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Optimal Decay Rate for the Indirect Stabilization of Systems of Hyperbolic Equations

Extended Abstract

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Abstract— **We consider the indirect stabilization of systems of hyperbolic-type equations, such as wave and plate equations with different boundary conditions. By energy method, we show that a single feedback allows to stabilize the full system at a polynomial rate. Furthermore, we exploit refined resolvent estimates deducing the optimal decay rate of the energy of the system. Numerical simulations show resonance effect between the components of the system and confirm optimality of the proven decay rate.**

I. INTRODUCTION

The interest of the scientific community in the stabilization and control of systems of partial differential equations has remarkably increased, in recent years. This is probably due to the fact that such systems arise in several applied mathematical models, such as those used for studying the vibrations of flexible structures and networks (see [12] and references therein), or fluids and fluid-structure interactions (see, for instance, [7], [6], [10], [14], [21], [23]).

When dealing with systems involving quantities described by several components, pretending to control or observe all the state variables might be irrealistic. In applications to mathematical models for the vibrations of flexible structures (see [3] and [5]), electromagnetism (see, for instance, [15]), or fluid control (see [11] and the references therein), it may happen that only part of such components can be observed. This is why it becomes essential to study whether controlling only a reduced number of state variables suffices to ensure the stability of the full system.

It turns out that certain systems possess an internal structure that compensates for the aforementioned lack of control variables. Such a phenomenon is referred to as *indirect stabilization* or *indirect control* (see [22]).

The issue of stabilizing a system of partial differential equations through suitable damping terms (possibly in feedback form) on each component of the system has its origin in the pioneering works of Lagnese and Lions [17] and Russell [22]. In their approach, the multiplier method is the main tool to reach the desired estimates on the energy of each component of the system. This techniques has been further developed in [16] and later in [1] for systems of hyperbolic equations.

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In particular, Russell [22] addressed the indirect stabilization problem, that occurs when the damping (or the control) acts on a reduced number of equations of the system. In this situation the (uniformly) exponential decay rate usually cannot be achieved, but weaker decay rates might hold. This is the case in [2] and [4], where (uniform) polynomial stability for the whole system is showed, under a suitable compatibility condition on the operators involved in the system, by means of multipliers properly adapted to the peculiar structure of the system under investigation. Indeed, it turns out that different compatibility conditions and multipliers are required to cope with systems with boundary conditions of similar type on each component [2] or *hybrid* [4].

An operator-theoretical approach has been recently proposed in [8] and [9], addressing the issue of optimality of the decay rate. Both references exploit the relation between the decay rate of the system ruled by the dynamics $\mathscr{A}$ and the asymptotic behaviour of the norm of the resolvent operator of $\mathscr{A}$ on the imaginary axis (see, for example, [18]).

In this talk we intend to combine both energy methods and resolvent estimates, the former to give an abstract criterion to show polynomial stabilization for large classes of systems, the latter to prove optimality of the decay rate of stabilization.

A. Examples

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be open and bounded, with sufficiently smooth boundary Γ. We denote by $\Delta = \sum_{i=1}^d \partial^2_{x_i x_i}$ the Laplacian operator; moreover, for sake of simplicity, we use the subscript u_t to denote the partial derivative of u with respect of the variable t.

Many systems of hyperbolic equations, such as coupled wave-wave, or wave-plate or plate-plate equations, with several different boundary conditions, are considered in [2], [4]. However, in this short note we will focus on a single example, the Petrowsky-wave system

$$\begin{cases} u_{tt} - \Delta u + \beta u_t + \alpha v = 0 & \text{in } \Omega \times (0, +\infty) \\ v_{tt} + \Delta^2 v + \alpha u = 0 & \text{in } \Omega \times (0, +\infty), \end{cases} \tag{I.1}$$

for some constants $\alpha \in \mathbb{R}, \beta > 0$, with Robin boundary conditions

$$\frac{\partial u}{\partial \nu}(\cdot, t) + \sigma u(\cdot, t) = 0 \quad \text{on } \Gamma,\ t > 0, \quad \sigma > 0, \tag{I.2}$$

on u and either hinged

$$v(\cdot, t) = 0 = \Delta v(\cdot, t) \quad \text{on } \Gamma,\ t > 0 \tag{I.3}$$

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or clamped boundary conditions

$$v(\cdot,t) = 0 = \frac{\partial v}{\partial \nu}(\cdot,t) \quad \text{on } \Gamma,\ t>0 \tag{I.4}$$

on v, with initial conditions

$$u(0)=u^0\,,\ u_t(0)=u^1\,,\ v(0)=v^0\,,\ v_t(0)=v^1 \quad \text{in } \Omega\,, \tag{I.5}$$

for some functions u^0, u^1, v^0, v^1 in suitable spaces (see section II). Let us notice that, in system (I.1), only the first component, in u, is damped, through the feedback term βu_t, while the component in v is undamped. A general result in [2] ensures that system (I.1) cannot be exponentially stable. We then sight for weaker decay rates, such as of polynomial type. We start introducing a general framework to cope with the stabilization of systems of evolution equations.

II. ABSTRACT SETTING

Let $(H, \langle\ ,\ \rangle)$ be a Hilbert space on the field $\mathbb{C}$ of complex numbers with associated norm $|\ |_H$. We consider the abstract system of evolution equations

$$\begin{cases} u''(t)+A_1u(t)+Bu'(t)+\alpha v=0 & \text{in } H\\ v''(t)+A_2v(t)+\alpha u=0 & \text{in } H \end{cases} \tag{II.1}$$

where

(H1) $A_i : D(A_i)\subset H \to H$ $(i=1,2)$ are densely defined closed linear selfadjoint operators such that

$$\langle A_iu,u\rangle \ge \omega_i|u|^2 \qquad \forall u\in D(A_i)\,,\ i=1,2\,,$$

for some $\omega_1,\omega_2>0$.

(H2) B is a bounded linear selfadjoint operator on H such that

$$\langle Bu,u\rangle \ge \beta|u|^2 \qquad \forall u\in H\,,\ \text{for some } \beta>0.$$

(H3) α is a real number such that $0<|\alpha|<\sqrt{\omega_1\omega_2}$.

We associate to the operator A_i, $i=1,2$, the energy

$$E_i(u,p)=\frac12\left(|A_i^{1/2}u|_H^2+|p|_H^2\right)$$

for all $(u,p)\in D(A_i^{1/2})\times H$, $i=1,2$. System (II.1), with initial conditions

$$\begin{cases} u(0)=u^0\in D(A_1^{1/2}), & u'(0)=u^1\in H\,,\\ v(0)=v^0\in D(A_2^{1/2}), & v'(0)=v^1\in H\,, \end{cases} \tag{II.2}$$

can be formulated as a first order Cauchy problem in the space

$$\mathscr{H}=D(A_1^{1/2})\times H\times D(A_2^{1/2})\times H\,,$$

that becomes a Hilbert space on $\mathbb{C}$ endowed with the scalar product

$$\begin{aligned}(U,\hat U) &:= \langle A_1^{1/2}u, A_1^{1/2}\hat u\rangle + \langle p,\hat p\rangle + \langle A_2^{1/2}v, A_2^{1/2}\hat v\rangle\\ &\quad + \langle q,\hat q\rangle + \alpha\langle u,\hat v\rangle + \alpha\langle v,\hat u\rangle \end{aligned} \tag{II.3}$$

for every $U=(u,p,v,q), \hat U=(\hat u,\hat p,\hat v,\hat q)\in\mathscr{H}$, that also induces the norm $|U|_{\mathscr{H}} := (U,U)^{1/2}$ on $\mathscr{H}$. Indeed, introducing the operator

$$\begin{cases} D(\mathscr{A}) = D(A_1)\times D(A_1^{1/2})\times D(A_2)\times D(A_2^{1/2})\\ \mathscr{A}U=(p,-A_1u-Bp-\alpha v, q, -A_2v-\alpha u)\,, \end{cases} \tag{II.4}$$

for every $U=(u,p,v,q)\in D(\mathscr{A})$, problem (II.1)-(II.2) can be recast as

$$\begin{cases} U'(t)=\mathscr{A}U(t)\\ U(0)=U_0=(u^0,u^1,v^0,v^1)\in\mathscr{H}\,. \end{cases} \tag{II.5}$$

The operator $\mathscr{A}$ generates a C_0-semigroup $e^{t\mathscr{A}}$ on $\mathscr{H}$ (see [4, Lemma 4.2]), that satisfies $e^{t\mathscr{A}}U_0=(u(t),p(t),v(t),q(t))$, where (u,v) is the solution of problem (II.1) with initial conditions (II.2) and $(p,q)=(u',v')$. Finally, for every $U\in\mathscr{H}$, we define the total energy of system (II.5) by

$$\mathscr{E}(U(t)) := E_1(u,p)+E_2(v,q)+2\alpha\,\mathtt{Re}\langle u,v\rangle\,,$$

where $\mathtt{Re}(z)$ denotes the real part of $z\in\mathbb{C}$.

III. STABILIZATION RESULT

In [4] a general compatibility condition is given, in order to ensure polynomial decay rate of the total energy associated to system (II.5), namely,

$$D(A_2)\subset D(A_1^{1/2}) \text{ and } |A_1^{1/2}u|\le c|A_2u| \quad \forall u\in D(A_2) \tag{III.1}$$

for some constant $c>0$. Under assumption (H1), condition (III.1) is equivalent to the following formulations:

(a) $A_1^{1/2}A_2^{-1}\in\mathscr{L}(H)$.

(b) For some constant $c>0$ and for every $u\in D(A_1)$, $v\in D(A_2)$

$$|\langle A_1u,v\rangle|\le c|A_2v|\langle A_1u,u\rangle^{1/2}.$$

We refer to [19], [20] for the definition of the interpolation space $(X,Y)_{\theta,2}$ between two Banach spaces X and Y.

Theorem 3.1 ([4]): Assume (H1), (H2), (H3) and (III.1).

i) If $U_0\in D(\mathscr{A}^n)$ for some $n\ge1$, then the energy of the solution of (II.5) satisfies

$$\mathscr{E}(U(t))\le\frac{c_n}{t^{n/4}}\sum_{k=0}^{n}\mathscr{E}(U^{(k)}(0)) \qquad \forall t>0$$

for some constant $c_n>0$.

ii) If $U_0\in(\mathscr{H},D(\mathscr{A}^n))_{\theta,2}$ for some $n\ge1$ and $0<\theta<1$, then the solution of (II.5) satisfies

$$|U(t)|^2_{\mathscr{H}}\le\frac{c_{n,\theta}}{t^{n\theta/4}}\|U_0\|^2_{(\mathscr{H},D(\mathscr{A}^n))_{\theta,2}} \qquad \forall t>0$$

for some constant $c_{n,\theta}>0$.

iii) If $U_0\in D((-\mathscr{A})^\theta)$ for some $0<\theta<1$, then the solution of problem (II.5) satisfies

$$|U(t)|^2_{\mathscr{H}}\le\frac{c_\theta}{t^{\theta/4}}\|U_0\|^2_{D((-\mathscr{A})^\theta)} \qquad \forall t>0$$

for some constant $c_\theta>0$.

IV. APPLICATION OF RESOLVENT ESTIMATES

In the following, we set $H := L^2(\Omega)$ and we define the bounded operator $B : H \to H$ by $Bu = \beta u$ for all $u \in H$, for some positive constant $\beta > 0$. We first point out that the systems (I.1)-(I.2)-(I.3) and (I.1)-(I.2)-(I.4) can be rewritten as (II.5), introducing the operators

$$D(A_1) := \{u \in H^2(\Omega) : \left(\frac{\partial u}{\partial \nu} + \sigma u\right)(\cdot,t) = 0 \text{ on } \Gamma,\, t > 0\}, \quad A_1 u = -\Delta u$$

and

$$D(A_2) := \{v \in H^4(\Omega) : v(\cdot,t) = 0 = \frac{\partial v}{\partial \nu}(\cdot,t) \text{ on } \Gamma,\ t > 0\}, \quad A_2 v = \Delta^2 v$$

or

$$D(A_2) := \{v \in H^4(\Omega) : v(\cdot,t) = 0 = \Delta v(\cdot,t) \text{ on } \Gamma,\ t > 0\}, \quad A_2 v = \Delta^2 v,$$

and defining $(\mathscr{A}, D(\mathscr{A}))$ as in (II.4). Moreover, systems (I.1)-(I.2)-(I.3) and (I.1)-(I.2)-(I.4) fulfill the compatibility condition (III.1), implying the estimate on the total energy

$$\mathscr{E}(U(t)) \le \frac{C}{(1+t)^{1/4}} |U_0|^2_{\mathscr{H}}$$

for every initial condition $U_0 = (u^0, u^1, v^0, v^1) \in D(\mathscr{A})$ and for some constant $C > 0$.

In order to complete the analysis of the optimal decay rate for the total energy of the systems under consideration, we recall a characterization of polynomial stability of semigroup norm given in [9]. Given two functions f and g, we use the notations $f(t) = O(g(t))$ as $t \to \infty$ when the function $|f(t)/g(t)|$ is bounded for large t; and $f(t) = o(g(t))$ as $t \to \infty$ if the function $f(t)/g(t)$ tends to 0 as $t \to \infty$.

Proposition 4.1: Let $T(t)$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that the imaginary axis $i\mathbb{R}$ lies in the resolvent set $\rho(A)$ of A. For every fixed $\gamma > 0$, the following conditions are equivalent:

1. $\|R(ib,A)\|_{\mathscr{L}(H)} = O(|b|^\gamma) \quad$ as $b \to +\infty$;
2. $\|T(t)A^{-1}\|_{\mathscr{L}(H)} = O(|t|^{-1/\gamma}) \quad$ as $t \to +\infty$;
3. $\|T(t)A^{-1}x\|_H = o(|t|^{-1/\gamma}) \quad$ as $t \to +\infty \quad \forall x \in H$.

In particular, we are interested in the case $\gamma = 4$, the decay rate we show for the systems addressed in Subsection I-A. Indeed, the next result holds true.

Theorem 4.2 ([13]): The operators $(\mathscr{A}, D(\mathscr{A}))$ associated with systems (I.1)-(I.2)-(I.3) and (I.1)-(I.2)-(I.4) satisfy

$$\|R(ib,\mathscr{A})\|_{\mathscr{L}(\mathscr{H})} = O(|b|^4) \quad \text{as } b \to +\infty.$$

Then, for every integer $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$|U(t)|_{\mathscr{H}} = |e^{t\mathscr{A}} U_0|_{\mathscr{H}} \le \frac{C_m}{(1+t)^{m/4}} |U_0|_{D(\mathscr{A}^m)}$$

for every $t \ge 0$ and $U_0 \in D(\mathscr{A}^m)$. In particular, for every $U_0 = (u^0, u^1, v^0, v^1) \in D(\mathscr{A})$,

$$\mathscr{E}(U(t)) \le \frac{C}{(1+t)^{1/2}} |U_0|^2_{\mathscr{H}} \quad \forall t \ge 0.$$

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