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The linear wave equation on N-dimensional spatial domains

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***Abstract*— We study the wave equation on a bounded Lipschitz set, characterizing all homogeneous boundary conditions for which this partial differential equation generates a contraction semigroup in the energy space $L^2(\Omega)^{n+1}$. The proof uses boundary triplet techniques.**

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***Index Terms*— Port-Hamiltonian system, contraction semigroup, boundary triplet**

I. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded set with Lipschitz-continuous boundary and let Γ_0 and Γ_1 be open subsets of $\partial\Omega$, such that $\Gamma_1 \cap \Gamma_0 = \emptyset$ and $\overline{\Gamma}_1 \cup \overline{\Gamma}_0 = \partial\Omega$. The divergence and gradient on Ω are defined in the distribution sense via

$$\operatorname{div} v = \frac{\partial v_1}{\partial x_1} + \ldots + \frac{\partial v_n}{\partial x_n} \quad \text{and}$$
$$\operatorname{grad} w = \left(\frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_n}\right)^\top .$$

The *Laplacian* is the operator $\Delta z := \operatorname{div}(\operatorname{grad} z)$.

The following PDE describes a wave equation with a viscous damper on the part Γ_1 of $\partial\Omega$ and a reflecting boundary condition on Γ_0:

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2}(\xi,t) &= (\Delta z)(\xi,t) \quad \text{on } \Omega \times \mathbb{R}_+, \\ 0 &= \nu \cdot \operatorname{grad}\ z(\xi,t) + k(\xi)\frac{\partial z}{\partial t}(\xi,t) \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \qquad (1) \\ 0 &= \frac{\partial z}{\partial t}(\xi,t) \quad \text{on } \Gamma_0 \times \mathbb{R}_+ \end{aligned}$$

Here $\nu \in L^\infty(\partial\Omega;\mathbb{R}^n)$ is the outward unit normal of $\partial\Omega$ and the non-negative real-valued function k describes the amount of damping in almost every point $\xi \in \Gamma_1$.

In this paper we show that the PDE (1) possesses a unique solution for all initial data in $L^2(\Omega)^{n+1}$. However, our result is much more general. Namely, we characterize all boundary conditions for which the wave equation possesses a unique solution that is contractive with respect to the energy. In the full article [6] underlying this paper, the results are formulated for arbitrary boundary triplets, and the wave equation is merely an example.

We follow the port-Hamiltonian approach as has been done for the one-dimensional wave equation in [2], [3]. The first step is to rewrite $\frac{\partial^2 z}{\partial t^2}(\cdot,t) = (\Delta z)(\cdot,t)$ on Ω in the energy variables, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \dot z(t) \\ \operatorname{grad} z(t)\end{bmatrix} = \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0\end{bmatrix}\begin{bmatrix} \dot z(t) \\ \operatorname{grad} z(t)\end{bmatrix}, \qquad (2)$$

where $\dot z(t) = \frac{\mathrm{d}z}{\mathrm{d}t}(t)$. Note that the position can be recovered from (2) by simply integrating the first state component. Next we want to characterize those domains of the operator $\left[\begin{smallmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0\end{smallmatrix}\right]$ for which it is the infinitesimal generator of a contraction semigroup in $L^2(\Omega)^{n+1}$. From Lemma 7.2.3 of [3] it is clear that this also characterizes existence of a contraction semigroup on the energy space, i.e., when (1) contains the physical parameters.

II. Background and setting

The necessary background for the present article has been compiled in [4]. Here we only fix the notation very briefly and the reader is referred to [4] for more details.

We define

$$H^{\operatorname{div}}(\Omega) := \{v \in L^2(\Omega)^n \mid \operatorname{div} v \in L^2(\Omega)\},$$

equipped with the graph norm of div. This is the maximal domain for which div can be considered as operator between L^2 spaces. We will consider grad as an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega)^n$ with domain contained in $H^1(\Omega)$.

Theorem 2.1: For a bounded Lipschitz set Ω the following hold:

1) The boundary trace mapping $g \mapsto g|_{\partial\Omega} : C^1(\overline{\Omega}) \to C(\partial\Omega)$ has a unique continuous extension γ_0 that maps $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$. The space $H^1_0(\Omega)$ equals $\{g \in H^1(\Omega) \mid \gamma_0 g = 0\}$.
2) The normal trace mapping $u \mapsto \nu \cdot \gamma_0 u : H^1(\Omega)^n \to L^2(\partial\Omega)$ has a unique continuous extension $\gamma_\perp$ that maps $H^{\operatorname{div}}(\Omega)$ onto $H^{-1/2}(\partial\Omega)$. Here the dot $\cdot$ denotes the inner product in $\mathbb{R}^n$, $p \cdot q = q^\top p$ without complex conjugate. Furthermore, the space $H^{\operatorname{div}}_0(\Omega)$ equals

$$H^{\operatorname{div}}_0(\Omega) = \{f \in H^{\operatorname{div}}(\Omega) \mid \gamma_\perp f = 0\}.$$

We call γ_0 the *Dirichlet trace map* and $\gamma_\perp$ the *normal trace map*. Note that $\gamma_\perp$ is *not* the Neumann trace γ_N; the relation between the two is $\gamma_N f = \gamma_\perp \operatorname{grad} f$, for f smooth enough.

Theorem 2.2: Let Ω be a bounded Lipschitz set in $\mathbb{R}^n$. For all $f \in H^{\operatorname{div}}(\Omega)$ and $g \in H^1(\Omega)$ it holds that

$$\begin{aligned} \langle \operatorname{div} f, g\rangle_{L^2(\Omega)} + \langle f, \operatorname{grad} g\rangle_{L^2(\Omega)^n} \qquad (3) \\ = (\gamma_\perp f, \gamma_0 g)_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}. \end{aligned}$$

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In particular, we have the following Green's formula:

$$\langle \Delta h, g\rangle_{L^2(\Omega)} + \langle \operatorname{grad} h, \operatorname{grad} g\rangle_{L^2(\Omega)^n} = (\gamma_\perp \operatorname{grad} h, \gamma_0 g)_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)},$$

which is valid for all $h, g \in H^1(\Omega)$, such that $\Delta h \in L^2(\Omega)$.

III. Duality of the divergence and the gradient

Since $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ are each others duals with pivot space $L^2(\partial\Omega)$, we can make the following definition: The *annihilator* in $H^{-1/2}(\partial\Omega)$ of a subspace $R \subset H^{1/2}(\partial\Omega)$ is the (closed) subspace

$$R^{(\perp)} := \{v \in H^{-1/2}(\partial\Omega) \mid (v, r) = 0 \quad \forall r \in R\},$$

Where (v, r) denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. The following result formulates an exact duality between the divergence and the gradient:

Theorem 3.1: Let Ω be a bounded Lipschitz set in $\mathbb{R}^n$ and let $H_0^1(\Omega) \subset G \subset H^1(\Omega)$. Consider $\operatorname{grad}\big|_G$ as an unbounded operator from the dense subspace $G \subset L^2(\Omega)$ into $L^2(\Omega)^n$. Then its adjoint is given by $\left(\operatorname{grad}\big|_G\right)^* = -\operatorname{div}\big|_D$ with

$$D := \{f \in H^{\operatorname{div}}(\Omega) \mid \gamma_\perp f \in (\gamma_0 G)^\perp\}. \tag{4}$$

Furthermore, the set D is a closed subspace of $H^{\operatorname{div}}(\Omega)$ that contains $H_0^{\operatorname{div}}(\Omega)$, i.e., $H_0^{\operatorname{div}}(\Omega) \subset D \subset H^{\operatorname{div}}(\Omega)$.

Assume that G is closed in $H^1(\Omega)$. Then $D = H^{\operatorname{div}}(\Omega)$ if and only if $G = H_0^1(\Omega)$, and $D = H_0^{\operatorname{div}}(\Omega)$ if and only if $G = H^1(\Omega)$.

Theorem 3.1 follows essentially from the "integration by parts formula" (3). For a given domain G of the gradient operator, (4) says that the corresponding domain D of the adjoint divergence operator is the inverse image under $\gamma_\perp$ of the annihilator $(\gamma_0 G)^{(\perp)}$.

We proceed by specialising Theorem 3.1 to the case where the functions in the domain of the gradient operator vanish on an open subset $\Gamma_0 \subset \partial\Omega$.[1] Following [9, Chap. 13], we will identify $L^2(\Gamma_0)$ with the space of functions in $L^2(\partial\Omega)$ that vanish almost everywhere on $\partial\Omega \setminus \Gamma_0$. Hence we have

$$L^2(\partial\Omega) = L^2(\Gamma_0) \oplus L^2(\partial\Omega \setminus \Gamma_0),$$

and we denote the corresponding orthogonal projection onto $L^2(\Gamma_0)$ by π_0. If Γ_1 is as described in the introduction and the common boundary $\partial\Omega \setminus (\Gamma_0 \cup \Gamma_1)$ of Γ_0 and Γ_1 has surface measure zero, then $L^2(\partial\Omega\setminus\Gamma_0) = \Gamma_1$, but this seems to be unimportant in our setting.

In [9, §13.6] the following space of functions in $H^1(\Omega)$, whose boundary trace vanish on Γ_0, was introduced:

$$H^1_{\Gamma_0}(\Omega) := \{g \in H^1(\Omega) \mid (\gamma_0 g)|_{\Gamma_0} = 0 \text{ in } L^2(\Gamma_0)\}.$$

The space $H^1_{\Gamma_0}(\Omega)$ is closed, because it can be viewed as the kernel of the bounded operator $\pi_0\gamma_0 : H^1(\Omega) \to L^2(\Gamma_0)$; recall that γ_0 is bounded from $H^1(\Omega)$ into $H^{1/2}(\partial\Omega)$ by Theorem 2.1 and that the latter is continuously embedded in $L^2(\partial\Omega)$ by its definition.

By Theorem 3.1, $\operatorname{grad}\big|^*_{H^1_{\Gamma_0}(\Omega)} = -\operatorname{div}\big|_{H^{\operatorname{div}}_{\Gamma_0}(\Omega)}$, where

$$H^{\operatorname{div}}_{\Gamma_0}(\Omega) := \{f \in H^{\operatorname{div}}(\Omega) \mid \gamma_\perp f \in \left(\gamma_0 H^1_{\Gamma_0}(\Omega)\right)^{(\perp)}\}, \tag{5}$$

and it follows that $H^{\operatorname{div}}_{\Gamma_0}(\Omega)$ is closed in $H^{\operatorname{div}}(\Omega)$. In particular, $H_0^1(\Omega) = H^1_{\partial\Omega}(\Omega)$ corresponds to $H^{\operatorname{div}}_{\partial\Omega}(\Omega) = H^{\operatorname{div}}(\Omega)$, and this case was used extensively in [7], [10], [11], [5]. The other extreme case is $H^1(\Omega) = H^1_\emptyset(\Omega)$, which corresponds to $H^{\operatorname{div}}_\emptyset(\Omega) = H^{\operatorname{div}}_0(\Omega)$.

As a consequence of the Riesz representation theorem, there exists a unitary operator $\Psi : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$, such that

$$\begin{aligned}(x, z)_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)} &= \langle \Psi x, z\rangle_{H^{1/2}(\partial\Omega)} \\ &= \langle x, \Psi^* z\rangle_{H^{-1/2}(\partial\Omega)}\end{aligned}$$

for all $x \in H^{-1/2}(\partial\Omega)$ and $z \in H^{1/2}(\partial\Omega)$; see [8, p. 288–289] or [9, p. 57]. This Ψ is called *the duality operator* [8].

We have the following practical description of the annihilator in (5):

Proposition 3.2: It holds that

$$\left(\gamma_0 H^1_{\Gamma_0}(\Omega)\right)^{(\perp)} = \overline{L^2(\Gamma_0)}^{H^{-1/2}(\partial\Omega)}$$

and

$$\left(\gamma_0 H^1_{\Gamma_0}(\Omega)\right)^{(\perp)} \cap L^2(\partial\Omega) = L^2(\partial\Omega) \ominus \left(\gamma_0 H^1_{\Gamma_0}(\Omega)\right).$$

IV. Tools for existence proofs for PDEs

The operator A defined as

$$\begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix}\Bigg|_{\mathcal{D}} : \begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix} \supset \mathcal{D} \to \begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix} \tag{6}$$

with domain $\mathcal{D} = \left[\begin{smallmatrix} H_0^1(\Omega) \\ H^{\operatorname{div}}(\Omega)\end{smallmatrix}\right]$ is skew-adjoint by Theorem 3.1. We shall next characterize all domains $\mathcal{D}$ (in practice we characterise the boundary conditions),

$$\begin{bmatrix} H_0^1(\Omega) \\ H_0^{\operatorname{div}}(\Omega) \end{bmatrix} \subset \mathcal{D} \subset \begin{bmatrix} H^1(\Omega) \\ H^{\operatorname{div}}(\Omega) \end{bmatrix}, \tag{7}$$

which make A in (6) maximal dissipative or skew-adjoint on $L^2(\Omega)^{n+1}$. We achieve this by associating a boundary triplet to A in (6).

The first step is to adapt the definition [1, p. 155] of a boundary triplet for a symmetric operator to the case of a skew-symmetric operator. It is based on the observation that an operator iA_0 is skew-symmetric if and only if A_0 is symmetric; see also [8, §5].

Definition 4.1: Let A_0 be a densely defined, skew-symmetric, and closed linear operator on a Hilbert space X. By a *boundary triplet* for A_0^* we mean a triple $(\mathcal{B}; B_1, B_2)$ consisting of a Hilbert space $\mathcal{B}$ and two bounded linear operators $B_1, B_2 : \operatorname{dom}(A_0^*) \to \mathcal{B}$, such that

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \operatorname{dom}(A_0^*) = \begin{bmatrix} \mathcal{B} \\ \mathcal{B} \end{bmatrix}$$

[1] By saying that Γ_0 is open in $\partial\Omega$, we mean that Γ_0 is the intersection of $\partial\Omega$ and some open set in $\mathbb{R}^n$.

and for all $x, \widetilde{x} \in \operatorname{dom}(A_0^*)$ it holds that

$$\begin{aligned}\langle A_0^* x, \widetilde{x}\rangle_X + &\langle x, A_0^* \widetilde{x}\rangle_X \\ &= \langle B_1 x, B_2 \widetilde{x}\rangle_{\mathcal{B}} + \langle B_2 x, B_1 \widetilde{x}\rangle_{\mathcal{B}}.\end{aligned} \tag{8}$$

The analogue of (8) is written as follows in [1, p. 155]:

$$\langle \mathcal{A}^* x, \widetilde{x}\rangle - \langle x, \mathcal{A}^* \widetilde{x}\rangle = \langle \Gamma_1 x, \Gamma_2 \widetilde{x}\rangle_{\mathcal{B}} + \langle \Gamma_2 x, \Gamma_1 \widetilde{x}\rangle_{\mathcal{B}},$$

and setting $A_0^* = (i\mathcal{A})^*$, $B_1 = \Gamma_1$, and $B_2 = i\Gamma_2$ in (8), we obtain exactly this.

Theorem 4.2: Let Ω be a bounded Lipschitz set. The operator

$$A_0 := \begin{bmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & 0 \end{bmatrix}, \quad \operatorname{dom}(A_0) := \begin{bmatrix} H_0^1(\Omega) \\ H_0^{\operatorname{div}}(\Omega) \end{bmatrix},$$

is closed, skew-symmetric, and densely defined on $\begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix}$. Its adjoint is

$$A_0^* = \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix}, \ \operatorname{dom}(A_0^*) := \begin{bmatrix} H^1(\Omega) \\ H^{\operatorname{div}}(\Omega) \end{bmatrix}. \tag{9}$$

Setting $B_0 := \begin{bmatrix} \gamma_0 & 0 \end{bmatrix}$ and $B_\perp := \begin{bmatrix} 0 & \gamma_\perp \end{bmatrix}$, we obtain that $(H^{1/2}(\partial\Omega); B_0, \Psi B_\perp)$ is a boundary triplet for A_0^*.

One can now prove the following n-dimensional analogue of [3, Thm 7.2.4]:

Theorem 4.3: Let $\mathcal{H}$ be a Hilbert space and let $W_B = \begin{bmatrix} W_1 & W_2 \end{bmatrix} : \begin{bmatrix} H^{1/2}(\partial\Omega) \\ H^{-1/2}(\partial\Omega) \end{bmatrix} \to \mathcal{H}$ be a bounded linear operator, such that

$$\operatorname{ran}(W_1 - W_2\Psi^*) \subset \operatorname{ran}(W_1 + W_2\Psi^*). \tag{10}$$

Then the restriction $A := A_0^*\big|_{\operatorname{dom}(A)}$ of A_0^* in (9) to $\operatorname{dom}(A) := \ker\left(W_B \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix}\right)$ is a closed operator on $L^2(\Omega)^{n+1}$ and the following conditions are equivalent:

1) A generates a contraction semigroup on $L^2(\Omega)^{n+1}$.
2) A is dissipative: $\operatorname{Re}\langle Ax, x\rangle \le 0$ for all $x \in \operatorname{dom}(A)$.
3) The operator $W_1 + W_2\Psi^*$ is injective and the following operator inequality holds in $\mathcal{H}$:

$$W_1 \Psi W_2^* + W_2 \Psi^* W_1^* \ge 0. \tag{11}$$

The inequality (11) can equivalently be written as follows, with $J = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$:

$$\begin{bmatrix} W_1 & W_2\Psi^* \end{bmatrix} J \begin{bmatrix} W_1 & W_2\Psi^* \end{bmatrix}^* \ge 0.$$

This inequality in fact means that A^* is dissipative, and this in turn implies that the range inclusion (10) is a maximality condition. Indeed, if (10) holds, then Theorem 4.3 essentially says that A is dissipative if and only if A^* is dissipative. If $W_1 + W_2\Psi^*$ is invertible, then W_B is automatically surjective and (10) holds, but this can be the case only for "minimal" choices of $\mathcal{H}$.

We finish this section with our main result.

Theorem 4.4: Make the assumptions and use the notation in Theorem 4.2. Let $V_B = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ be a bounded everywhere defined operator from $L^2(\partial\Omega)^2$ into some Hilbert space $\mathcal{H}$ and define

$$\begin{aligned}\mathcal{A} := \big\{ a \in \operatorname{dom}(A_0^*) \;\big|\; & B_\perp a \in L^2(\partial\Omega) \\ & \wedge \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix} a = 0 \big\}.\end{aligned} \tag{12}$$

Then the following two conditions are together sufficient for the closure A of the operator $A_0^*\big|_{\mathcal{A}}$ to generate a contraction semigroup on $L^2(\Omega)^{n+1}$:

1) The kernel of V_B is a dissipative relation in $L^2(\partial\Omega)$, i.e., $\operatorname{Re}\langle u, v\rangle_{L^2(\partial\Omega)} \le 0$ for all $u, v \in L^2(\partial\Omega)$ such that $V_1 u + V_2 v = 0$.
2) The following operator inequality holds in $\mathcal{H}$:

$$V_1 V_2^* + V_2 V_1^* \ge 0. \tag{13}$$

The operator A generates a unitary group if $\operatorname{Re}\langle u, v\rangle_{L^2(\partial\Omega)} = 0$ for all $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in \ker(V_B)$ and $V_1 V_2^* + V_2 V_1^* = 0$.

Condition 2 is also necessary for A to generate a contraction semigroup (unitary group).

The strength in the preceding result, as compared to Theorem 4.3, lies in the fact that we only need to investigate the kernel of $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ which is a relation in $L^2(\partial\Omega)$. If we decided to use Theorem 4.3, then we would need to study a significantly less practical subspace of $\left[\begin{smallmatrix} H^{-1/2}(\partial\Omega) \\ H^{1/2}(\partial\Omega) \end{smallmatrix}\right]$.

Corollary 4.5: Under the following additional assumptions, condition 1 in Theorem 4.4 becomes necessary too:

1) The operator V_2 is injective with a closed range.
2) Denoting the orthogonal projection in $\mathcal{H}$ onto $\operatorname{ran}(V_2)$ by P, the intersection $\ker\big((I-P)V_1\big) \cap H^{1/2}(\partial\Omega)$ is dense in $\ker\big((1-P)V_1\big)$.

V. Application to the wave equation

In this final section, we apply Theorem 4.4 to our example in the introduction:

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2}(\xi, t) &= (\Delta z)(\xi, t) \quad \text{on } \Omega \times \mathbb{R}_+, \\ 0 &= \nu \cdot \operatorname{grad}\ z(\xi, t) + k(\xi)\, \frac{\partial z}{\partial t}(\xi, t) \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \\ 0 &= \frac{\partial z}{\partial t}(\xi, t) \quad \text{on } \Gamma_0 \times \mathbb{R}_+. \end{aligned} \tag{14}$$

We want to show that the operator associated to this PDE generates a contraction semigroup on the energy space $L^2(\Omega)^{n+1}$. For that we write the wave equation in the form (2); hence we have that our state vector is $x(t) = \left[\begin{smallmatrix} \dot{z}(t) \\ \operatorname{grad} z(t) \end{smallmatrix}\right]$. Furthermore, the system operator A is A_0^*, from equation (9), restricted to some domain. This domain is determined by the boundary conditions in (14).

We assume that the two parts Γ_0 and Γ_1 of $\partial\Omega$ are such that $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\overline{\Gamma}_0 \cup \overline{\Gamma}_1 = \partial\Omega$, and that Γ_0 and Γ_1 have a common boundary of surface measure zero. These assumptions are not restrictive; the last assumption is satisfied, e.g., if Γ_0 and Γ_1 themselves have Lipschitz-continuous boundaries.

In order to apply Theorem 4.4, we first have to reformulate the boundary conditions of (1) as the kernel of

$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix}$ for some bounded operators V_1 and V_2. As range space of V_1 and V_2 we take $\mathcal{H} := \begin{bmatrix} L^2(\Gamma_1) \\ L^2(\Gamma_0) \end{bmatrix}$. Recall that π_0 is the orthogonal projection in $L^2(\partial\Omega)$ onto $L^2(\Gamma_0)$, and we denote the corresponding projection onto $L^2(\Gamma_1)$ by π_1. Now we define:

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} := \begin{bmatrix} \pi_1 M_k & \pi_1 \\ \pi_0 & 0 \end{bmatrix}, \tag{15}$$

where M_k is the bounded operator of multiplication by k in $L^2(\partial\Omega)$. (The function $k \in L^2(\Gamma_1; \mathbb{R})$, $k(\cdot) \geq 0$, is extended by zero on Γ_0.)

Next we check if the kernel of $\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix}$ corresponds to our boundary conditions. Since the state is $x(t) = \begin{bmatrix} \dot{z}(t) \\ \mathrm{grad}\, z(t) \end{bmatrix}$, we have that

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix} x = \begin{bmatrix} \pi_1 M_k & \pi_1 \\ \pi_0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_0 \dot{z} \\ \gamma_\perp \mathrm{grad}\, z \end{bmatrix},$$

and we see that $x = \begin{bmatrix} \dot{z} \\ \mathrm{grad}\, z \end{bmatrix}$, with $\gamma_\perp \mathrm{grad}\, z \in L^2(\partial\Omega)$, lies in $\ker\left(\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix}\right)$ if and only if $\pi_0 \gamma_0 \dot{z} = 0$ and

$$\pi_1 M_k\, \gamma_0 \dot{z} + \pi_1\, \gamma_\perp \mathrm{grad}\, z = 0, \tag{16}$$

which indeed agrees with the boundary conditions in (14).

We show that $\ker\left(\begin{bmatrix} V_1 & V_2 \end{bmatrix}\right)$ is a dissipative relation in $L^2(\partial\Omega)$ as follows. It holds that $\begin{bmatrix} u \\ v \end{bmatrix} \in \ker\left(\begin{bmatrix} V_1 & V_2 \end{bmatrix}\right)$ if and only if $\pi_1 v = -M_k \pi_1 u$ and $\pi_0 u = 0$. For any such $\begin{bmatrix} u \\ v \end{bmatrix}$, we have

$$\begin{aligned} &\mathrm{Re}\,\langle u, v\rangle_{L^2(\partial\Omega)} = \\ &\mathrm{Re}\,\langle \pi_0 u, \pi_0 v\rangle_{L^2(\Gamma_0)} + \mathrm{Re}\,\langle \pi_1 u, \pi_1 v\rangle_{L^2(\Gamma_1)} = \\ &-\mathrm{Re}\,\langle \pi_1 u, M_k \pi_1 u\rangle_{L^2(\Gamma_1)} \leq 0. \end{aligned}$$

We still need to verify that $V_1 V_2^* + V_2 V_1^* \geq 0$. For all $p \in L^2(\Gamma_1)$ and $q \in L^2(\Gamma_0)$ it holds that

$$\begin{aligned} &2\mathrm{Re}\,\left\langle V_1 V_2^* \begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} p \\ q \end{bmatrix} \right\rangle_{\begin{bmatrix} L^2(\Gamma_1) \\ L^2(\Gamma_0) \end{bmatrix}} = \\ &2\mathrm{Re}\,\left\langle \begin{bmatrix} M_k \pi_1 \\ \pi_0 \end{bmatrix} \begin{bmatrix} \mathcal{I}_1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} p \\ q \end{bmatrix} \right\rangle_{\begin{bmatrix} L^2(\Gamma_1) \\ L^2(\Gamma_0) \end{bmatrix}} = \\ &2\mathrm{Re}\,\langle M_k p, p\rangle_{L^2(\Gamma_1)} \geq 0, \end{aligned}$$

where $\mathcal{I}_1 : L^2(\Gamma_1) \to L^2(\partial\Omega)$ is the injection operator; hence $\pi_0 \mathcal{I}_1 = 0$.

By Theorem 4.4, we conclude that the closure A of the operator $\mathcal{A}$ defined in (12), with $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ given by (15), generates a contraction semigroup on $L^2(\Omega)^{n+1}$.

Using the results of [6], this operator closure can be directly characterised as $A = A_0^*|_{\mathrm{dom}\,(A)}$, where

$$\mathrm{dom}\,(A) = \ker\left(\begin{bmatrix} \Pi_1 M_k & \Pi_1 \\ \pi_0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix}\right),$$

with Π_1 the orthogonal projection in $H^{-1/2}(\partial\Omega)$ onto $H^{-1/2}(\partial\Omega) \ominus L^2(\Gamma_0)$. Here $\mathcal{H}$ needs to be chosen differently from above, since $\mathrm{ran}\,(\Pi_1) \not\subset L^2(\Gamma_1)$; take for instance $\mathcal{H} := \begin{bmatrix} \mathrm{ran}(\Pi_1) \\ L^2(\Gamma_0) \end{bmatrix}$. One could also prove that A generates a contraction semigroup on $L^2(\Omega)^{n+1}$ using this representation and Theorem 4.3, but that leads to more complicated calculations than those above.

By Proposition 3.2, we can also write $\mathrm{dom}\,(A)$ as

$$\mathrm{dom}\,(A) = \left\{ \begin{bmatrix} g \\ f \end{bmatrix} \in \begin{bmatrix} H^1_{\Gamma_0}(\Omega) \\ H^{\mathrm{div}}(\Omega) \end{bmatrix} \;\middle|\; M_k \gamma_0 g + \gamma_\perp f \in \left(\gamma_0 H^1_{\Gamma_0}(\Omega)\right)^{(\perp)} \right\}.$$

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