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A Bridge Between Stochastic and Deterministic Asymptotic Stability With Convergence Rates and Intensity Rates

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***Abstract*— In this paper, we improve the notion of stochastic stability by clarifying convergence rates of sample paths and intensity rates of diffusion terms, based on the shapes of stochastic Lyapunov functions. Using this notion, we clarify conditions that scaling operations for stochastic Lyapunov functions are available. Then, we create a measure of the distance between stochastic and deterministic stability properties by considering the inclusive relation between various stochastic stability properties.**

I. INTRODUCTION

The stochastic Lyapunov stability theory is widely discussed in the field of stochastic control theory during the last half-century [6], [7], [13], [15], [16], [20]. An important concern of this debate is the construction of stochastic stability properties compatible with Lyapunov stability properties for deterministic systems. Many basic results are introduced and compared with the deterministic Lyapunov theory such as direct and indirect methods, especially, in [13]. However, the stochastic stability theory continues to be different from the deterministic one; a wide variety of stochastic stability notions have been proposed because of the randomness of Wiener processes [1], [2], [8], [14], [4], [9]. Therefore, bridges between stochastic and deterministic stability notions are required for many researches on dynamical systems, such as stability theory for nonlinear controlled systems.

To compare stochastic and deterministic stability properties, we should first consider the notion of *almost sure asymptotic stability (ASAS)* properties. Recently, the authors have clarified the difference in conditions between ASAS and deterministic asymptotic stability properties [18], [19]. They suggest that the ASAS properties should be subdivided into several groups; more clearly, the most popular *global asymptotic stability in probability (global ASIP)* [13] is distinctly weaker than Bardi and Cesaroni's *uniform almost sure asymptotic stability (UASAS)* [4], which is "almost the same" as deterministic asymptotic stability.

However, to the best of our knowledge, the measure of the distance between stochastic and deterministic stability properties has not been proposed yet. In this paper, we build such a measure taking into account the convergence rates of sample paths and intensity rates of disturbance terms.

Convergence rates for the sample paths have been first proposed by Kozin [14] and Khasminskii [13] with exponential convergence under ASAS properties, which is considered the analogy of exponential stability for deterministic systems. The property also provides the intensity rates of disturbance terms; i.e., exponential p-stability ensures that the expectation of the p-th power of the Euclidean norm of the state has a finite limit as time goes infinity. The property has been developed using the notion of Lyapunov exponents [1], [13], and the method is well known to be useful; however, applying this to nonlinear stochastic systems is problematic because of difficulty in calculations [8], [13].

Recently, Yin et al. [21] have proposed the notion of *finite-time stability in probability*, which is considered the analogy of finite-time stability for deterministic systems [3]; however, it does not provide the intensity rates of disturbance terms. Hence, we should find a way to organically combine finite-time stability in probability with exponential p-stability.

Thus, our main aim is to develop the notion of exponential p-stability into other convergence rates by clarifying the correspondence relationship with deterministic stability, based on stochastic Lyapunov functions.

In addition, we should consider additive noises for wide use of our notion. There are many cases that disturbance coefficients do not vanish at the origin although "stochastic Lyapunov functions" for such systems exist. We briefly discuss such cases as a basis for our future works.

This is accomplished, in this paper, by working through the following stages. In Section II, we briefly summarize the various notions of stochastic stability properties, such as global ASIP, exponential p-stability, finite-time stability in probability and UASAS. In Section III, we define our basic notion of *stability with respect to V* by clarifying stability theory. In Section IV, we build a bridge between stochastic and deterministic systems by clarifying the conditions for scaling stochastic Lyapunov functions. In Section V, we confirm that the bridge is active as the measures of the distances of the two notions. Furthermore, in Section VI, we try extending our discussion to the case of additive noises. Section VII summarizes the conclusions of this study.

In this paper, $\mathbb{R}^n$ denotes an n-dimensional Euclidean space; in particular, $\mathbb{R}$ denotes $\mathbb{R}^1$. If $\gamma : [0,\infty) \to [0,\infty)$ satisfies $\gamma \in K_\infty$, then $\gamma(t)$ is monotone increasing, $\gamma(0) \equiv 0$, and $\gamma(r) \to \infty$ as $r \to \infty$. The conditional probability and the conditional expectation of some event A, under some condition B, are written as $\mathbb{P}\{A|B\}$ and $\mathbb{E}\{A|B\}$, respectively. Further, $w(t) := (w_1, w_2, \dots, w_d)^T \in \mathbb{R}^d$ is a d-dimensional, independent, and standard Wiener process; i.e., all the values of the variances are t, and all the covariances are constantly zero. The differential forms of Itô and Stratonovich integrals of $\sigma : \mathbb{R}^n \to \mathbb{R}^d$ in $w(t) \in \mathbb{R}^d$ are denoted by $\sigma(x)dw$ and

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$\sigma(x)\circ dw$, respectively. For $k_1,k_2:\mathbb{R}^n\to\mathbb{R}^n$, a Lie derivative is defined as $(\mathrm{L}_{k_1}k_2)(x) := \frac{\partial k_2}{\partial x}(x)k_1(x)$. Further,

$$\mathrm{sgn}\,(y) = \begin{cases} 1 & \text{if} \quad y>0, \\ 0 & \text{if} \quad y=0, \\ -1 & \text{if} \quad y<0. \end{cases} \tag{1}$$

II. Basic Results of Almost Sure Asymptotic Stability

This section provides a preliminary discussion on stochastic Lyapunov theory for a stochastic system:

$$dx = f(x)dt + \sum_{\alpha=1}^{d} \sigma_\alpha(x)\circ dw_\alpha \tag{2}$$

with convergence rates, where $x\in\mathbb{R}^n$ is a state vector and $f,\sigma_1,\sigma_2,\dots,\sigma_d:\mathbb{R}^n\to\mathbb{R}^n$ are assumed to provide local solutions for all initial states $x(0)=x_0\in\mathbb{R}^n$ and satisfy $f(0)=\sigma_1(0)=\cdots=\sigma_d(0)=0$.

To derive stochastic Lyapunov stability properties, the following notion is important:

Definition 1 (Infinitesimal Operator): Let a Markov process $x(t)\in\mathbb{R}^n$, $t\geq 0$ and $x(t)=x_t\in\mathbb{R}^n$ be considered. If

$$(\mathscr{L}v)(x(t)) = \lim_{h\downarrow 0}\frac{\mathbb{E}\{v(x(t+h))|x(t)=x_t\}-v(x(t))}{h} \tag{3}$$

can be defined for a function $v:\mathbb{R}^n\to\mathbb{R}$, then $\mathscr{L}$ is said to be *an infinitesimal operator* for $x(t)$. □

For (2), the infinitesimal operator is calculated as follows:

Theorem 1 ([13]): The infinitesimal operator for (2) is represented as

$$\mathscr{L}(\cdot) = \mathrm{L}_f(\cdot) + \frac{1}{2}\sum_{\alpha=1}^{d}\mathrm{L}_{\sigma_\alpha}\mathrm{L}_{\sigma_\alpha}(\cdot). \tag{4}$$

A. Global Asymptotic Stability in Probability

To begin with, we show the most basic stochastic asymptotic stability with almost sure convergence to the origin:

Definition 2 (Stability in Probability [13]): The origin of (2) is *stable in probability* if

$$\lim_{x_0\to 0}\mathbb{P}\left\{\sup_{0\leq t}|x(t)|>\varepsilon\,\middle|\,x(0)=x_0\right\}=0 \tag{5}$$

for any $\varepsilon>0$. □

Definition 3: (Global Asymptotic Stability in Probability [13]): The origin of (2) is *globally asymptotically stable in probability (globally ASIP)* if it is stable in probability, and if

$$\mathbb{P}\left\{\lim_{t\uparrow\infty}|x(t)|=0\,\middle|\,x(0)=x_0\right\}=1 \tag{6}$$

for all $x_0\in\mathbb{R}^n$. □

Using the foregoing definitions, we obtain the following theorem:

Theorem 2 ([13]): The origin of (2) is globally ASIP if there exists a C^2, positive definite and proper function $V:\mathbb{R}^n\to\mathbb{R}$ such that $(\mathscr{L}V)(x)$ is negative definite in $\mathbb{R}^n$. ⧫

Remark 1: In general, the functions satisfying all the conditions of Theorem 2 are called *stochastic Lyapunov functions* [13], [15]. In this paper, however, we appropriate the name for the functions satisfying Definition 8 below, because our focus is on stochastic stability stronger than globally ASIP property. ⧫

B. Almost Sure Asymptotic Stability With Convergence Rates

Here we consider the stochastic stability with convergence rates.

Definition 4 (Exponential p-Stability [13]): The origin of (2) is called *exponentially p-stable* if there exist $p,A,\alpha>0$ such that

$$\mathbb{E}\{|x(t)|^p|x(0)=x_0\}\leq A|x_0|^p\exp(-\alpha t) \tag{7}$$

for all $x_0\in\mathbb{R}^n$. □

The following theorem is similar to the result for exponential stability of deterministic systems [12]:

Theorem 3 ([13]): The origin of (2) is exponentially p-stable if there exist constants $k_1,k_2,k_3>0$ and a C^2 function $V:\mathbb{R}^n\to\mathbb{R}$ such that

$$k_1|x|^p \leq V(x) \leq k_2|x|^p \tag{8}$$
$$(\mathscr{L}V)(x) \leq -k_3|x|^p. \tag{9}$$

for all $x_0\in\mathbb{R}^n$. □

Recently, Yin et al. have introduced the notion of *finite-time stability in probability* [21]:

Definition 5 (Finite-Time Stability in Probability [21]): The origin of (2) is called *finite-time stable in probability* if it is stable in probability, and if there exists the first hitting time τ satisfying

$$\mathbb{P}\left\{\tau=\inf_{0\leq t}\{t|x(t)=0\}<\infty\,\middle|\,x(0)=x_0\right\}=1 \tag{10}$$

for all $x_0\in\mathbb{R}^n$. □

The finite-time stability in probability also has the similar result to that for finite-time stability of deterministic systems [3]:

Theorem 4 ([21]): The origin of (2) is finite-time stable in probability if there exist class K_∞ functions $k_1,k_2:\mathbb{R}\to\mathbb{R}$, constants $k_3,p>0$, $a\in(0,1)$, and a C^2 function $V:\mathbb{R}^n\to\mathbb{R}$ such that

$$k_1(|x|)\leq V(x)\leq k_2(|x|) \tag{11}$$
$$(\mathscr{L}V)(x)\leq -k_3V^a(x). \tag{12}$$

□

The foregoing results are also similar to those for deterministic systems. However, Theorem 3 is based on p-stability, which is widely discussed as "pth moment stability" [1], [8], [13], [17]; in contrast, Theorem 4 is based on stability in probability [13], [15]. These two notions are generally different in stochastic stability theory, as will be shown in the following sections.

C. Uniform Almost Sure Asymptotic Stability

Finally, in this section, we consider the result of uniform almost sure asymptotic stability (UASAS) [4], whose stability is considered "almost the same" as that of asymptotic stability for deterministic systems.

First, we define the UASAS property:

Definition 6 ([4]): The origin of (2) is said to be *globally uniformly almost surely asymptotically stable (globally UASAS)* if there exists $\gamma \in \mathscr{K}_\infty$, and if $|x(t)| \le \gamma(|x|)$ and

$$\lim_{t\uparrow\infty} |x(t)| = 0 \tag{13}$$

holds almost surely for all $x \in \mathbb{R}^n$ and $t \ge 0$. □

The Lyapunov stability theory for UASAS property is as follows:

Theorem 5 ([4], [5]): The origin of (2) is globally UASAS if and only if there exists a lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying all the following conditions:

1) for all $x \in \mathbb{R}^n$, there exist functions class K_∞ functions $k_1, k_2 : \mathbb{R} \to \mathbb{R}$ satisfying (11);
2) there exists a positive definite and Lipshitz function $l : \mathbb{R}^n \to \mathbb{R}$ such that $V(x)$ is a viscosity supersolution to

$$-(\mathscr{L}V)(x) \ge l(x); \tag{14}$$

3) for all $x \in \mathbb{R}^n$ and for all $\alpha \in \mathbb{N}_1^d$, it is also a viscosity supersolution to

$$(\mathrm{L}_{\sigma_\alpha}V)(x) = 0. \tag{15}$$

♦

Obviously, if the origin is globally UASAS, then it is also globally ASIP. What is important is that all sublevel sets of $V(x)$ for Theorem 5 are invariance sets [4]; i.e., Itô's formula implies that the stochastic differentiation of $V(x)$ is represented as

$$dV(t) = (\mathscr{L}V)(x(t))dt + \sum_{\alpha=1}^{d} (\mathrm{L}_{\sigma_\alpha}V)(x(t))dw_\alpha(t). \tag{16}$$

Thus, if $V(x)$ satisfies the condition (15), then

$$dV(t) = (\mathscr{L}V)(x(t))dt \tag{17}$$

holds with probability one. That is why we consider UASAS property as almost the same as asymptotic stability for deterministic systems. More detailed discussion on this issue appears in our previous work [18].

Remark 2: In all theorems of this section, excepting Theorem 5, V is defined as C^2 for all $x \in \mathbb{R}^n$; however, we can reduce the restriction as C^2 for all $x \in \mathbb{R}^n \backslash \{0\}$, provided it is continuous and zero at the origin, via the formulation similar to Theorem 5. The result is also confirmed by the results of the positive supermartingale property [13]. ♦

III. Generalized Notion of p-Stability With Ensuring Convergence Rates

In this section, we propose stochastic stability notions based on the shape of stochastic Lyapunov functions. In what follows, $V : \mathbb{R}^n \to \mathbb{R}$ is assumed to be continuous, positive definite and proper for all $x \in \mathbb{R}^n$ and C^2 in $x \in \mathbb{R}^n \backslash \{0\}$; $V_0 := V(x(0))$.

Definition 7: The origin of (2) is said to be

1) *exponentially stable with respect to V*, if there exist positive constants M and α such that

$$\mathbb{E}\{V(x(t))|x(0) = x_0\} \le MV_0 \exp\{-\alpha t\} \tag{18}$$

for all $t \ge 0$ and for all $x_0 \in \mathbb{R}^n$;

2) *rationally stable with respect to V*, if there exist positive constants M and α such that

$$\mathbb{E}\{V(x(t))|x(0) = x_0\} \le MV_0(1 + V_0^\alpha t)^{-\frac{1}{\alpha}} \tag{19}$$

for all $t \ge 0$ and for all $x_0 \in \mathbb{R}^n$;

3) *finite-time stable with respect to V*, if there exist positive constants A and α such that

$$\mathbb{E}\{V(x(t))|x(0) = x_0\} \le \begin{cases} V_0(1 + V_0^{-\alpha} t)^{\frac{1}{\alpha}}, & \forall t \in [0, V_0^\alpha A], \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

for all $x_0 \in \mathbb{R}^n$.

Finally, if one of the three conditions holds, the origin of (2) is also said to be *stable with respect to V*. □

Here we define stochastic Lyapunov functions as follows:

Definition 8: If there exist $k > 0$ and $a > 0$ such that

$$\mathscr{L}V(x) \le -kV^a(x) \tag{21}$$

holds for all $x \in \mathbb{R}^n$, then V is said to be a *stochastic Lyapunov function*. □

Using the foregoing notions, we obtain the following theorem:

Theorem 6: Assume that there exists a stochastic Lyapunov function V for (2). If $a = 1$, the origin is exponentially stable w.r.t. V; if $a > 1$, it is rational stable w.r.t. V; and if $a \in (0,1)$, it is finte-time stable w.r.t. V. ♦

Proof: If (21) holds for $k > 0$ and $a > 0$, we obtain

$$\mathbb{E}\{V(x(t))|x(0) = x_0\} - V_0 = \int_0^t \mathbb{E}\{\mathscr{L}V(x(s))\}ds. \tag{22}$$

Differentiating the inequality with respect to t, we obtain

$$\frac{d}{dt}\mathbb{E}\{V(x(t))|x(0) = x_0\} \le -k\mathbb{E}\{V^a(x(t))|x(0) = x_0\}. \tag{23}$$

Thus, we obtain

$$\mathbb{E}\{V(x(t))|x(0) = x_0\} \le V_0 \exp(-kt) \tag{24}$$

for $a = 1$, and

$$\mathbb{E}\{V(x(t))|x(0) = x_0\} \le \left\{V_0^{1-a} - (1-a)kt\right\}^{\frac{1}{1-a}} \tag{25}$$

for $a \ne 1$.

1) Let $a = 1$. (24) with $M = 1$ and $\alpha = k$ implies (18).

2) Let $a>1$. If $A>0$, there exists $m_1>0$ such that
$$0<m_1\leq\frac{A+B}{1+B}\tag{26}$$
for any $B\geq 0$; substituting $A=1/(a-1)k$ and $B=V_0^{a-1}t$, we obtain
$$0<m_1\left(1+V_0^{-(1-a)}t\right)\leq\frac{1}{-(1-a)}+V_0^{-(1-a)}t.\tag{27}$$
This implies that (25) is calculated as
$$\begin{aligned}&\mathbb{E}\{V(x(t))|x(0)=x_0\}\\&\leq\{-(1-a)k\}^{\frac{1}{1-a}}mV_0\left\{1+V(x_0)^{-(1-a)}t\right\}^{\frac{1}{1-a}}.\end{aligned}\tag{28}$$
Therefore, (28) with $M=\{-(1-a)k\}^{\frac{1}{1-a}}m$ and $\alpha=-(1-a)$ implies (19).
3) Let $a\in(0,1)$. If $0<B\leq A$,
$$\frac{A-B}{1+B}\leq A\tag{29}$$
holds; substituting $A=1/(1-a)k$ and $B=V_0^{a-1}t$, we obtain (20).

■

If we add condition (8), then the exponential stability w.r.t. V becomes exponential p-stability, which is compatible with exponential stability for deterministic systems [12], [3]. Similarly, if we add a condition $k_1|x|^{p_1}\leq V(x)\leq k_2|x|^{p_2}$ with $k_1>0$, $k_2>0$, $p_2\geq p_1>0$, then rational stability w.r.t. V is compatible with rational stability [3]. Finite-time stability w.r.t. V is, however, somewhat different from finite-time stability [3]; instead, the finite-time stability in probability of Definition 5 is the similar definition to deterministic sense. Nevertheless, we should consider the notion of finite-time stability w.r.t. V, because stochastic Lyapunov functions are generally not capable of scaling liberally. The restriction also makes us consider the importance of shapes of stochastic Lyapunov functions; further discussion on this is presented in the following section.

Remark 3: Obviously, any norm is capable of being used for stability w.r.t. V; of course, it also covers ρ-stability defined for stochastic weighted homogeneous systems [10]. Why we define such a large class of stability is because we focus our attention on only two properties: convergence rates in this section and scaling problems in the following section. We consider that the restriction for norms or weighted homogeneous norms may create unnecessary confusion in our discussion. ♦

IV. Stability Under Scaling Operation

In this section, we consider conditions for stochastic stability under scaling stochastic Lyapunov functions. In what follows, $W(x)=V^p(x)$ for $p>0$ is considered a stochastic Lyapunov candidate function for (2). The following definition emphasizes the scaling operation for stochastic Lyapunov functions:

Definition 9: The origin of (2) is said to be exponentially, rationally, or finite-time p-stable w.r.t. V, if the origin is exponentially, rationally, or finite-time stable w.r.t. V^p, respectively. □

The following result shows that we are capable of scaling down all stochastic Lyapunov functions:

Theorem 7: Assume that there exists a stochastic Lyapunov function V for (2). Let us consider that $p\leq 1$ assuming V^p to be C^2, except the origin. If $a=1$, the origin of (2) is exponentially p-stable w.r.t. V; if $a>1$, it is rationally p-stable w.r.t. V; and if $a<1$, it is finite-time p-stable w.r.t. V. ♦

Proof: Considering a function $W(x)$, we obtain
$$\begin{aligned}(\mathscr{L}W)(x)=&pV^{p-1}(x)(\mathscr{L}V)(x)\\&+\frac{p(p-1)}{2}V^{p-2}(x)\sum_{\alpha=1}^{d}(\mathrm{L}_{\sigma_\alpha}V)^2(x).\end{aligned}\tag{30}$$
Because $p\leq 1$, the second term on the right-hand side of this inequality is non-positive; thus, we obtain
$$(\mathscr{L}W)(x)\leq-pkW^{\frac{a+p-1}{p}}(x).\tag{31}$$
The rest of the proof is the same as that for Theorem 6, provided we replace V of Theorem 6 by W. ■

In contrast, the following result shows that we are capable of scaling up if the drift terms and the diffusion terms of (2) have good properties with well-formed V:

Theorem 8: Assume that there exists a stochastic Lyapunov function V for (2). The origin is p-stable w.r.t. V for some $p>1$ if there exist constants $c_1,c_2,\ldots,c_d$ such that
$$(\mathrm{L}_{\sigma_\alpha}V)^2(x)\leq c_\alpha^2V^{a+1}(x)\tag{32}$$
and
$$k>\frac{p-1}{2}\sum_{\alpha=1}^{d}c_\alpha^2\tag{33}$$
hold for all $x\in\mathbb{R}^n$ and for all $\alpha\in\mathbb{N}_1^d$. Furthermore, if $a=1$, the origin of (2) is exponentially p-stable w.r.t. V; if $a>1$, rationally p-stable w.r.t. V; and if $a<1$, finite-time p-stable w.r.t. V. ♦

Proof: By (21), (30), (32) and (33), there exists $k'>0$ such that
$$\begin{aligned}(\mathscr{L}W)(x)\leq&-p\left(k-\frac{p-1}{2}\right)W^{\frac{a+p-1}{p}}(x)\\\leq&-k'W^{\frac{a+p-1}{p}}(x).\end{aligned}\tag{34}$$
The rest of sufficiency of the proof is the same as that for Theorem 6, provided we replace V of Theorem 6 by W. ■

V. Distance Between Stochastic and Deterministic Stability Properties

In this section, we emphasize that p-stability w.r.t. V builds a bridge with asymptotic stability for deterministic systems.

First, we consider the inclusive relation between stochastic stability properties defined above. Theorem 8 implies that stability w.r.t. V is tighter than global ASIP property:

Corollary 1: Let $p>0$ be fixed. If the origin of (2) is p-stable w.r.t. V, then it is also globally ASIP. ♦

This corollary is obvious because any V^p satisfies the condition in Theorem 2; however, the converse is not always true; a simple example is shown below:

Example 1: Let us consider

$$dx = -Axdt + Sx \circ dw, \tag{35}$$

where $x, w \in \mathbb{R}$ and $A, S > 0$. If we choose a Lyapunov candidate function as $V(x) = (1/2)x^2$, we obtain

$$(\mathscr{L}V)(x) = -(A - S^2)V^a(x) = -kV^a(x). \tag{36}$$

Thus, if $k > 0$, $V(x)$ is a stochastic Lyapunov function. Furthermore,

$$\left(\frac{\partial V}{\partial x}(x)Sx\right) = (2S)^2 V^2(x) = c^2 V^2(x) \tag{37}$$

implies that a condition to be satisfied for (33) is

$$p < \frac{2(A - S^2) + 4S^2}{4S^2}. \tag{38}$$

For example, if $A = 4S^2$, the inequality is

$$p < \frac{5}{2} < 3. \tag{39}$$

In this case, the origin is 2-stable w.r.t. V; hence, it is also globally ASIP. However, there is no guarantee that it would be 3-stable w.r.t. V. ♦

What is important is that the origin is globally ASIP even if p is quite small; in such cases, however, the transient states may vibrate with quite a large amplitude. For practical purposes, at least 1-stable w.r.t. $|x|$, generally said to be *stability in the mean*, is necessary. Of course, this discussion has already appeared in the earlier works [13], [14], though implicitly; the novel contribution of this study is that it clarifies the expectation of the disturbance amplitudes by ensuring convergence rates:

Corollary 2: Let $p > 0$ be fixed. If the origin of (2) is finite-time p-stable w.r.t. V, then it is also finite-time stable in probability. ♦

The converse is not always true; a simple example is presented below:

Example 2: Let us consider

$$dx = -A\mathrm{sgn}(x)|x|^{2a-1}dt + S\mathrm{sgn}(x)|x|^a \circ dw, \tag{40}$$

where $x, w \in \mathbb{R}$, $A, S > 0$ and $a \in (1/2, 1)$. If we choose a Lyapunov candidate function as $V(x) = (1/2)x^2$, we obtain

$$(\mathscr{L}V)(x) = -2^a \left\{A - \frac{1}{2}S^2(1+a)\right\} V^a(x) = -kV^a(x). \tag{41}$$

Thus, if $k > 0$, $V(x)$ is a stochastic Lyapunov function. Furthermore,

$$\left(\frac{\partial V}{\partial x}(x)S\mathrm{sgn}(x)|x|^a\right) = 2^{a+1}S^2V^{a+1}(x) = c^2V^{a+1}(x) \tag{42}$$

implies that a condition to be satisfied for (33) is

$$p < \frac{2A + S^2(1-a)}{2S^2}. \tag{43}$$

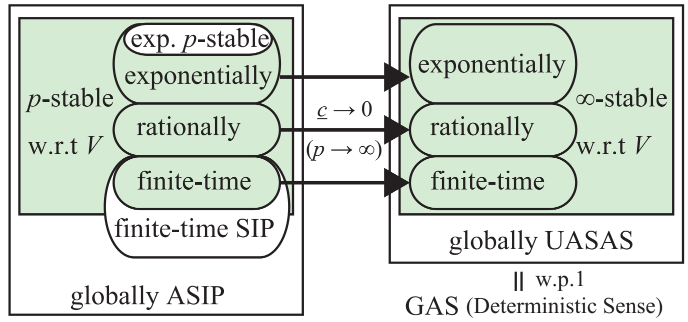

Fig. 1. Relationship between stochastic and deterministic asymptotic stability properties. The colored areas show our definitions of stochastic stability properties. The left side boxes denote "purely stochastic" stability properties, and those on the right side global UASAS, which is the same as global asymptotic stability (GAS) for deterministic systems with probability one (w.p.1). Note, however, that even if the origins of stochastic systems are globally UASAS, i.e., their diffusion terms have no effect on the dynamics of stochastic Lyapunov functions, the terms still have effects on the dynamics of the systems. The distance between the left and the right side boxes is measured by the norm $||\underline{c}||$ because it denotes the influence of diffusion terms on stochastic Lyapunov functions.

For example, if $A = 3S^2$, the inequality is

$$p < \frac{5-a}{2} < 3. \tag{44}$$

In this case, the origin is finite-time 2-stable w.r.t. V; however, there is no guarantee that it would be finite-time 3-stable w.r.t. V. ♦

Thus, even if the origin is finite-time stable in probability, we should try to construct a stochastic Lyapunov function V such that V^p also becomes a stochastic Lyapunov function with sufficiently large $p > 0$.

Translating the result in the words of control engineering, globally and finite-time ASIP properties ensure the almost sure existence of steady states; in contrast, p-stability and (p-)stability w.r.t. V describe the measure of the influence of Gaussian white noises for transient states. This implies that p-stability w.r.t. V almost surely converges with stability for deterministic systems, as p tends to infinity.

Corollary 3: If all the conditions for Theorem 8 are satisfied with $c_1 = c_2 = \cdots = c_d = 0$, the origin of (2) is p-stable w.r.t. V for any large $p > 0$; furthermore, it is also globally UASAS. ♦

This corollary is immediately obtained by Definition 6 and Theorem 8. Consequently, for

$$\underline{c}_\alpha = \min\{|c_\alpha| \in [0,\infty) | (32) \text{ holds}\}, \ \alpha \in \mathbb{N}_1^d, \tag{45}$$

$\underline{c} = \sqrt{\sum_{\alpha=1}^d \underline{c}_\alpha^2}$ indicates the distance from UASAS property; i.e., it is considered as "an intensity rate" of diffusion terms. Of course the rate is deeply concerned with the shape of a stochastic Lyapunov function.

The relationship between stochastic and deterministic stability properties is also illustrated in Fig. 1.

VI. Consideration on Additive Noise

Finally, we consider the situation that the origin of (2) becomes non-equilibrium point by Gaussian white noises;

i.e., $\sigma(0) \neq 0$. Of course the situation has been widely discussed such as stability of random sets [2] or noise-to-state stability [6], [11]; in contrast, our view point is simpler than those previous works.

Definition 10: Let (2) be considered, provided that $\sigma_1 = \sigma_2 = \cdots = \sigma_d = 0$ are not necessary to hold. The origin is said to be *stable w.r.t.* $\mathbb{E}V$ if there exist a stochastic Lyapunov function. □

Actually, Theorem 6 holds even if $\sigma_1 = \sigma_2 = \cdots = \sigma_d = 0$ are not satisfied; (23) implies that the expectation of V converges to zero even if the origin is not equilibrium. Instead, the expectation of the state is "an equilibrium." Of course the discussion is immediately confirmed by Itô's stochastic analysis; however, for stochastic control problems, it is important to emphasis the origin is "stable" because such systems have steady states in the sense of expectation.

To measure the influence of additive noises, we should compare our results with ones for stochastic disturbance attenuation such as Theorem 4.1 in [6]. Thus, we obtain the following result:

Theorem 9: Let V^p be continuous, positive definite and proper in $\mathbb{R}^n$ and C^2 in $\mathbb{R}^n\backslash\{0\}$ with fixed $p \in (0,1)$. Assume that there exist $c_1, c_2, \ldots, c_d \in \mathbb{R}$ and $k, a, \gamma > 0$ such that

$$(\mathscr{L}V)(x) \leq -kV^a(x) + \gamma, \tag{46}$$

$$(\mathrm{L}_{\sigma_\alpha}V)^2 \leq \sum_{\alpha=1}^{d} c_\alpha^2 V, \tag{47}$$

$$\gamma \leq \frac{1-p}{2}\sum_{\alpha=1}^{d} c_\alpha^2. \tag{48}$$

Then, the origin of (2) is stable w.r.t. $\mathbb{E}V^p$. ♦

Proof: Considering $W = V^p$ with (30), (46) and (47), we obtain

$$(\mathscr{L}W)(x) \leq -pkW^{\frac{a+p-1}{p}} + p\left(\gamma + \frac{p-1}{2}\sum_{\alpha=1}^{d} c_\alpha^2\right)W^{\frac{p-1}{p}}. \tag{49}$$

The condition (48) implies that the second term of the right hand side of (49) is negative definite. Therefore, the theorem is proven. ■

What is important is that, for small γ, we are capable of choosing large $p \in (0,1)$; if γ is too large, we may not choose any p satisfying V^p as C^2 except the origin. Hence, the value of γ is preferable as small as possible;

$$\underline{\gamma} = \min\{\gamma \in [0,\infty) | (46) \text{ holds}\} \tag{50}$$

indicates intensity rates of steady states.

VII. Concluding Remarks

In this paper, we have clarified the appropriate measure of distance from stochastic stability property to deterministic one with convergence rates of sample paths and intensity rates of diffusion terms via the scaling problems of stochastic Lyapunov functions.

The derivation of the conditions necessary for stability w.r.t. V will be quite interesting for future work on dynamical system theory. Also, developing the method of additive noises, described in Section VI, will be an important theme for future research on industrial applications.

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