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Regulator equations for boundary control systems

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Abstract— **In this paper we address the state feedback regulator problem for regular boundary control systems (a special class of infinite-dimensional linear systems). The plant is assumed to be exponentially stable and is driven by a linear (possibly infinite-dimensional) exosystem via a disturbance signal. The exosystem has its spectrum in the closed right half-plane and also generates the reference signal for the plant output. The regulator problem is to design a controller that, while guaranteeing the stability of the closed-loop system without the exosystem, drives the tracking error to zero. A particular version of this problem is the state feedback regulator problem in which the states of the exosystem and the plant are known to the controller. Under suitable assumptions, we show that the latter problem is solvable if and only if a system of three algebraic equations, called the regulator equations, is solvable. We derive conditions, in terms of the transfer function of the plant and eigenvalues of the exosystem, for the solvability of the regulator equations. An example illustrating our theory is presented.**

I. INTRODUCTION

In this paper we study the tracking and disturbance rejection problem, also called the *regulator problem*, for plants in a special class of linear infinite-dimensional systems called the regular boundary control systems. We assume that the reference and disturbance signals are produced by a linear unstable signal generator called the *exosystem*. There are two standard versions of the regulator problem: In the first, called the *state feedback regulator problem*, the controller is provided with full information of the state of the plant and the exosystem, while in the second version, called the *error feedback regulator problem*, only the tracking error is available to the controller. In this work we will focus on the state feedback version alone, under the assumption that the plant is exponentially stable.

Pioneering work on the regulator problems for linear finite-dimensional systems is in Francis [7], where the solvability of these problems is shown to be equivalent to the solvability of a pair of linear matrix equations called the *regulator equations*. Similar results have been established for finite-dimensional nonlinear systems in Byrnes and Isidori [2] under the assumption that the plant is locally exponentially stabilizable and the exosystem has a Lyapunov stable equilibrium at the origin with each initial condition in a neighborhood of this origin being Poisson stable. It is shown in [2] that the solvability conditions given in [7] can be generalized naturally in terms of the solvability of a pair of nonlinear equations – still called the regulator equations. A passivity-based approach to the nonlinear regulator problem has been explored in Jayawardhana and Weiss [13], [14].

In Byrnes *et al* [3], building on the results in [7], a geometric theory of output feedback regulation for infinite-dimensional linear plants with bounded control and observation operators driven by finite-dimensional exosystems has been developed. In particular in [3] the solvability of both the state and error feedback regulator problems has been characterized in terms of the solvability of certain equations referred to, once again, as the regulator equations. Also, simple criteria for the solvability of the regulator equations have been derived.

Regulator theory for infinite-dimensional linear systems with bounded control and observation operators has been significantly advanced by a group of researchers at Tampere University of Technology (Finland) who have developed a sophisticated theory of infinite-dimensional exosystems, see for instance [8], [11], [12], [10], [16]. The state feedback regulator problem for exponentially stabilizable linear plants driven by infinite-dimensional exosystems generating periodic signals was addressed in [11]. The results in [11] were generalized in [12] by considering strongly stabilizable plants and a broader class of exosystems, and addressing both the state and error feedback regulator problems. The recent paper Boulite *et al* [1] builds on the above works to address the state feedback regulator problem for polynomially stabilizable linear plants driven by infinite-dimensional exosystems.

In the literature on the regulator problem, including the works mentioned above, it is usually assumed (to avoid technical difficulties) that the control and observation operators of the plant are bounded. In our recent paper [15] we eliminated this limitation and extended the key results in [3] on the state feedback regulator problem to exponentially stable plants that are regular linear systems. Regular linear systems model many physical systems that have unbounded control and observation operators. Under some assumptions, we showed that the regulator problem is solvable for a regular linear system if and only if a pair of algebraic equations, called the regulator equations, is solvable. Using the solution to the regulator equations (when it exists), we can design a controller that solves the state feedback regulator problem.

In this paper we focus on exponentially stable plants belonging to a special class of regular linear systems called the regular boundary control systems. There is considerable interest in plants with boundary control and/or boundary observation, for which the control and/or observation operators

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are unbounded, see for instance Staffans [17], Tucsnak and Weiss [19], [20]. For plants that are regular boundary control systems, it follows from [15] that the state feedback regulator problem is solvable if and only if the pair of regulator equations is solvable. In the present work we will establish that this pair of regulator equations is solvable if and only if a system of three algebraic equations is solvable. The solution to the pair of regulator equations (when it exists) is also a solution to the system of three algebraic equations and vice versa. The value of this result is that it significantly simplifies the process of solving the regulator equations. We will illustrate this with an example. In this work, the system of three algebraic equations will also be referred to as the regulator equations.

We assume that the plant is exponentially stable and not just stabilizable, the latter assumption being customary in regulator theory. This is not limiting, since in regulator theory the problems of stabilization and regulation can be decoupled and addressed sequentially. Hence we shall assume that the plant has been stabilized via a suitable feedback and we shall solve the regulator problem for the exponentially stable plant. We also assume that the state operator of the linear, unstable and possibly infinite-dimensional exosystem is bounded.

Following [3], we characterize the solvability of the regulator equations in terms of the zeros of the plant transfer function and the natural frequencies of the exosystem. Under some reasonable additional assumptions on the exosystem, we give an explicit formula for the feedback operator that solves the state feedback regulator problem. In the last section we show how the theory developed in this work can be applied to solve the tracking problem for a Rayleigh beam with structural damping. The control is the torque applied at one end-point and the output is the angular velocity at the same point. This output is required to track a sinusoidal reference signal.

II. Background

A. Regular linear systems

This subsection is a very brief overview of regular systems theory, mostly following [18], [19], [21]. For a Hilbert space Y and $\alpha \in \mathbb{R}$ we define the weighted function space

$$L^2_\alpha([0,\infty);Y) = \left\{ \phi \in L^2_{loc}([0,\infty);Y) \,\middle|\, \int_0^\infty e^{-2\alpha t}\|\phi(t)\|^2 \mathrm{d}t < \infty \right\},$$

with the norm being the square-root of the integral appearing above. For any $a \in \mathbb{R}$ we define the open and closed right half-planes bounded by a, by

$$\mathbb{C}^+_a = \left\{ s \in \mathbb{C} \,\middle|\, \mathrm{Re}\, s > a \right\}, \qquad \overline{\mathbb{C}^+_a} = \left\{ s \in \mathbb{C} \,\middle|\, \mathrm{Re}\, s \geq a \right\}.$$

Let Z be a Hilbert space and A the generator of an operator semigroup (also called strongly continuous semigroup of operators) $\mathbb{T}$ on Z. We denote by $\rho(A)$ the resolvent set of A. We define two new Hilbert spaces as follows: for $\beta \in \rho(A)$,

$$Z_1 = \mathcal{D}(A) \quad \text{with} \quad \|z\|_1 = \|(\beta I - A)z\|$$

and the space Z_{-1} is the completion of Z with respect to the norm

$$\|z\|_{-1} = \|(\beta I - A)^{-1}z\|.$$

These spaces are independent of the choice of β and we have the dense embeddings

$$Z_1 \hookrightarrow Z \hookrightarrow Z_{-1}. \tag{2.1}$$

The operators $\mathbb{T}_t$ extend to Z_{-1}, and the generator of the extended semigroup is an extension of A to an operator in $\mathcal{L}(Z, Z_{-1})$. We use the same notation $\mathbb{T}_t$ and A for these extended operators. We denote by $\omega_0(\mathbb{T})$ the *growth bound* of the semigroup $\mathbb{T}$. Recall that $\mathbb{T}$ (or A) is called *exponentially stable* if $\omega_0(\mathbb{T}) < 0$.

If $C \in \mathcal{L}(Z_1, Y)$, where Y is another Hilbert space, then the Λ-*extension* of C (with respect to A), denoted C_Λ, is defined as follows (see [22]):

$$C_\Lambda z = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} z \tag{2.2}$$

and its domain $\mathcal{D}(C_\Lambda)$ consists of those $z \in Z$ for which the above limit exists.

We call C an *admissible observation operator* for $\mathbb{T}$ if for some (hence, for every) $\tau > 0$ there exists $m_\tau > 0$ such that

$$\int_0^\tau \|C\mathbb{T}_t z\|^2 \mathrm{d}t \leq m_\tau \|z\|^2 \qquad \forall\, z \in \mathcal{D}(A). \tag{2.3}$$

In this case, for every $z \in Z$, the formula $y(t) = C_\Lambda \mathbb{T}_t z$ makes sense for almost every $t \geq 0$ and it defines a function $y \in L^2_\alpha([0,\infty);Y)$, for every $\alpha > \omega_0(\mathbb{T})$. Also, (2.3) becomes valid for all $z \in Z$ if we replace C with C_Λ. The dual of the above admissibility concept can be expressed as follows: if U is a Hilbert space and $B \in \mathcal{L}(U, Z_{-1})$, then B is called an *admissible control operator* for $\mathbb{T}$ if for some (hence, for every) $\tau > 0$ and for every $u \in L^2([0,\infty);U)$,

$$\int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma) \mathrm{d}\sigma \in Z.$$

Note that this integral gives the strong solution of $\dot z(t) = Az(t) + Bu(t)$ at time τ, if $z(0) = 0$. In this case, $z(\tau)$ depends continuously on u and on τ, hence there exists $\kappa_\tau > 0$ such that

$$\left\| \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma) \mathrm{d}\sigma \right\| \leq \kappa_\tau \|u\|_{L^2([0,\tau];U)}.$$

Definition 2.1: Consider the generator A of a strongly continuous semigroup $\mathbb{T}$ on Z, an admissible control operator $B \in \mathcal{L}(U, Z_{-1})$ and an admissible observation operator $C \in \mathcal{L}(Z_1, Y)$, as defined earlier. The triple (A, B, C) is called *regular*, in the sense of [21], [22], if in addition the following conditions hold:
(1) $C_\Lambda(sI - A)^{-1}B$ exists for some (hence, for every) $s \in \rho(A)$ (this means that we have $(sI - A)^{-1}BU \subset \mathcal{D}(C_\Lambda)$).
(2) The mapping $\mathbf{G}_0(s) = C_\Lambda(sI - A)^{-1}B$, called the *transfer function* associated to the triple (A, B, C), is bounded on some right half-plane.

The fact that (A, B, C) is a regular triple is equivalent to the fact that for some (hence, for every) $D \in \mathcal{L}(U, Y)$, the equations

$$\dot z(t) = Az(t) + Bu(t), \qquad y(t) = C_\Lambda z(t) + Du(t), \tag{2.4}$$

define a *regular linear system* Σ. This system has input space U, state space Z and output space Y. The signals u, z and

y are called the *input, state trajectory* and *output* of Σ. A is called the *semigroup generator* of Σ, B is called the *control operator* of Σ, C is called the *observation operator* of Σ and D is called the *feedthrough operator* of Σ. For any initial state $z(0) = z_0 \in Z$ and for any $u \in L^2_\alpha([0,\infty);U)$, the equations (2.4) describing Σ have unique solutions z and y such that z is continuous, $y \in L^2_\gamma([0,\infty);Y)$ for all $\gamma \geq \alpha$ with $\gamma > \omega_0(\mathbb{T})$ and both equations hold for almost every $t \geq 0$. The *transfer function* of Σ is $\mathbf{G}(s) = \mathbf{G}_0(s) + D$, which means that

$$\hat{y}(s) = C(sI - A)^{-1} z_0 + \mathbf{G}(s)\hat{u}(s),$$

where a hat is used to denote the Laplace transformation, and this formula holds for all s in the right half-plane $\mathbb{C}^+_\gamma$. The *generating operators* of Σ are (A, B, C, D) and every regular linear system is determined by its four generating operators.

B. Boundary control systems

The operators $\tilde{A} \in \mathcal{L}(\underset{\sim}{Z}, Z)$ and $G \in \mathcal{L}(\underset{\sim}{Z}, \tilde{U})$, where $\underset{\sim}{Z}$, Z and $\tilde{U}$ are Hilbert spaces such that $\underset{\sim}{Z} \subset Z$ with continuous embedding, define a boundary control system in the sense of [19, Section 10.1] (with input space $\tilde{U}$ and state space Z) if there exists a $\beta \in \mathbb{C}$ such that the following properties hold:

(i) G is onto,
(ii) $\operatorname{Ker} G$ is dense in Z,
(iii) $\beta I - \tilde{A}$ restricted to $\operatorname{Ker} G$ is onto,
(iv) $\operatorname{Ker}(\beta I - \tilde{A}) \cap \operatorname{Ker} G = 0$.

Define $A = \tilde{A}\big|_{\operatorname{Ker} G}$. Then $A \in \mathcal{L}(\operatorname{Ker} G, Z)$. There is a unique $\tilde{B} \in \mathcal{L}(\tilde{U}, Z_{-1})$ such that $\tilde{A} = A + \tilde{B}G$ where A is regarded as an operator in $\mathcal{L}(Z, Z_{-1})$. Moreover, for every $s \in \rho(A)$ we have $(sI - A)^{-1}\tilde{B} \in \mathcal{L}(\tilde{U}, \underset{\sim}{Z})$ and

$$G(sI - A)^{-1}\tilde{B} = I.$$

Notice that we have $G\phi = 0$ for all $\phi \in \mathcal{D}(A)$.

Indeed, all this follows from Proposition 10.1.2 in [19] and the text around it, if we use the following correspondence of the notation: what is called X, Z, U, L, G, A, B in [19] is called here (in the same order) $Z, \underset{\sim}{Z}, \tilde{U}, \tilde{A}, G, A, \tilde{B}$.

III. The plant, the exosystem and the error

In this section we describe the basic assumptions about the plant to be controlled and the exosystem, and we present some simple consequences of these assumptions without proof (for proofs see [15]). The *plant* is a *regular boundary control system* described by the following equations ($t \geq 0$):

$$\begin{cases} \dot{z}(t) = \tilde{A}z(t), \qquad Gz(t) = \mathcal{B}u(t) + \mathcal{B}^1 d(t), \\ y(t) = C_\Lambda z(t) + Du(t) + D^1 d(t). \end{cases} \tag{3.1}$$

The state of this system is $z(t)$, its input signal is $\left[\begin{smallmatrix} u \\ d \end{smallmatrix}\right]$ and its output signal is y. We regard u as the control input (to be generated by a controller) while d is a disturbance. We have $z(t) \in Z$, where the state space Z is assumed to be a Hilbert space. We have $u(t) \in U$, $d(t) \in U^1$ and $y(t) \in Y$, where U, U^1 and Y are Hilbert spaces. The operators $\tilde{A} \in \mathcal{L}(\underset{\sim}{Z}, Z)$ and $G \in \mathcal{L}(\underset{\sim}{Z}, \tilde{U})$, as introduced in Section II-B, define a boundary control system. We assume that A, defined as the restriction of $\tilde{A}$ to $\operatorname{Ker} G$, is the generator of an *exponentially stable* operator semigroup $\mathbb{T}$ on Z and that $\mathcal{B} \in \mathcal{L}(U, \tilde{U})$ and $\mathcal{B}^1 \in \mathcal{L}(U^1, \tilde{U})$. Denote $B = \tilde{B}\mathcal{B}$ and $B^1 = \tilde{B}\mathcal{B}^1$. The control operator $B \in \mathcal{L}(U, Z_{-1})$ is admissible for $\mathbb{T}$, while $B^1 \in \mathcal{L}(U^1, Z_{-1})$ (not necessarily admissible). The observation operator $C \in \mathcal{L}(Z_1, Y)$ is admissible for $\mathbb{T}$, $D \in \mathcal{L}(U, Y)$ and $D^1 \in \mathcal{L}(U^1, Y)$. We assume that the triple (A, B, C) is regular and for some (hence, for every) $s \in \rho(A)$, the product $C_\Lambda(sI - A)^{-1}B^1$ exists (which is weaker than demanding (A, B^1, C) to be regular). When $\operatorname{Ran}[\mathcal{B}\ \mathcal{B}^1] = \tilde{U}$, it can be shown that

$$\underset{\sim}{Z} = \mathcal{D}(A) + (\lambda I - A)^{-1}BU + (\lambda I - A)^{-1}B^1U^1 \subset \mathcal{D}(C_\Lambda), \tag{3.2}$$

where $\lambda \in \rho(A)$. The plant in (3.1) can be equivalently described by the equations

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) + B^1 d(t), \text{ (state equation)} \\ y(t) = C_\Lambda z(t) + Du(t) + D^1 d(t). \text{ (output)} \end{cases} \tag{3.3}$$

We will, for most part, work with this equivalent description for the plant. For this plant, the solvability of the state feedback regulator problem was characterized in terms of the solvability of the regulator equations in [15]. In Section IV we will derive a simplified form of the regulator equations for boundary control systems. This form will be much easier to apply when we design a state feedback controller for a boundary controlled beam equation in Section VI.

We assume that there exists a linear system with no input, referred to as the *exosystem* (sometimes called the exogenous system), that produces both the reference output r and the disturbance signal d: for all $t \geq 0$

$$\dot{w}(t) = Sw(t), \quad r(t) = Q^1 w(t), \quad d(t) = C^1 w(t). \tag{3.4}$$

Here $S \in \mathcal{L}(W)$, where W is a Hilbert space, and its spectrum $\sigma(S)$ is a subset of $\overline{\mathbb{C}^+_0}$, i.e., the exosystem is completely unstable. In the applications that we have in mind, $\sigma(S)$ is on the imaginary axis. We have $Q^1 \in \mathcal{L}(W, Y)$ and $C^1 \in \mathcal{L}(W, U^1)$. We refer to the difference between the measured and reference outputs as the *error*:

$$\begin{aligned} e(t) = y(t) - r(t) &= C_\Lambda z(t) + Du(t) + D^1 d(t) - Q^1 w(t) \\ &= C_\Lambda z(t) + Du(t) + Qw(t), \end{aligned}$$

where $Q \in \mathcal{L}(W, Y)$ is defined by $Q = D^1C^1 - Q^1$.

We will also need to consider the *combined plant* Σ_p representing the plant and the exosystem together, on the combined state space $X = Z \times W$, with the state

$$x(t) = \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \in X = Z \times W,$$

with input space U and output space Y, described by

$$\dot{x}(t) = A_p x(t) + B_p u(t), \ \ A_p = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \ B_p = \begin{bmatrix} B \\ 0 \end{bmatrix}, \tag{3.5}$$

$$e(t) = C_{p\Lambda} x(t) + D_p u(t), \quad C_p = \begin{bmatrix} C_\Lambda & Q \end{bmatrix}, \quad D_p = D, \tag{3.6}$$

where $P = B^1C^1$ and

$$\mathcal{D}(A_p) = \mathcal{D}(C_p) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in X \;\middle|\; Az + Pw \in Z \right\}. \quad (3.7)$$

Lemma 3.1: A_p defined in (3.5), (3.7) generates an operator semigroup $\mathbb{T}^p$ on X.

Consider the spaces X_1 and X_{-1} introduced in Section II. We have $X_1 = \mathcal{D}(A_p)$ and $X_{-1} = Z_{-1} \times W$. The domain of C_p is (by definition) $\mathcal{D}(A_p)$ and $C_{p\Lambda}$ in (3.6) is the Λ-extension of C_p.

Proposition 3.2: The combined plant Σ_p from (3.5)–(3.7) is regular. In particular, B_p and C_p are admissible for $\mathbb{T}^p$ and the transfer function of Σ_p is

$$\begin{aligned} \mathbf{G}_p(s) &= C_{p\Lambda}(sI - A_p)^{-1}B_p + D_p \\ &= C_\Lambda(sI - A)^{-1}B + D. \end{aligned} \quad (3.8)$$

The operator $C_{p\Lambda}$ can be described as follows:

$$\mathcal{D}(C_{p\Lambda}) = \mathcal{D}(C_\Lambda)\times W \text{ and } C_{p\Lambda}\begin{bmatrix} z \\ w \end{bmatrix} = C_\Lambda z + Qw. \quad (3.9)$$

The combined plant is partially stable (since A is stable) but not stabilizable, because there is no way to influence the component w of the state. The problem we want to solve in this paper is to make the output signal e of Σ_p small, meaning that it belongs to a weighted L^2 space, see Section IV for details.

The *Sylvester* equation

$$\Pi S = A\Pi + P + BL, \quad (3.10)$$

which must be solved for Π, when $L \in \mathcal{L}(W, U)$ is given, will play an important role in the sequel.

Lemma 3.3: The Sylvester equation (3.10) has a unique solution $\Pi \in \mathcal{L}(W, Z)$, moreover $\operatorname{Ran}\Pi \subset \underset{\sim}{Z}$, so that the product $C_\Lambda \Pi$ exists and is in $\mathcal{L}(W, Y)$.

IV. The state feedback regulator problem

We continue to use the assumptions and the notation from Section III. In particular, recall that $Q = D^1C^1 - Q^1$ and $P = B^1C^1$. In the state feedback regulator problem, stated below, we consider the state of the combined plant Σ_p to be accessible to the controller, which is a static linear feedback.

Problem 4.1: **The state feedback regulator problem:** For the combined plant Σ_p from (3.5)–(3.7), find a feedback control law in the form $u = Lw$, with $L \in \mathcal{L}(W, U)$, such that for the resulting closed-loop system with no input, described by

$$\begin{bmatrix} \dot z \\ \dot w \end{bmatrix} = \begin{bmatrix} A & P + BL \\ 0 & S \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} := A_p^L \begin{bmatrix} z \\ w \end{bmatrix}, \quad (4.1)$$

$$e = \begin{bmatrix} C_\Lambda & Q + DL \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}, \quad (4.2)$$

we have $e \in L^2_\alpha([0, \infty); Y)$ for some $\alpha < 0$ and for all initial conditions $z(0) = z_0 \in Z$ and $w(0) = w_0 \in W$ (i.e., for any initial state in X).

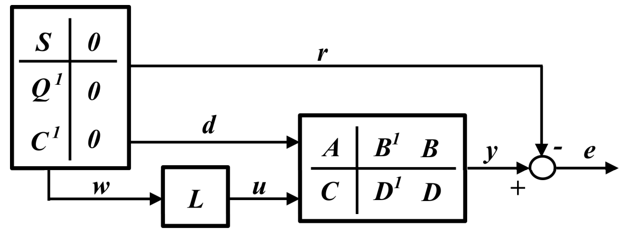

Figure 1. The closed-loop system corresponding to the state feedback regulator problem. The closed-loop system is not asymptotically stable, but the error is in $L^2_\alpha([0,\infty); Y)$ with $\alpha < 0$, like the output of an exponentially stable system.

The next theorem was established in [15] and it gives necessary and sufficient conditions for the solvability of the state feedback regulator problem.

Theorem 4.2: Suppose that there exist operators $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, U)$ satisfying the *regulator equations*

$$\Pi S = A\Pi + B\Gamma + P, \quad (4.3)$$

$$0 = C_\Lambda \Pi + D\Gamma + Q. \quad (4.4)$$

The first regulator equation holds in $\mathcal{L}(W, Z)$ and the second holds in $\mathcal{L}(W, Y)$. In this case a feedback law solving the linear state feedback regulator problem is

$$u(t) = \Gamma w(t). \quad (4.5)$$

Conversely, if an operator $L \in \mathcal{L}(W, U)$ solves the linear state feedback regulator problem, then there exists $\Pi \in \mathcal{L}(W, Z)$ such that, taking $\Gamma = L$, the equations (4.3)-(4.4) are satisfied.

When solving the regulator problem for a plant that is a boundary control system, it may be advantageous to use an alternative form of (4.3), as described below.

Proposition 4.3: Consider the plant (3.1) written as a regular boundary control system. The first regulator equation (4.3) can be rewritten equivalently as the following two equations:

$$\Pi S = \tilde A \Pi, \qquad G\Pi = \mathcal{B}\Gamma + \mathcal{B}^1 C^1. \quad (4.6)$$

Proof: Assume that the first regulator equation holds. We have explained in Section III that the plant (3.1) can be written in the standard form (3.3) if we denote $B = \tilde B \mathcal{B}$ and $B^1 = \tilde B \mathcal{B}^1$. Hence the first regulator equation is

$$\Pi S = A\Pi + \tilde B \left(\mathcal{B}\Gamma + \mathcal{B}^1 C^1\right). \quad (4.7)$$

We apply GA^{-1} to both sides and use the fact that GA^{-1} is zero on Z:

$$G\Pi + GA^{-1}\tilde B \left(\mathcal{B}\Gamma + \mathcal{B}^1 C^1\right) = 0.$$

Now we recall from Section II-B that for every $s \in \rho(A)$, $G(sI - A)^{-1}\tilde B = I$, so that $GA^{-1}\tilde B = -I$. Hence, the first regulator equation implies that $G\Pi = \mathcal{B}\Gamma + \mathcal{B}^1 C^1$, as claimed in the second part of (4.6). If we substitute this formula into (4.7), we get $\Pi S = A\Pi + \tilde B G \Pi$. Since (as mentioned in Section II-B) we have $\tilde{A} = A + \tilde{B}G$ (where $A \in \mathcal{L}(Z, Z_{-1})$), we obtain from here the first part of (4.6).

Conversely, suppose that (4.6) holds. From the first equation we obtain, using that $\tilde{A} = A + \tilde{B}G$, that $\Pi S = A\Pi + \tilde{B}G\Pi$. Express here $G\Pi$ using the second equation from (4.6), to obtain $\Pi S = A\Pi + \tilde{B}\left(\mathcal{B}\Gamma + \mathcal{B}^1 C^1\right)$. Using that $\tilde{B}\mathcal{B} = B$, $\tilde{B}\mathcal{B}^1 = B^1$ and $P = B^1 C^1$, we get (4.3). ■

V. Solvability of the regulator equations

In Section IV we have characterized the solvability of the state feedback regulator problem in terms of the solvability of the regulator equations. In this section, following Byrnes *et al* [3], we characterize the solvability of the regulator equations in terms of the nonresonance condition between the system transmission zeros and the natural frequencies of the exosystem. Under some reasonable additional assumptions on S, we also give an explicit formula for the feedback operator L that solves the state feedback regulator problem.

Assumptions. We continue to use the assumptions and the notation of Section III. Thus, the plant to be controlled is described by (3.3) and the exosystem by (3.4). In addition, *we assume that W is finite-dimensional and certain eigenvectors of S are an (algebraic) basis in W* (i.e., S has no Jordan blocks). The basis assumption is made to simplify our presentation and can be dropped.

Recall that since A is exponentially stable and S is completely unstable, $\sigma(A) \cap \sigma(S) = \emptyset$. Denote $\omega_0 = \omega_0(\mathbb{T}) < 0$. We denote $\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B + D$, defined on $\mathbb{C}^+_{\omega_0}$, so that $\mathbf{G}$ is the transfer function of the plant from u to y.

Definition 5.1: $s_0 \in \mathbb{C}^+_{\omega_0}$ is a *transmission zero* of $\mathbf{G}$ if $\mathbf{G}(s_0)$ is not onto.

The following theorem is the main result of this section.

Theorem 5.2: The regulator equations (4.3) and (4.4) are solvable for any $P \in \mathcal{L}(W, Z_{-1})$ and any $Q \in \mathcal{L}(W, Y)$ such that $C_\Lambda(sI - A)^{-1}P$ exists for some (hence, for every) $s \in \rho(A)$, if and only if each $\lambda \in \sigma(S)$ is not a transmission zero of $\mathbf{G}$.

In this case, a feedback operator L that solves Problem 4.1 is defined by its action on a basis of eigenvectors w_i of S as follows:

$$\begin{aligned} Lw_i = &-\mathbf{G}^*(\lambda_i)\left[\mathbf{G}(\lambda_i)\mathbf{G}^*(\lambda_i)\right]^{-1} \\ &\cdot \left[C_\Lambda(\lambda_i I - A)^{-1}Pw_i + Qw_i\right], \end{aligned} \tag{5.1}$$

where λ_i is the eigenvalue corresponding to w_i.

Proof: Suppose that the regulator equations are solvable for any P and Q such that $C_\Lambda(sI - A)^{-1}P$ exists for some $s \in \rho(A)$. Then from Theorem 4.2, for each such P and Q, there exists a feedback law $u = Lw$ that solves Problem 4.1. With this feedback, the equations of the closed-loop system are (4.1) and (4.2). It is easy to see that

$$\hat{e}(s) = C(sI - A)^{-1}z(0) + \mathbf{H}(s)(sI - S)^{-1}w(0), \tag{5.2}$$

where, for all $s \in \mathbb{C}^+_{\omega_0}$,

$$\mathbf{H}(s) = C_\Lambda(sI - A)^{-1}(P + BL) + Q + DL \tag{5.3}$$

and the formula for $\hat{e}(s)$ holds on any right half-plane to the right of $\sigma(S)$. Choosing $z(0) = 0$ and $w(0) = w_i$, we obtain

$$\hat{e}(s) = \mathbf{H}(s)\frac{w_i}{s - \lambda_i}. \tag{5.4}$$

By analytic continuation, this remains valid on $\mathbb{C}^+_{\omega_0}$, except at the point λ_i. Since the feedback $u = Lw$ solves the state feedback regulator problem, $e \in L^2_\alpha([0, \infty); Y)$ for some $\alpha < 0$, so that $\hat{e}$ is analytic on $\mathbb{C}^+_\alpha$. Comparing this with (5.4), we get

$$\mathbf{H}(\lambda_i)w_i = 0 \qquad \forall\, \lambda_i \in \sigma(S). \tag{5.5}$$

Since this equality must hold for any P and Q as in the theorem, we get

$$\operatorname{Ran}\mathbf{G}(\lambda_i) = Y \qquad \forall\, \lambda_i \in \sigma(S). \tag{5.6}$$

Indeed, this follows from the fact that when $P = 0$, then $\mathbf{H} = \mathbf{G}L + Q$.

Conversely, suppose that the condition (5.6) holds, so that $\mathbf{G}(\lambda_i)\mathbf{G}^*(\lambda_i)$ is bijective and therefore, using the bounded inverse theorem, invertible. On the set of eigenvectors of S, which is a basis in W, define L using (5.1). It follows that

$$-\mathbf{G}(\lambda_i)Lw_i = C_\Lambda(\lambda_i I - A)^{-1}Pw_i + Qw_i$$

and from here we can easily derive (5.5). From (5.2) we see that the component of e due to $z(0)$ is in $L^2_\alpha([0, \infty); Y)$ for any α such that $\omega_0 < \alpha < 0$. By using superposition in (5.2) we see that it is enough to verify that $e \in L^2_\alpha([0, \infty); Y)$ (with $\alpha < 0$) when $z(0) = 0$ and $w(0) = w_i$. In this case,

$$\hat{e}(s) = \mathbf{H}(s)\frac{w_i}{s - \lambda_i} \qquad \forall\, s \in \mathbb{C}^+_\gamma, \tag{5.7}$$

for some $\gamma > 0$ with $\mathbb{C}^+_\gamma \cap \sigma(S) = \emptyset$. Using (5.5) and analytic continuation, we can rewrite (5.7):

$$\hat{e}(s) = [\mathbf{H}(s) - \mathbf{H}(\lambda_i)]\frac{w_i}{s - \lambda_i} \qquad \forall\, s \in \mathbb{C}^+_{\omega_0}.$$

Using (5.3) and the resolvent identity, this becomes

$$\hat{e}(s) = -C(sI - A)^{-1}(\lambda_i I - A)^{-1}(P + BL)w_i.$$

Notice that the vector $z_i = -(\lambda_i I - A)^{-1}(P + BL)w_i$ is in Z. Therefore, for almost every $t \geq 0$, $e(t) = C_\Lambda \mathbb{T}_t z_i$, which shows (as explained in Section II after (2.3)) that $e \in L^2_\alpha([0, \infty); Y)$ for any α such that $\omega_0 < \alpha < 0$.

Thus, the linear state feedback regulator problem, and consequently also the regulator equations (see Theorem 4.2) can be solved using L defined in (5.1). ■

Remark 5.3: The feedback operator L in Theorem 5.2 is not unique, in general. Indeed, everything in this theorem and its proof remains valid if we use some other right inverse of $\mathbf{G}(\lambda_i)$ instead of $\mathbf{G}^*(\lambda_i)[\mathbf{G}(\lambda_i)\mathbf{G}^*(\lambda_i)]^{-1}$.

From Theorem 5.2 it follows that for a given pair P and Q, (5.6) is a sufficient condition for the regulator equations to be solvable. When $U = \mathbb{C}$ and $Y = \mathbb{C}$, under an additional hypothesis, this condition also becomes necessary.

Corollary 5.4: Let $U = Y = \mathbb{C}$. Assume that the pair (A_p, C_p) is detectable in the sense of [23] and $H_p \in \mathcal{L}(U, X_{-1})$ detects this pair (this implies that (A_p, H_p, C_p) is a regular triple and $A_p + H_p C_{p\Lambda}$ generates an exponentially stable semigroup). Then the regulator equations (4.3) and

(4.4) have a solution for a given $P \in \mathcal{L}(W, Z_{-1})$ and $Q \in \mathcal{L}(W, Y)$ such that $C_\Lambda(sI - A)^{-1}P$ exists for some (hence, for every) $s \in \rho(A)$, if and only if for each $\lambda_i \in \sigma(S)$, $\mathbf{G}(\lambda_i) \neq 0$.

Proof: The sufficiency of the condition $\mathbf{G}(\lambda_i) \neq 0$ follows from Theorem 5.2. To establish its necessity assume that a feedback law $u = Lw$ solves the state feedback regulator problem. Each $\lambda_i \in \sigma(S)$ is also an eigenvalue of A_p with the corresponding eigenvector being

$$v_i = \begin{bmatrix} (\lambda_i I - A)^{-1} P w_i \\ w_i \end{bmatrix},$$

where w_i is the eigenvector of S corresponding to λ_i. It follows from (3.2) and (3.9) that $v_i \in \mathcal{D}(C_{p\Lambda})$. We will show that $C_{p\Lambda} v_i \neq 0$ for each $i \in \{1, 2, \ldots, k\}$.

Fix i and consider the exponentially stable system

$$\dot{\Theta} = (A_p + H_p C_{p\Lambda})\Theta, \quad \Theta(0) = v_i \,.$$

Assume that $C_{p\Lambda} v_i = 0$. Then $C_{p\Lambda}\mathbb{T}^p_t v_i = e^{\lambda_i t} C_{p\Lambda} v_i = 0$. Clearly the function $\Theta(t) = \mathbb{T}^p_t v_i$ is the unique classical solution to the above exponentially stable system and $\mathbb{T}^p_t v_i \to 0$ as $t \to \infty$. Since A_p is upper triangular, so is $\mathbb{T}^p_t$. In particular, $\mathbb{T}^p_t v_i \to 0$ implies that $e^{St} w_i \to 0$, as $t \to \infty$, which is impossible. Therefore $C_{p\Lambda} v_i \neq 0$ for all i and for each $\lambda_i \in \sigma(S)$,

$$C_\Lambda(\lambda_i I - A)^{-1} P w_i + Q w_i \neq 0 \,.$$

This fact along with (5.5), which holds here for reasons similar to those in the proof of Theorem 5.2, implies that $C_\Lambda(\lambda_i I - A)^{-1} B + D \neq 0$, i.e., $\mathbf{G}(\lambda_i) \neq 0$ for each $\lambda_i \in \sigma(S)$. ∎

VI. Boundary control of Rayleigh beam

We consider a harmonic tracking problem for a damped Rayleigh beam, in the presence of structural damping (see [9]). We denote the transverse displacement of the beam at the position $x \in [0, \pi]$ and the time $t \geq 0$ by $q(x, t)$. The beam equation, influenced by a boundary control $u(t)$, is:

$$\frac{\partial^2 q(x,t)}{\partial t^2} - \alpha \frac{\partial^4 q(x,t)}{\partial x^2 \partial t^2} - a \frac{\partial^3 q(x,t)}{\partial x^2 \partial t} + \frac{\partial^4 q(x,t)}{\partial x^4} = 0, \quad (6.1)$$

$$q(0,t) = q(\pi,t) = \frac{\partial^2 q}{\partial x^2}(\pi,t) = 0, \; -\frac{\partial^2 q}{\partial x^2}(0,t) = u(t), \quad (6.2)$$

$$y(t) = \frac{\partial^2 q}{\partial x \partial t}(0,t). \quad (6.3)$$

Here $\alpha > 0$ is proportional to the moment of inertia of the cross section of the beam and $a > 0$ is the damping coefficient. This equation models a single-input-single-output boundary control system with u being the torque applied at $x = 0$ and the output y being the angular velocity at the same point.

We now briefly discuss the state space formulation for the Rayleigh beam and refer to Weiss and Curtain [24] for more details. Let $H = \mathcal{H}^1_0(0,\pi)$ and $V = \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi)$. Define the inner product on H such that

$$\langle \varphi, \psi \rangle_H = \left\langle \left(I - \alpha \frac{\mathrm{d}^2}{\mathrm{d}x^2} \right) \varphi, \psi \right\rangle_{L^2(0,\pi)} \quad \forall\, \varphi, \psi \in V.$$

Consider the operator $\mathcal{R} : L^2[0,\pi] \to V$ defined as

$$\mathcal{R} = \left(I - \alpha \frac{\mathrm{d}^2}{\mathrm{d}x^2} \right)^{-1}.$$

As a bounded operator on $L^2[0,\pi]$, $\mathcal{R}$ is strictly positive and it leaves both H and V invariant. We define the operator $A_0 : \mathcal{D}(A_0) \to H$ by

$$\mathcal{D}(A_0) = \left\{ \varphi \in \mathcal{H}^3(0,\pi) \,\middle|\, \varphi(0) = \varphi(\pi) = 0, \; \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(0) = \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(\pi) = 0 \right\},$$

$$A_0 \varphi = \frac{\mathrm{d}^4}{\mathrm{d}x^4}(\mathcal{R}\varphi) \qquad \forall\, \varphi \in \mathcal{D}(A_0).$$

The operator A_0 is strictly positive, self-adjoint and commutes with $\mathcal{R}$. We will use the following notation: $H_1 = \mathcal{D}(A_0)$, $H_{\frac{1}{2}} = V$, $H_{-\frac{1}{2}} = L^2[0,\pi]$ and $H_{-1} = \mathcal{H}^{-1}(0,\pi)$. A_0 can be extended to a bounded operator from $H_{\frac{1}{2}}$ to $H_{-\frac{1}{2}}$ that commutes with $\mathcal{R}$ (hence, also with $\mathcal{R}^{-1}$).

We now rewrite (6.1)–(6.2) as a boundary control system. We consider the transverse displacement q and the velocity $\dot{q}$ to be the state variables. Let $Z = H_{\frac{1}{2}} \times H$ and $U = \mathbb{C}$ be the state space and the input space. It is now easy to see that the following operator $A : \mathcal{D}(A) \to Z$ is m-dissipative, hence a generator:

$$\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}, \quad A = \begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix},$$

where $A_1 = -a\mathcal{R}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \in \mathcal{L}(V)$. Note that $A_1 \geq 0$ on V and on H. Define $\tilde{A}_0$, the obvious extension of A_0 to

$$\mathcal{D}(\tilde{A}_0) = \left\{ \varphi \in \mathcal{H}^3(0,\pi) \,\middle|\, \varphi(0) = \varphi(\pi) = 0, \; \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(\pi) = 0 \right\},$$

and we define $\tilde{A} : \underset{\sim}{Z} \to Z$, an extension of A, and the boundary trace operator $G : \underset{\sim}{Z} \to U$ by

$$\underset{\sim}{Z} = \mathcal{D}(\tilde{A}_0) \times V, \quad \tilde{A} = \begin{bmatrix} 0 & I \\ -\tilde{A}_0 & -A_1 \end{bmatrix},$$

$$G \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = -\frac{\mathrm{d}^2 z_1}{\mathrm{d}x^2}(0).$$

Then the equations (6.1)–(6.2) can be written exactly as in (3.1), with $\mathcal{B} = I$ and $\mathcal{B}^1 = 0$. Notice that the restriction of $\tilde{A}$ to $\operatorname{Ker} G$ is A, as required.

Define the operator $C_0 \in \mathcal{L}(H_{\frac{1}{2}}, \mathbb{C})$ by $C_0\varphi = \frac{\mathrm{d}\varphi}{\mathrm{d}x}(0)$ and the observation operator $C : \mathcal{D}(A) \to \mathbb{C}$ by $C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$, which corresponds to the output equation (6.3). When $a = 0$ (no structural damping), it is established in [24] that the above boundary control system with the observation operator C is regular. Since A_1 is a bounded operator, it follows from [22], that the same is true when $a \neq 0$. In order to see that A is exponentially stable one can, with straightforward modifications, apply Proposition 3.14 and Theorem 3.18 in [6] (in their notation, set $A = -A_0$ and $B = -A_1$). We mention that the control operator is $B = C^*$.

We denote the two components of the state z by z_1 (displacement) and z_2 (velocity). We want to design a control u such that the output y in (6.3) tracks a prescribed sinusoidal trajectory $r(t) = M\sin(\omega t + \psi)$ of known frequency $\omega > 0$, amplitude $M > 0$, and phase ψ, i.e., the error $e = y - r$ is in $L^2_\delta[0, \infty)$ for some $\delta < 0$. For any M and ψ, the signal r can be generated by the exosystem in (3.4) with

$$S = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \qquad Q^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Using the notation $\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix}$ and $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix}$, the first regulator equation (4.3) rewritten in the equivalent form (4.6) becomes

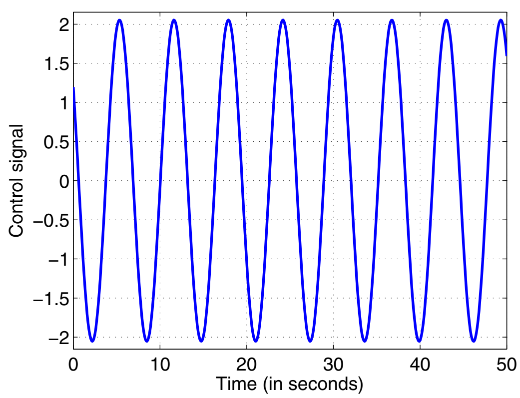

Figure 2. The control signal $u = \Gamma w$ for the Rayleigh beam.

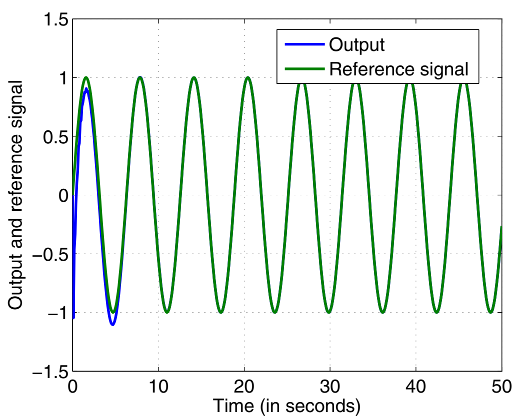

Figure 3. The sinusoidal reference signal and the plant output.

$$-\omega^2\Pi_1 + \tilde{A}_0\Pi_1 - \omega A_1\Pi_2 = 0, \tag{6.4}$$

$$-\omega^2\Pi_2 + \tilde{A}_0\Pi_2 + \omega A_1\Pi_1 = 0, \tag{6.5}$$

$$\omega\Pi_1 = \Pi_4, \qquad -\omega\Pi_2 = \Pi_3, \tag{6.6}$$

$$-\frac{\mathrm{d}^2\Pi_1}{\mathrm{d}x^2}(0) = \Gamma_1, \qquad -\frac{\mathrm{d}^2\Pi_2}{\mathrm{d}x^2}(0) = \Gamma_2. \tag{6.7}$$

From (6.4) and (6.5), using the definitions of $\tilde{A}_0$, A_1 and $\mathcal{R}$, we get that

$$\Pi_1'''' + \alpha\omega^2\Pi_1'' + a\omega\Pi_2'' - \omega^2\Pi_1 = 0, \tag{6.8}$$

$$\Pi_2'''' + \alpha\omega^2\Pi_2'' - a\omega\Pi_1'' - \omega^2\Pi_2 = 0. \tag{6.9}$$

Since $\operatorname{Ran}\Pi \subset \underset{\sim}{Z}$ (see Lemma 3.3), we have $\Pi_1, \Pi_2 \in \mathcal{D}(\tilde{A}_0)$ and therefore

$$\Pi_1(0) = \Pi_2(0) = \Pi_1(\pi) = \Pi_2(\pi) = 0, \tag{6.10}$$

$$\Pi_1''(\pi) = \Pi_2''(\pi) = 0. \tag{6.11}$$

The second regulator equation (4.4) and (6.6) give

$$\Pi_1'(0) = 0, \quad \Pi_2'(0) = -1/\omega. \tag{6.12}$$

Remark 6.1: The transfer function for the beam is

$$\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B = C_0 s(s^2 + sA_1 + A_0)^{-1}C_0^*.$$

From Theorem 5.2, the system of regulator equations (6.4)–(6.12) has a solution if $\mathbf{G}(j\omega) \neq 0$. For any $\omega > 0$ and non-zero $u \in \mathbb{C}$,

$$\begin{aligned} \langle \mathbf{G}(j\omega)u, u\rangle_{\mathbb{C}} &= \langle C_0 j\omega(-\omega^2 + j\omega A_1 + A_0)^{-1}C_0^* u, u\rangle_{\mathbb{C}} \\ &= -j\omega^3\langle v, v\rangle_{H^{\frac{1}{2}}} + \omega^2\langle v, A_1 v\rangle_{H^{\frac{1}{2}}} + j\omega\langle v, A_0 v\rangle_{H^{\frac{1}{2}}}, \end{aligned}$$

where $v = (-\omega^2 + j\omega A_1 + A_0)^{-1}C_0^* u$. Since A_0 and A_1 are self-adjoint operators and $A_1 v \neq 0$, it follows that $\mathbf{G}(j\omega) \neq 0$. Indeed, note that $A_1 v = 0$ implies that $\mathcal{R}^{-1}v = v$, which has no non-zero solutions.

The ordinary differential equations in (6.8), (6.9), along with the boundary conditions (6.10), (6.11) and (6.12), are first solved for Π_1 and Π_2. The functions Π_3, Π_4, Γ_1 and Γ_2 can then be computed from (6.6) and (6.7).

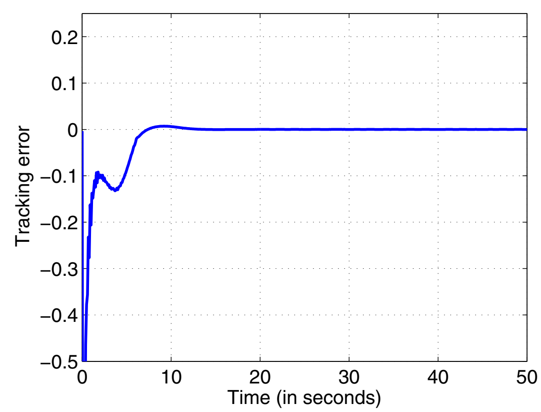

Figure 4. The tracking error in this example. This error tends to zero.

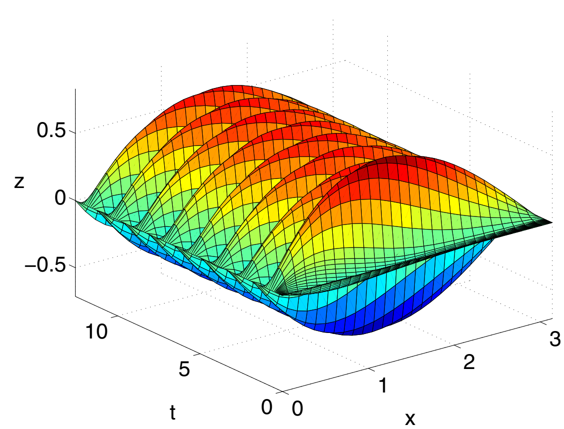

Figure 5. Displacement profile on the interval $[0, \pi]$ as a function of time.

For our numerical simulations we choose $\alpha = 1$, $a = 2$, $M = 1$ and $\omega = 1$. Hence the signal to be tracked is $r(t) = \sin(t)$. We set the initial conditions to be $z_1(x, 0) = 0$, $z_2(x, 0) = 0$. The system (6.7)–(6.12) was solved using the finite element package COMSOL [4] on the time interval $0 < t < 50$. The simulation results are presented in Figures 2, 3, 4 and 5.

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