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# A note on the input-output structure of linear periodic continuous-time systems with complex-valued coefficients

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*Abstract*— **This paper focuses on the preservation of the period in the periodic Kalman canonical decomposition with complex-valued coefficients. This clarifies essential difference between linear periodic systems with real-valued coefficients and with complex-valued coefficients.**

## I. INTRODUCTION

Motivated by theoretical interests, this paper focuses on the preservation of the period in the periodic Kalman canonical decomposition. To this end, we extend the class of linear periodic system from the one with the real-valued coefficients [1] to the one with the complex-valued coefficients. Then, we prove that it is always possible to construct a periodic coordinate transformation with the same period of the given periodic system. The result of this paper clarifies essential difference between linear periodic systems with real-valued coefficients and with complex-valued coefficients. We notice here that proofs in this paper are similar to [2], [3]; however, an alternative block diagonalization method relying on Sibuya [4] is introduced because a block diagonalization method relying on Dolezal [5] is not directly applicable in the periodic decomposition of periodic matrix-valued functions.

We use the following notations. $\mathbb{R}$ (respectively, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N}$) denotes the set of all real numbers (respectively, complex numbers, integers, natural numbers). $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $C^k(\mathbb{R}, \mathbb{K}^{n\times m})$ denotes the set of all $C^k$-functions, i.e., $k$-times continuously differentiable functions, from $\mathbb{R}$ to $\mathbb{K}^{n\times m}$. $C^k_{\mathrm{inv}}(\mathbb{R}, \mathbb{K}^{n\times n})$ denotes the set of all invertible functions in $C^k(\mathbb{R}, \mathbb{K}^{n\times n})$. If the function $P(t)$ is periodic with a period $T > 0$, i.e., $P(t + T) = P(t)$ for $t \in \mathbb{R}$, it is called $T$-periodic. The set of all $T$-periodic functions in $C^k(\mathbb{R}, \mathbb{K}^{n\times m})$ is denoted by $C^k_T(\mathbb{R}, \mathbb{K}^{n\times m})$. The set of all invertible $T$-periodic functions in $C^k(\mathbb{R}, \mathbb{K}^{n\times m})$ is denoted by $C^k_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{K}^{n\times m})$.

## II. PROBLEM FORMULATION

Consider a linear $T$-periodic continuous-time systems

$$\dot{x} = A(t)x + B(t)u, \quad \dot{x} := \frac{dx}{dt} \tag{1}$$

$$y = C(t)x, \tag{2}$$

where $t \in \mathbb{R}$ is a time, $x(t) \in \mathbb{K}^n$ is a state vector, $u(t) \in \mathbb{K}^m$ is an input, $y(t) \in \mathbb{K}^p$ is an output, and $A \in C^0_T(\mathbb{R}, \mathbb{K}^{n\times n})$, $B(t) \in C^0_T(\mathbb{R}, \mathbb{K}^{n\times m})$, $C(t) \in C^0_T(\mathbb{R}, \mathbb{K}^{p\times n})$ denote the coefficient matrices for certain nonnegative integers $n, m, p$. Let $\Phi$ denote the state transition matrix of (1) with $u = 0$ for $t \in \mathbb{R}$, i.e., a unique solution of the following equations: $\frac{\partial}{\partial s}\Phi(s,t) = A(s)\Phi(s,t), \Phi(t,t) = I_n$. $\Phi$ satisfies the following properties: $\Phi(s,t) = \Phi(s,\tau)\Phi(\tau,t)$, $\Phi(s,t)^{-1} = \Phi(t,s)$, $\det\Phi(s,t) \neq 0$, $\Phi(s,t) = \Phi(s + kT, t + kT)$ for $s, \tau, t \in \mathbb{R}$, $k \in \mathbb{Z}$.

Consider a coordinate transformation

$$\xi = Z(t)x \tag{3}$$

where $Z(t) \in C^1_{\mathrm{inv}}(\mathbb{R}, \mathbb{K}^{n\times n})$ is called a coordinate transformation matrix. The system in (1)-(2) is transformed to

$$\dot{\xi} = F(t)\xi + G(t)u \tag{4}$$

$$y = H(t)\xi \tag{5}$$

where

$$F(t) := (\dot{Z}(t) + Z(t)A(t))Z(t)^{-1} \tag{6}$$

$$G(t) := Z(t)B(t) \tag{7}$$

$$H(t) := C(t)Z(t)^{-1}. \tag{8}$$

The problem is to find a periodic coordinate transformation matrix $Z(t)$ such that the triplet $(H, F, G)$ has a certain block structure.

*Problem 1:* Consider the $T$-periodic system (1)-(2) with the $\mathbb{K}$-valued coefficients $A \in C^0_T(\mathbb{R}, \mathbb{K}^{n\times n})$, $B(t) \in C^0_T(\mathbb{R}, \mathbb{K}^{n\times m})$, $C(t) \in C^0_T(\mathbb{R}, \mathbb{K}^{p\times n})$. Find a $kT$-periodic coordinate transformation matrix $Z \in C^1_{kT,\mathrm{inv}}(\mathbb{R}, \mathbb{K}^{n\times n})$, where $k \in \mathbb{N}$, satisfying the following three conditions (i)-(iii):
The first condition (i): the triplet $(H, F, G)$ in (6)-(8) takes on the form

$$F(t) = \begin{bmatrix} F_{11}(t) & F_{12}(t) & F_{13}(t) & F_{14}(t) \\ 0 & F_{22}(t) & 0 & F_{24}(t) \\ 0 & 0 & F_{33}(t) & F_{34}(t) \\ 0 & 0 & 0 & F_{44}(t) \end{bmatrix} \tag{9}$$

$$G(t) = \begin{bmatrix} G_1(t) \\ G_2(t) \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

$$H(t) = \begin{bmatrix} 0 & H_2(t) & 0 & H_4(t) \end{bmatrix} \tag{11}$$

at each $t \in \mathbb{R}$, where the block diagonal components of (9) are square with appropriate dimensions.

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The second condition (ii): The pair $(F_c, G_c)$ is controllable at each $t \in \mathbb{R}$, where $F_c(t)$ and $G_c(t)$ are defined by

$$F_c(t) := \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix}, \quad G_c(t) := \begin{bmatrix} G_1(t) \\ G_2(t) \end{bmatrix}, \quad (12)$$

where the submatrices in the right hand sides are composed of (9)-(10).

The condition (iii): The pair $(H_o, F_o)$ is observable at each $t \in \mathbb{R}$, where $H_o(t)$ and $F_o(t)$ are defined by

$$H_o(t) := \begin{bmatrix} H_2(t) & H_4(t) \end{bmatrix}, \quad F_o(t) := \begin{bmatrix} F_{22}(t) & F_{24}(t) \\ 0 & F_{44}(t) \end{bmatrix}, \quad (13)$$

where the submatrices in the right hand sides are composed of (9) and (11).

If such a $Z(t)$ exists, the transformed system (4)-(5) or the triplet $(H, F, G)$ in (9)-(11) is called the $kT$-periodic Kalman canonical decomposition of the system (1)-(2) in $\mathbb{K}$.

## III. Structure of Subspaces

In this section, we study the structure of subspaces resulting from the controllability and the observability for $\mathbb{K} = \mathbb{C}$. Similar arguments has been proved for $\mathbb{K} = \mathbb{R}$ in [1].

### A. Controllable Subspace

Let us recall the definition of the controllable subspace for linear time-varying systems.

*Definition 2:* A state $x_0 \in \mathbb{C}^n$ of the system (1) is said to be controllable at time $t$ if there exist a finite $s \geq t$, depending on $x_0$ and $t$, and a piecewise continuous function $u \in \mathbf{U}$, depending on $x_0$, $t$ and $s$ and defined on the closed interval $[t, s]$, which satisfy the integral equation

$$\Phi(s,t)x_0 + \int_t^s \Phi(s,\tau)B(\tau)u(\tau)d\tau = 0.$$

The system (1) is said to be controllable at time $t$, or the pair $(A, B)$ is said to be controllable at time $t$, if all the states in $\mathbb{C}^n$ are controllable at time $t$.

*Definition 3:* The controllability Gramian is defined by

$$W_c(t,s) := \int_t^s \Phi(t,\tau)B(\tau)B(\tau)^*\Phi(t,\tau)^* d\tau. \quad (14)$$

*Definition 4:* The set of states controllable at time $t$ is denoted by

$$\mathcal{C}(t) := \bigcup_{p\in[t,\infty)} \left\{ \int_t^p \Phi(t,\tau)B(\tau)u(\tau)d\tau : u \in \mathbf{U} \right\},$$

which is said to be a *controllable subspace* at time $t$.

Since linear periodic systems are time-varying, Definitions 2-4 are also valid for linear periodic systems for $\mathbb{K} = \mathbb{C}$.

Let us recall the properties of $\mathcal{C}(t)$ for linear time-varying systems for $\mathbb{K} = \mathbb{R}$.

*Theorem 5 (Theorems 1 and Theorem 2 in [2]):* Consider the system (1) which is not necessarily periodic. Then $\mathcal{C}(t)$ satisfies the following conditions. (i) $\mathcal{C}(t) = \bigcup_{t\leq p<\infty} \operatorname{Im} W_c(t,p)$. (ii) There exists a positive scalar function $p_c(t)$ such that $\mathcal{C}(t) = \operatorname{Im} W_c(t, t + p_c(t))$. (iii) $\mathcal{C}(t)$ is backward $\Phi$-invariant, i.e., $\mathcal{C}(t) \supset \Phi(t,s)\,\mathcal{C}(s)$ for $t \leq s$. (iv) The pair $(A, B)$ is controllable at time $t$ if and only if $\dim \mathcal{C}(t) = n$ at time $t$.

The backward $\Phi$-invariance in Theorem 5 (iii) is a bottleneck for constructing the Kalman canonical decomposition which is globally valid in time for linear time-varying systems. Hence, for $\mathbb{K} = \mathbb{R}$, the authors have introduced the concept of interval-wise $\Phi$-invariance and have studied the various Kalman canonical decompositions depending on the variants of controllability and the observability in [3]. Then, for $\mathbb{K} = \mathbb{R}$, the authors have obtained stronger assertions for linear periodic systems in [1].

Here, we obtain compatible results for $\mathbb{R} = \mathbb{C}$.

*Proposition 6:* Consider the $T$-periodic system (1). Then $\mathcal{C}(t)$ satisfies the following conditions. (i) $\mathcal{C}(t) = \operatorname{Im} W_c(t, t + nT)$ for $t \in \mathbb{R}$. (ii) $\mathcal{C}(t)$ is $T$-periodic, i.e., $\mathcal{C}(t) = \mathcal{C}(t + T)$ for $t \in \mathbb{R}$. (iii) $\mathcal{C}(t)$ is $\Phi$-invariant, i.e., $\mathcal{C}(t) = \Phi(t,s)\,\mathcal{C}(s)$ for $t, s \in \mathbb{R}$.

### B. Observable Subspace

Let us recall the definition and the properties of the observability subspace for linear time-varying systems.

*Definition 7:* A state $z_0 \in \mathbb{R}^n$ of the system (1)-(2) is said to be *observable* at time $t$ if the state $z_0$ of the adjoint system $\dot{z} = -A(t)^*z + C(t)^*\tilde{u}$ is controllable at time $t$. The system (1)-(2) is observable at time $t$, or the pair $(C, A)$ is said to be observable at time $t$, if all the states are observable at time $t$.

*Definition 8:* The observability Gramian is defined by

$$W_o(t,s) := \int_t^s \Phi(\tau,t)^*C(\tau)^*C(\tau)\Phi(\tau,t)d\tau. \quad (15)$$

*Definition 9:* The set of states observable at time $t$ is denoted by

$$\mathcal{O}(t) := \bigcup_{p\in[t,\infty)} \left\{ \int_t^p \Phi(t,\tau)^*C(\tau)^*\tilde{u}(\tau)d\tau : \tilde{u} \in \tilde{\mathbf{U}} \right\},$$

which is said to be an *observable subspace* at time $t$.

*Theorem 10 (Theorem 3 and Theorem 4 in [2]):* Consider the system (1)-(2) which is not necessarily periodic. Then $\mathcal{O}(t)$ satisfies the following properties. (i) $\mathcal{O}(t) = \bigcup_{t\leq p<\infty} \operatorname{Im} W_o(t,p)$. (ii) There exist a positive scalar function $p_o(t)$ such that $\mathcal{O}(t) = \operatorname{Im} W_o(t, t + p_o(t))$. (iii) $\mathcal{O}(t)$ is backward $\Phi^*$-invariant, i.e., $\mathcal{O}(t) \supset \Phi(s,t)^*\,\mathcal{O}(s)$ for $t \leq s$. (iv) The pair $(C, A)$ is observable at time $t$ if and only if $\dim \mathcal{O}(t) = n$ at time $t$.

The backward $\Phi^*$-invariance in Theorem 10 (iii) is not convenient for constructing the Kalman canonical decomposition. Hence, the authors have shown that the annihilator

$\mathcal{O}(t)^\perp$ is forward $\Phi$-invariant for linear time-varying systems in [3]. Then, for $\mathbb{K}=\mathbb{R}$, the authors have obtained stronger properties for linear periodic systems in [1].

Here, we obtain compatible results for $\mathbb{K}=\mathbb{C}$.

*Proposition 11:* Consider the $T$-periodic system (1)-(2). Then $\mathcal{O}(t)$ satisfies the following conditions. (i) $\mathcal{O}(t)^\perp = \mathrm{Ker}\, W_o(t, t+nT)$ for $t \in \mathbb{R}$. (ii) $\mathcal{O}(t)$ is $T$-periodic, i.e., $\mathcal{O}(t)^\perp = \mathcal{O}(t+T)^\perp$ for $t \in \mathbb{R}$. (iii) $\mathcal{O}(t)^\perp$ is $\Phi$-invariant, i.e., $\Phi(s,t)\,\mathcal{O}(t)^\perp = \mathcal{O}(s)^\perp$ for $t, s, \in \mathbb{R}$.

### C. Intersection Subspace

The $\Phi$-invariance properties of $\mathcal{C}(t)$ and $\mathcal{O}(t)^\perp(t)$ take over to their intersection subspace $\mathcal{C}(t)\cap\mathcal{O}(t)^\perp$.

*Proposition 12:* Consider the $T$-periodic system (1)-(2). Then $\mathcal{C}(t)\cap\mathcal{O}(t)^\perp$ satisfies the following conditions. (i) $\mathcal{C}(t)\cap\mathcal{O}(t)^\perp$ is $T$-periodic, i.e., $\mathcal{C}(t)\cap\mathcal{O}(t)^\perp = \mathcal{C}(t+T)\cap\mathcal{O}(t+T)^\perp$ for $t\in\mathbb{R}$. (ii) $\mathcal{C}(t)\cap\mathcal{O}(t)^\perp$ is $\Phi$-invariant, i.e., $\Phi(s,t)\{\mathcal{C}(t)\cap\mathcal{O}(t)^\perp\} = \{\mathcal{C}(s)\cap\mathcal{O}(s)^\perp\}$ for $s,t\in\mathbb{R}$.

The $\Phi$-invariance properties of $\mathcal{C}(t)$, $\mathcal{O}(t)^\perp$, $\mathcal{C}(t)\cap\mathcal{O}(t)^\perp$ in Proposition 6 (iii), Proposition 11 (iii), Proposition 12 (ii) does not imply the $\Phi$-invariance properties of the other intersection subspaces $\mathcal{C}(t)\cap\mathcal{O}(t)$, $\mathcal{C}(t)^\perp\cap\mathcal{O}(t)^\perp$, $\mathcal{C}(t)^\perp\cap\mathcal{O}(t)$. But their dimensions are shown to be constant.

*Proposition 13:* Consider the $T$-periodic system (1)-(2). Then the intersection subspaces satisfy the following conditions

$$\dim\{\mathcal{C}(t)\cap\mathcal{O}(t)^\perp\} = const. =: n_1 \tag{16}$$
$$\dim\{\mathcal{C}(t)\cap\mathcal{O}(t)\} = const. =: n_2 \tag{17}$$
$$\dim\{\mathcal{C}(t)^\perp\cap\mathcal{O}(t)^\perp\} = const. =: n_3 \tag{18}$$
$$\dim\{\mathcal{C}(t)^\perp\cap\mathcal{O}(t)\} = const. =: n_4 \tag{19}$$

for $t\in\mathbb{R}$.

The conditions (16)–(19) are equivalent to the existence of the Kalman canonical decomposition which is valid global in time. It follows that, for any given $T$-periodic system (1)-(2), it is always possible to construct a coordinate transformation matrix $Z \in C^1_{\mathrm{inv}}(\mathbb{R}, \mathbb{C}^{n\times n})$ such that the transformed system (4)-(5) takes on the Kalman canonical decomposition. However, it is not clear whether it is possible to find a $T$-periodic transformation matrix $Z \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{C}^{n\times n})$.

## IV. Periodic Kalman canonical decomposition

### A. Motivating Example in $\mathbb{R}$

Firstly we recall an illustrative example which is a counterexample to the existence of the $T$-periodic Kalman canonical decomposition in $\mathbb{R}$.

*Example 14:* Let $A \in C^0_T(\mathbb{R},\mathbb{R}^{2\times 2})$, $B \in C^0_T(\mathbb{R},\mathbb{R}^{2\times 1})$, and $C \in C^0_T(\mathbb{R},\mathbb{R}^{1\times 2})$ be given by

$$A(t) := \begin{bmatrix} 0 & \frac{\pi}{T} \\ -\frac{\pi}{T} & 0 \end{bmatrix}, \tag{20}$$
$$B(t) := \begin{bmatrix} \sin\left(\frac{\pi t}{T}\right)\left(\cos\left(\frac{\pi t}{T}\right)+\sin\left(\frac{\pi t}{T}\right)\right) \\ \sin\left(\frac{\pi t}{T}\right)\left(\cos\left(\frac{\pi t}{T}\right)-\sin\left(\frac{\pi t}{T}\right)\right) \end{bmatrix}, \tag{21}$$
$$C(t) = 0_{1\times 2}. \tag{22}$$

Then the controllability Gramian over $[t, t+2T]$ satisfies $\mathrm{rank}\, W_c(t, t+2T) = 1 < 2$ for all $t\in\mathbb{R}$; therefore, the pair $(A, B)$ is uncontrollable. It is clear that all nonzero states in $\mathbb{R}^2$ are not observable. Suppose that there exists a $T$-periodic coordinate transformation matrix $Z \in C^1_{T,\mathrm{inv}}(\mathbb{R},\mathbb{R}^{2\times 2})$ such that the triplet $(C, A, B)$ is transformed to $(H, G, F)$ of the forms (9)-(11). If such $Z(t)$ exists, we have $n_1 = 1$, $n_2 = 0$, $n_3 = 1$, $n_4 = 0$. It follows that $F(t)$ in (9) takes on the form

$$F(t) = \begin{bmatrix} F_{11}(t) & F_{13}(t) \\ 0 & F_{33}(t) \end{bmatrix} \tag{23}$$

where $F_{11}, F_{13}, F_{33} \in C^0_T(\mathbb{R},\mathbb{R})$. By the $T$-periodicity of $Z(t)$, the monodromy matrices in $x$-coordinate and in $\xi$-coordinate are similar; in other words, the characteristic multipliers are invariant with respect to a coordinate transformation $\xi = Z(t)x$. Since $e^{A(0)T} = -I_2$, the characteristic multipliers in $x$-coordinate are $-1$ with multiplicity 2; therefore, the characteristic multipliers in $\xi$-coordinate is also $-1$ with multiplicity 2. On the contrary, it follows from the upper triangular structure of (23) that the characteristic multipliers in $\xi$-coordinate are given by $\exp(\int_0^T F_{11}(\tau)d\tau)$ and $\exp(\int_0^T F_{33}(\tau)d\tau)$. Hence we have a contradiction to the existence of $T$-periodic coordinate transformation matrix $Z \in C^1_{T,\mathrm{inv}}(\mathbb{R},\mathbb{R}^{2\times 2})$.

### B. $T$-periodic Kalman canonical decomposition in $\mathbb{R}$

Next we recall the necessary and sufficient condition for the existence of the $T$-periodic Kalman canonical decomposition in $\mathbb{R}$.

*Theorem 15:* Consider the $T$-periodic system (1)-(2) with the real-valued coefficients $A \in C^0_T(\mathbb{R},\mathbb{R}^{n\times n})$, $B(t) \in C^0_T(\mathbb{R},\mathbb{R}^{n\times m})$, $C(t) \in C^0_T(\mathbb{R},\mathbb{R}^{p\times n})$. Then there exists a $T$-periodic coordinate transformation $Z \in C^1_{T,\mathrm{inv}}(\mathbb{R},\mathbb{R}^{n\times n})$ such that the transformed system (4)-(5) takes on the $T$-periodic Kalman canonical decomposition in $\mathbb{R}$ if and only if there exists a $T$-periodic $Z(t) \in C^1_{T,\mathrm{inv}}(\mathbb{R},\mathbb{R}^{n\times n})$ satisfying

$$Z(t)W_c(t,t+nT)Z(t)^{\mathrm{T}} = \begin{bmatrix} \tilde{P}_{11}(t) & \tilde{P}_{12}(t) & & \\ \tilde{P}_{12}(t)^{\mathrm{T}} & \tilde{P}_{22}(t) & & \\ & & 0_{n_3\times n_3} & \\ & & & 0_{n_4\times n_4} \end{bmatrix}, \tag{24}$$

$$Z(t)^{-\mathrm{T}}W_o(t,t+nT)Z(t)^{-1} = \begin{bmatrix} 0_{n_1\times n_1} & & & \\ & \tilde{Q}_{22}(t) & & \tilde{Q}_{24}(t) \\ & & 0_{n_3\times n_3} & \\ & \tilde{Q}_{24}(t)^{\mathrm{T}} & & \tilde{Q}_{44}(t) \end{bmatrix}, \tag{25}$$

for certain matrix-valued functions $\tilde{P}_{11} \in C^1(\mathbb{R}, \mathbb{R}^{n_1 \times n_1})$, $\tilde{P}_{12} \in C^1(\mathbb{R}, \mathbb{R}^{n_1 \times n_2})$, $\tilde{P}_{22} \in C^1(\mathbb{R}, \mathbb{R}^{n_2 \times n_2})$, $\tilde{Q}_{22} \in C^1(\mathbb{R}, \mathbb{R}^{n_2 \times n_2})$, $\tilde{Q}_{24} \in C^1(\mathbb{R}, \mathbb{R}^{n_2 \times n_4})$, $\tilde{Q}_{44} \in C^1(\mathbb{R}, \mathbb{R}^{n_4 \times n_4})$, where the submatrices

$$\tilde{P}_{sub} := \begin{bmatrix} \tilde{P}_{11}(t) & \tilde{P}_{12}(t) \\ \tilde{P}_{12}(t)^{\mathrm{T}} & \tilde{P}_{22}(t) \end{bmatrix},\ \tilde{Q}_{sub} := \begin{bmatrix} \tilde{Q}_{22}(t) & \tilde{Q}_{24}(t) \\ \tilde{Q}_{24}(t)^{\mathrm{T}} & \tilde{Q}_{44}(t) \end{bmatrix}$$

are positive definite for $t \in \mathbb{R}$.

The simultaneous block diagonalization (24)-(25) in Theorem 15 is not always possible. Indeed, it can be shown that the simultaneous block diagonalization is not possible in Example 14.

In contrast, we have proved the existence of periodic Kalman canonical decomposition in $\mathbb{R}$ by relaxing a class of coordinate transformation matrices.

*Theorem 16:* Consider the $T$-periodic system (1)-(2) with the real-valued coefficients $A \in C_T^0(\mathbb{R}, \mathbb{R}^{n \times n})$, $B(t) \in C_T^0(\mathbb{R}, \mathbb{R}^{n \times m})$, $C(t) \in C_T^0(\mathbb{R}, \mathbb{R}^{p \times n})$. Then there exists a $8T$-periodic coordinate transformation matrix $Z(t) \in C^1_{8T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n \times n})$ such that the transformed system (4)-(5) takes on the $8T$-periodic Kalman canonical decomposition in $\mathbb{R}$.

We notice that the periods of the coordinate transformation matrix $Z(t)$ and the transformed system (4)-(5) in Theorem 16 is $8T$. This $8T$ originates in the thrice repetition of the double periodic factorization. In the case of $(A, B)$-pair or $(C, A)$-pair, the periods of the coordinate transformation matrix $Z(t)$ and the transformed system (4)-(5) can be shortened to $2T$, i.e., the $2T$-periodic decomposition is possible.

Below, we prove an example for periodic decomposition in $\mathbb{R}$.

*Example 17:* Consider the same $T$-periodic $A(t)$, $B(t)$, $C(t)$ in Example 14. Then the $2T$-periodic coordinate transformation matrix

$$\begin{aligned} Z(t) &= \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ -Z_{12}(t) & Z_{11}(t) \end{bmatrix} \\ Z_{11}(t) &= \tfrac{1}{\sqrt{2}} \left( \cos\left(\tfrac{\pi t}{T}\right) + \sin\left(\tfrac{\pi t}{T}\right) \right) \\ Z_{12}(t) &= \tfrac{1}{\sqrt{2}} \left( \cos\left(\tfrac{\pi t}{T}\right) - \sin\left(\tfrac{\pi t}{T}\right) \right) \end{aligned}$$

transformed the system into

$$F(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad G(t) = \begin{bmatrix} \sqrt{2} \sin\left(\frac{\pi t}{T}\right) \\ 0 \end{bmatrix},$$
$$H(t) = 0_{1 \times 2},$$

which takes on the $2T$-periodic Kalman canonical decomposition in $\mathbb{R}$.

### C. Periodic Kalman canonical decomposition in $\mathbb{C}$

Now we consider the periodic Kalman canonical decomposition in $\mathbb{C}$.

*Theorem 18:* Consider the $T$-periodic system (1)-(2) with the complex-valued coefficients $A \in C_T^0(\mathbb{R}, \mathbb{C}^{n \times n})$, $B(t) \in C_T^0(\mathbb{R}, \mathbb{C}^{n \times m})$, $C(t) \in C_T^0(\mathbb{R}, \mathbb{C}^{p \times n})$. Then there exists a $T$-periodic coordinate transformation matrix $Z(t) \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{C}^{n \times n})$ such that the transformed system (4)-(5) takes on the $T$-periodic Kalman canonical decomposition in $\mathbb{C}$.

In contrast to Theorem 15, the simultaneous block diagonalization in the forms of (24)-(25) is always possible for complex-valued functions. This fact, which is based on Sibuya [4] instead of Dolezal [5], is crucial to prove the sufficiency part of Theorem 18. Indeed, it can be shown that the simultaneous block diagonalization is possible for complex-valued functions in Example 14.

*Example 19:* Consider the same $T$-periodic $A(t)$, $B(t)$, $C(t)$ in Example 14 as complex-valued functions. Then the $T$-periodic coordinate transformation matrix

$$\begin{aligned} Z(t) &= \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ -Z_{12}(t) & Z_{11}(t) \end{bmatrix} \\ Z_{11}(t) &= \tfrac{1}{2\sqrt{2}} \left( (1+i) + (1-i) e^{\frac{2\pi i t}{T}} \right) \\ Z_{12}(t) &= \tfrac{(-1)^{-\frac{1}{4}}}{2} \left( -i + e^{\frac{2\pi i t}{T}} \right) \end{aligned}$$

transformed the system into

$$F(t) = \begin{bmatrix} \frac{i\pi}{T} & 0 \\ 0 & \frac{i\pi}{T} \end{bmatrix},\ G(t) = \begin{bmatrix} -\frac{i}{\sqrt{2}} \left( -1 + e^{\frac{2\pi i t}{T}} \right) \\ 0 \end{bmatrix},$$
$$H(t) = 0_{1 \times 2},$$

which takes on the $T$-periodic Kalman canonical decomposition in $\mathbb{C}$.

## V. Conclusions

We have studied the problem of the Kalman canonical decomposition for linear periodic continuous-time systems with complex-valued coefficients. Compared with the results for real-valued coefficients, the properties of controllable and observable subspaces are compatible for both real-valued and complex-valued cases; however, the constructions of the periodic coordinate transformation matrices are incompatible for real-valued and complex valued cases. In this way, the result of this paper clarifies essential difference between linear periodic systems with real-valued coefficients and with complex-valued coefficients.

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