from c for $\mathscr{P}$.

IV. Stabilization in the presence of external signals

In the previous section we have addressed the stabilization problem by assuming that the only variables involved are the control variable c and the to be controlled variable w. A more realistic scenario is the one where also external signals, acting on the plant and the compensator (see Figure 2), appear in the system description. Specifically, we assume for the plant and the controller the descriptions:

$$\mathscr{P} = \left\{ \begin{pmatrix} w \\ c \\ v \end{pmatrix} \in \mathscr{F}^{l_w+l_c+l_v} : \ (R_w,\ R_c,\ R_v) \circ \begin{pmatrix} w \\ c \\ v \end{pmatrix} = 0 \right\} \text{ and}$$

$$\mathscr{C} = \left\{ \left(\begin{smallmatrix} c \\ u \end{smallmatrix}\right) \in \mathscr{F}^{l_c+l_u} : \ (C_c,\ C_u) \circ \left(\begin{smallmatrix} c \\ u \end{smallmatrix}\right) = 0 \right\}$$

with $(R_w,\ R_c,\ R_v) \in A^{k_p\times(l_w+l_c+l_v)}$ and $(C_c,\ C_u) \in A^{k_c\times(l_c+l_u)}$. In the following, we use the notation $l_1 := l_w + l_c$, $l_2 = l_v + l_u$ and $l = l_1 + l_2$.

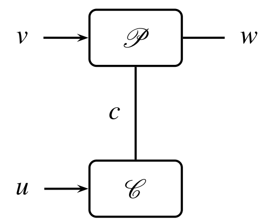

Fig. 2. The interconnection diagram for stabilization by partial interconnection with additional external inputs.

Definition 7. We say that $\mathscr{C}$ is a *$\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$* if

1. the interconnection $\mathscr{P} \wedge_c \mathscr{C}$ is regular,
2. $\left(\begin{smallmatrix} v \\ u \end{smallmatrix}\right)$ is free in $\mathscr{P} \wedge_c \mathscr{C}$, and
3. $\mathscr{N}_v(\mathscr{P}) \wedge_c \mathscr{N}_u(\mathscr{C}) \in \mathfrak{C}$.

The following technical lemma is of fundamental importance since it allows to extend the analysis of the previous section to the case of interconnection with exogenous signals.

Lemma 8. *The following facts are equivalent:*

1. $\mathscr{P} \wedge_c \mathscr{C}$ is regular and $\left(\begin{smallmatrix} v \\ u \end{smallmatrix}\right)$ is free in $\mathscr{P} \wedge_c \mathscr{C}$;
2. $\mathscr{N}_v(\mathscr{P}) \wedge_c \mathscr{N}_u(\mathscr{C})$ is regular, v is free in $\mathscr{P}$ and u is free in $\mathscr{C}$.

Proof. $2 \Longrightarrow 1$. In the computation

$$\begin{aligned} &\operatorname{rank}\left(\begin{smallmatrix} R_w & R_c & R_v & 0 \\ 0 & C_c & 0 & C_u \end{smallmatrix}\right) \\ &\quad \leqslant \operatorname{rank}(R_w,\ R_c,\ R_v,\ 0) + \operatorname{rank}(0,\ C_c,\ 0,\ C_u) \\ &\quad = \operatorname{rank}(R_w,\ R_c,\ R_v) + \operatorname{rank}(C_c,\ C_u) \\ &\quad \overset{*}{=} \operatorname{rank}(R_w,\ R_c) + \operatorname{rank}(C_c) \\ &\quad \overset{\dagger}{=} \operatorname{rank}\left(\begin{smallmatrix} R_w & R_c \\ 0 & C_c \end{smallmatrix}\right) \leqslant \operatorname{rank}\left(\begin{smallmatrix} R_w & R_c & R_v & 0 \\ 0 & C_c & 0 & C_u \end{smallmatrix}\right) \end{aligned}$$

the equality indicated by $*$ holds because v and u are free in $\mathscr{P}$ and $\mathscr{C}$, respectively, while the regularity of $\mathscr{N}_v(\mathscr{P}) \wedge_c \mathscr{N}_u(\mathscr{C})$ implies the equality †. As a consequence, all the relations are actually equalities. Since the expressions in the first two lines of the computation are equal, the interconnection $\mathscr{P} \wedge_c \mathscr{C}$ is regular. On the other hand, the fact that the expressions in the last line of the computation are equal implies that $\left(\begin{smallmatrix} v \\ u \end{smallmatrix}\right)$ is free in $\mathscr{P} \wedge_c \mathscr{C}$.

$1 \Longrightarrow 2$. Using a similar argument we obtain

$$\begin{aligned} &\operatorname{rank}\left(\begin{smallmatrix} R_w & R_c \\ 0 & C_c \end{smallmatrix}\right) \\ &\quad \leqslant \operatorname{rank}(R_w,\ R_c) + \operatorname{rank}(0,\ C_c) \\ &\quad \leqslant \operatorname{rank}(R_w,\ R_c,\ R_v,\ 0) + \operatorname{rank}(0,\ C_c,\ 0,\ C_u) \\ &\quad \overset{*}{=} \operatorname{rank}\left(\begin{smallmatrix} R_w & R_c & R_v & 0 \\ 0 & C_c & 0 & C_u \end{smallmatrix}\right) \overset{\dagger}{=} \operatorname{rank}\left(\begin{smallmatrix} R_w & R_c \\ 0 & C_c \end{smallmatrix}\right). \end{aligned}$$

Again, the equality caused by the regularity is indicated by $*$ and the one caused by the freeness by †. All relations are equalities, therefore $\mathscr{P} \wedge_c \mathscr{C}$ is regular. Furthermore, from

$$\begin{gathered} \operatorname{rank}(R_w,\ R_c) \leqslant \operatorname{rank}(R_w,\ R_c,\ R_v), \\ \operatorname{rank}(C_c) \leqslant \operatorname{rank}(C_c,\ C_u), \\ \operatorname{rank}(R_w, R_c) + \operatorname{rank}(C_c) = \operatorname{rank}(R_w, R_c, R_v) + \operatorname{rank}(C_c, C_u) \end{gathered}$$

we conclude that $\operatorname{rank}(R_w,\ R_c) = \operatorname{rank}(R_w,\ R_c,\ R_v)$ and $\operatorname{rank}(C_c) = \operatorname{rank}(C_c,\ C_u)$, i.e., the freeness of v and u in $\mathscr{P}$ and $\mathscr{C}$, respectively. ☐

A direct consequence of this lemma is that $\mathscr{C}$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$ in the present set-up if and only if v is free in $\mathscr{P}$, $\mathscr{N}_u(\mathscr{C})$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{N}_v(\mathscr{P})$ in the previous setting (without external signals) and u is free in $\mathscr{C}$. With the next lemma we concretize this relationship.

Lemma 9. *Assume that v is free in $\mathscr{P}$.*

1. *If $(A^{1\times k_c}C_c)^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{N}_v(\mathscr{P})$, then for every $D \in A^{l_c\times l_u}$ and $t \in T$ we have that $(A^{1\times k_c}(tC_c,\ C_cD))^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$.*
2. *If $(A^{1\times k_c}(C_c,\ C_u))^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$ then $(A^{1\times k_c}C_c)^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{N}_v(\mathscr{P})$, and there exists a matrix $D \in A_T^{l_c\times l_u}$ such that $C_u = C_cD$.*

Proof. 1. Since $t \in T$, t is an invertible element in A_T. This implies $A_T^{1\times k_c}tC_c = A_T^{1\times k_c}C_c$, and hence

$$\begin{aligned}\mathscr{Q}(A^{1\times k_c}tC_c) &= \mathscr{Q}(A_T^{1\times k_c}tC_c) = \mathscr{Q}(A_T^{1\times k_c}C_c)\\ &= \mathscr{Q}(A^{1\times k_c}C_c).\end{aligned}$$

This implies that also $(A^{1\times k_c}tC_c)^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{N}_v(\mathscr{P})$. Furthermore, since the columns of C_cD belong to the column space of C_c over the quotient field K, we have $\operatorname{rank}(tC_c) = \operatorname{rank}(tC_c,\ C_cD)$. In other words, the variables u (corresponding to the right block) are free in $(A^{1\times k_c}(tC_c,\ C_cD))^\perp$. The assertion follows from Lemma 8.

2. $(A^{1\times k_c}C_c)^\perp$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{N}_v(\mathscr{P})$ by point 3. of Definition 7 and Lemma 8. By assumption, $\mathscr{P}\wedge_c\mathscr{C}$ is a $\mathfrak{C}$-stable IO behaviour with input $\binom{v}{u}$ and output $\binom{w}{c}$. Thus by [8, Thm. and Def. 4.2] there exists a transfer matrix

$$\begin{pmatrix} H_1 & H_2\\ H_3 & H_4\end{pmatrix} \in A_T^{l_1\times l_2} \text{ such that}$$
$$\begin{pmatrix} R_w & R_c\\ 0 & C_c\end{pmatrix}\begin{pmatrix} H_1 & H_2\\ H_3 & H_4\end{pmatrix} = \begin{pmatrix} R_v & 0\\ 0 & C_u\end{pmatrix}.$$

This implies, in particular, that $C_cH_4 = C_u$. So, the statement holds for $D = H_4$. ☐

V. Regulation by Partial Interconnection

Consider a plant $\mathscr{P}$ and a compensator $\mathscr{C}$ described as in Section IV. In addition, assume that the external signals v and u are the trajectories of two behaviours

$$\begin{aligned}\mathscr{E}_\mathscr{P} &= \{v \in \mathscr{F}^{l_v}:\ V\circ v = 0\}, \quad V \in A^{k_1\times l_v}, \text{ and}\\ \mathscr{E}_\mathscr{C} &= \{u \in \mathscr{F}^{l_u}:\ U\circ u = 0\}, \quad U \in A^{k_2\times l_u},\end{aligned}$$

(see Figure 3). Introduce the matrix

$$K = (K_w,\ K_c,\ K_v,\ K_u) \in A^{k\times(l_w+l_c+l_v+l_u)}.$$

The control goal we investigate in this section is that of finding a compensator $\mathscr{C}$ such that $K\circ\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix}$ is negligible for all trajectories $\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix}$ of the interconnected behaviour

$$\begin{aligned}\mathscr{B} &:= \mathscr{E}_\mathscr{P}\wedge_v\mathscr{P}\wedge_c\mathscr{C}\wedge_u\mathscr{E}_\mathscr{C}\\ &= \left\{\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix} \in \mathscr{F}^l:\ \begin{pmatrix} R_w & R_c & R_v & 0\\ 0 & C_c & 0 & C_u\\ 0 & 0 & V & 0\\ 0 & 0 & 0 & U\end{pmatrix}\circ\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix} = 0\right\}\end{aligned}$$

i.e., they satisfy $\begin{pmatrix} w\\ c\\ v\end{pmatrix} \in \mathscr{P}$, $\binom{c}{u} \in \mathscr{C}$, $v \in \mathscr{E}_\mathscr{P}$ and $u \in \mathscr{E}_\mathscr{C}$. For example, the choice $K = (\mathrm{id},\ 0,\ 0,\ -\mathrm{id})$ corresponds to designing a compensator in such a way that, in the interconnected behaviour, the trajectories of the difference variable $K\circ\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix} = w - u$ are small. This means that the to be controlled variable w coincides (modulo a transient signal) with the tracking variable u, and is not affected by the other variables, in particular by the disturbance v.

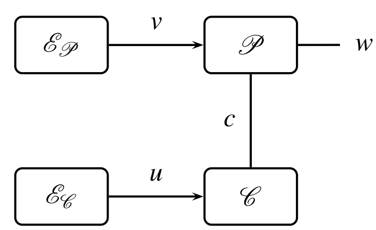

Fig. 3. The interconnection diagram for the regulation problem.

Definition 10. We call the behaviour $\mathscr{C}$ a $\mathfrak{C}$-*regulator* of $\mathscr{P}$, with respect to $\mathscr{E}_\mathscr{P}$, $\mathscr{E}_\mathscr{C}$ and K, if the following conditions are satisfied:

1. $\mathscr{C}$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$, in the sense of Section IV, and
2. $K\circ\mathscr{B}$, is $\mathfrak{C}$-negligible, where $\mathscr{B}$ is the interconnected behaviour.

Notice that from condition 1. it follows that the interconnection $\mathscr{P}\wedge_c\mathscr{C}$ is regular and $\binom{v}{u}$ is free in this behaviour. Consequently, $\mathscr{B} = \mathscr{E}_\mathscr{P}\wedge_v\mathscr{P}\wedge_c\mathscr{C}\wedge_u\mathscr{E}_\mathscr{C}$ is a regular interconnection, too.

Theorem 11 (Characterization of $\mathfrak{C}$-regulators)**.** *Assume that $\mathscr{C}$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$. Then $K\circ\mathscr{B} \in \mathfrak{C}$, namely $K\circ\mathscr{B}$ is $\mathfrak{C}$-negligible, if and only if*

$$\begin{aligned}&(\mathscr{E}_\mathscr{P}\times\mathscr{E}_\mathscr{C})\cap\mathscr{F}_2^{l_2} \subseteq \operatorname{proj}_{v,u}\left((\mathscr{P}\wedge_c\mathscr{C})\cap(A^{1\times k}K)^\perp\right)\cap\mathscr{F}_2^{l_2}\\ &= \left\{\binom{v}{u} \in \mathscr{F}_2^{l_2}:\ \exists\binom{w}{c} \in \mathscr{F}_2^{l_1}:\ \begin{pmatrix} R_w & R_c & R_v & 0\\ 0 & C_c & 0 & C_u\\ K_w & K_c & K_v & K_u\end{pmatrix}\circ\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix} = 0\right\}.\end{aligned}$$

Proof. **Necessity.** Assume that $K\circ\mathscr{B} \in \mathfrak{C}$. Let $\binom{v}{u} \in (\mathscr{E}_\mathscr{P}\times\mathscr{E}_\mathscr{C})\cap\mathscr{F}_2^{l_2}$. The behaviour $\mathscr{P}\wedge_c\mathscr{C}$ is an IO behaviour and this property is preserved when going over to $(\mathscr{P}\wedge_c\mathscr{C})\cap\mathscr{F}_2^l$. Let thus $\binom{w}{c} \in \mathscr{F}_2^{l_1}$ be such that $\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix} \in (\mathscr{P}\wedge_c\mathscr{C})\cap\mathscr{F}_2^l$. Then $\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix} \in \mathscr{B}\cap\mathscr{F}_2^l$. From $K\circ\mathscr{B} \in \mathfrak{C}$ we infer

$(K\circ\mathscr{B})\cap\mathscr{F}_2^l = K\circ(\mathscr{B}\cap\mathscr{F}_2^l)=0$ and conclude $K\circ\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix}\in K\circ(\mathscr{B}\cap\mathscr{F}_2^l)=0$, i.e., $K\circ\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix}=0$.

Sufficiency. Let $\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix}\in\mathscr{B}\cap\mathscr{F}_2^l$. Then $\binom{v}{u}\in(\mathscr{E}_\mathscr{P}\times\mathscr{E}_\mathscr{C})\cap\mathscr{F}_2^{l_2}$ and thus by assumption there exists $\binom{\widetilde{w}}{\widetilde{c}}\in\mathscr{F}_2^{l_1}$ such that

$$\begin{pmatrix} \widetilde{w}\\ \widetilde{c}\\ v\\ u\end{pmatrix}\in(\mathscr{P}\wedge_c\mathscr{C})\cap\mathscr{F}_2^l \quad\text{and}\quad K\circ\begin{pmatrix} \widetilde{w}\\ \widetilde{c}\\ v\\ u\end{pmatrix}=0.$$

But from the definition of $\mathscr{B}$ it follows that also $\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix}\in(\mathscr{P}\wedge_c\mathscr{C})\cap\mathscr{F}_2^l$. Thus the difference $\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix}-\begin{pmatrix} \widetilde{w}\\ \widetilde{c}\\ v\\ u\end{pmatrix}$ lies in $(\mathscr{P}\wedge_c\mathscr{C})\cap\mathscr{F}_2^l$ too and, since the inputs of this trajectory are zero, $\begin{pmatrix} w-\widetilde{w}\\ c-\widetilde{c}\end{pmatrix}\in\mathscr{N}_{v,u}((\mathscr{P}\wedge_c\mathscr{C}))\cap\mathscr{F}_2^l$ is an element of the steady states of the autonomous part. But since $\mathscr{P}\wedge_c\mathscr{C}$ is $\mathfrak{C}$-stable, i.e., the autonomous part is $\mathfrak{C}$-negligible, this trajectory must be zero. Therefore, $w=\widetilde{w}$, $c=\widetilde{c}$ and

$$K\circ\begin{pmatrix} w\\ c\\ v\\ u\end{pmatrix}=K\circ\begin{pmatrix} \widetilde{w}\\ \widetilde{c}\\ v\\ u\end{pmatrix}=0.$$

This ensures that $K\circ\mathscr{B}$ is $\mathfrak{C}$-negligible. □

The following algorithm provides a procedure to test whether the necessary and sufficient condition provided in the previous theorem is satisfied.

Algorithm 12. *Given a plant $\mathscr{P}$, a compensator $\mathscr{C}$, external behaviours $\mathscr{E}_\mathscr{P}$ and $\mathscr{E}_\mathscr{C}$, and a control goal K, in order to test whether $\mathscr{C}$ is a $\mathfrak{C}$-regulator for this set-up, the following steps need to be taken:*

1. *Verify that $\mathfrak{C}$ is a $\mathfrak{C}$-stabilizing compensator for $\mathscr{P}$. Check the conditions*
$$\operatorname{rank}(R_w,\ R_c,\ R_v)=\operatorname{rank}(R_w,\ R_c),$$
$$\operatorname{rank}(C_c,\ C_u)=\operatorname{rank}(C_c)\ \textit{and}$$
$$\operatorname{rank}\begin{pmatrix} R_w & R_c\\ 0 & C_c\end{pmatrix}=\operatorname{rank}(R_w,\ R_c)+\operatorname{rank}(C_c)$$
to ascertain the freeness of v and u and the regularity of the interconnection. The methods to test if $\mathscr{N}_v(\mathscr{P})\wedge_c\mathscr{N}_u(\mathscr{C})$ is $\mathfrak{C}$-negligible depend on the specific Serre subcategory $\mathfrak{C}$.
2. *Compute a universal left annihilator $X\in A^{m\times(k_p+k_c+k)}$ of $\begin{pmatrix} R_w & R_c\\ 0 & C_c\\ K_w & K_c\end{pmatrix}$ and matrices $V'\in A_T^{k_1'\times l_v}$ and $U'\in A_T^{k_2'\times l_u}$ such that*
$$\mathscr{Q}(A^{1\times k_1}V)=A_T^{1\times k_1'}V'\ \textit{and}\ \mathscr{Q}(A^{1\times k_2}U)=A_T^{1\times k_2'}U',$$
(see [11, Alg. 3.9]). Then, check if the inhomogeneous linear matrix equation
$$Y\begin{pmatrix} V' & 0\\ 0 & U'\end{pmatrix}=X\begin{pmatrix} R_v & 0\\ 0 & C_u\\ K_v & K_u\end{pmatrix}$$
has a solution $Y\in A_T^{m\times(k_1'+k_2')}$. If this is the case, $\mathscr{C}$ is a $\mathfrak{C}$-regulator in this setting.

Solving systems of inhomogeneous linear equations over quotient rings A_T can be difficult and it is still an unsolved problem in many cases. However, in [11, Sec. 7], an algorithm is given which performs this task in a number of important situations.

REFERENCES

[1] M. Belur and H. Trentelman, "On stabilization, pole placement and regular implementability," *IEEE Trans. Automat. Control*, vol. 47, no. 5, pp. 735–744, 2002.

[2] I. Blumthaler, "Stabilization and control design by partial output feedback and by partial interconnection," *Internat. J. Control*, vol. 85, no. 11, pp. 1717–1736, 2012.

[3] S. Fiaz, T. K., and H. Trentelman, "Tracking and regulation in the behavioral framework," *Automatica*, vol. 47, no. 11, pp. 2343–2354, 2011.

[4] S. Fiaz and H. Trentelman, "Regular implementability and stabilization using controllers with pre-specified input/output partition," *IEEE Trans. Automat. Control*, vol. 54, no. 7, pp. 1561–1568, 2009.

[5] D. Napp and P. Rocha, "Strongly autonomous interconnections and stabilization of 2D behaviors," *Asian J. Control*, vol. 12, no. 2, pp. 127–135, 2010.

[6] ——, "Stabilization of discrete 2D behaviors by regular partial interconnection," *Math. Control Signals Systems*, vol. 22, no. 4, pp. 295–316, 2011.

[7] U. Oberst, "Multidimensional constant linear systems," *Acta Appl. Math.*, vol. 20, pp. 1–175, 1990.

[8] ——, "Stability and stabilization of multidimensional input/output systems," *SIAM J. Control Optim.*, vol. 45, no. 4, pp. 1467–1507, 2006.

[9] C. Praagman, H. Trentelman, and R. Zavala Yoe, "On the parametrization of all regularly implementing and stabilizing controllers," *SIAM J. Control Optim.*, vol. 45, no. 6, pp. 2035–2053, 2007.

[10] P. Rocha, "Stabilization of multidimensional behaviors," *Multidimensional Syst. Signal Process.*, vol. 19, no. 3–4, pp. 273–286, 2008.

[11] M. Scheicher and U. Oberst, "Multidimensional discrete stability by Serre categories and the construction and parametrization of observers via Gabriel localizations," *SIAM J. Control Optim.*, vol. 51, no. 3, pp. 1873–1908, 2013.

[12] J.-P. Serre, "Groupes d'homotopie et classes de groupes abéliens," *Ann. of Math. (2)*, vol. 58, pp. 258–294, 1953.

[13] B. Stenström, *Rings of quotients*. New York: Springer-Verlag, 1975.

[14] H. Trentelman and D. Napp Avelli, "On the regular implementability of *n*D systems," *Systems Control Lett.*, vol. 56, no. 4, pp. 265–271, 2007.

[15] J. Trumpf, H. Trentelman, and J. Willems, "An internal model principle for observers," in *Proc. of the 2011 50th IEEE Conference on Decision and Control and European Control Conference*, Orlando (FL), 2011, pp. 3992–3999.

[16] J. Willems, "Control as interconnection," in *Feedback control, nonlinear systems and complexity*, B. Francis and A. Tannenbaum, Eds. Springer, 1995, vol. 202, pp. 261–275.

[17] ——, "On interconnections, control, and feedback," *IEEE Trans. Automat. Control*, vol. 42, no. 3, pp. 326–339, 1997.