21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

State-constrained stochastic optimal control problems via reachability approach*

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***Abstract*— This work is concerned with a general class of stochastic optimal control problems in presence of state-constraints. When state-constraints are taken into account and in absence of quite restrictive controllability assumptions on the dynamics, the continuity of the value function cannot be guaranteed and some well-known problems arise in its characterization as a viscosity solution of a Hamilton-Jacobi-Bellman equation. The approach proposed in this work leads to a characterization of the epigraph of the value function translating, at a first stage, the optimal control problem into a state-constrained stochastic target problem with unbounded controls. This new formulation of the problem has the advantage to allow to solve it by a level set approach, where the state-constraints can be managed by an exact penalization technique.**

Let $(\Omega, \mathbb{F}, \mathbb{F}_t, \mathbb{P})$ be a probability space, where the filtration $\{\mathbb{F}_t\}_{t\geq 0}$ is generated by a p-dimensional Brownian motion $\mathcal{B}(\cdot)$ (with $p \geq 1$).
Given $T > 0$ and $0 \leq t \leq T$, the following system of controlled stochastic differential equations (SDEs) in $\mathbb{R}^d$ ($d \geq 1$) is considered

$$\begin{cases} dX(s) = b(s, X(s), u_s)ds + \sigma(s, X(s), u_s)d\mathcal{B}_s \\ X(t) = x, \end{cases} \tag{1}$$

for $s \in (t, T]$, where $u \in \mathcal{U}$ set of square integrable predictable processes with values in a compact set $U \subset \mathbb{R}^m$ ($m \geq 1$). The following classical assumption will be considered on the coefficients b and σ:

(H1) $b : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times p}$ are continuous functions, Lipschitz continuous with respect to x, uniformly in u.

Let us consider a function (namely the cost function) ψ such that:

(H2) $\psi : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz continuous function.

Denoted by $X^u_{t,x}(\cdot)$ the strong solution of (1) associated with the control u, we aim to characterize and compute the value function v defined by the following optimal control problem

$$v(t, x) := \inf_{u\in\mathcal{U}} \left\{ \mathbb{E}\big[\psi(X^u_{t,x}(T))\big] : X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t, T] \text{ a.s. } \right\}. \tag{2}$$

In the unconstrained case $\mathcal{K} = \mathbb{R}^d$ by using the dynamic programming approach the function v can be characterized as the unique viscosity solution of a second order Hamilton-Jacobi-Bellman (HJB) equation (see [1],[2] for instance). However it is evident that many practical applications are concerned with the case $\mathcal{K} \subsetneq \mathbb{R}^d$ where, for instance, $\mathcal{K}$ can take into account the presence of an obstacle, economical/physical constraints etc.
In presence of state-constraints the characterization of v as a viscosity solution of an HJB equation becomes more complicated and it is essentially due to its loss of regularity. There is a rich literature dealing with state-constrained optimal control problems and the associated partial differential equation: we can refer for instance to [3] and [4] for the deterministic case and to [5] and [6] for the stochastic case. In absence of any controllability assumptions on the dynamics, the function v can only be characterized as a discontinuous viscosity solution of a bilateral HJB equation.
Our aim in this work is to provide an alternative way for characterizing and compute, in a very general setting, the value function associated with a state-constrained optimal control problem, trying to avoid the issues mentioned above. The approach we propose essentially relies on two ideas: the first one is to re-write problem (2) as a state-constrained stochastic target problem (essentially adapting to our case the arguments developed by Bouchard and Dang in [7]) and the second one is to solve this problem by a level-set approach by using an exact penalization technique for managing the state-constraints.

One can in fact easily verify that the following characterization of the value function v holds:

$$v(t, x) = \inf \left\{ z \in \mathbb{R} : \exists u \in \mathcal{U} \text{ s.t. } \mathbb{E}\big[\psi(X^u_{t,x}(T))\big] \leq z,\ X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t, T] \text{ a.s.} \right\}. \tag{3}$$

Our first important result is contained in the theorem below and it concerns the link between the right side of (3), that in literature is referred as a stochastic target problem with controlled-lost, and a suitable stochastic target problem,

*This work was partially supported by the EU under the 7th Framework Programme Marie Curie Initial Training Network "FP7-PEOPLE-2010-ITN", SADCO project, GA number 264735-SADCO

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ISBN: 978-90-367-6321-9

where the target is asked to be reached almost surely.
The idea at the basis of the result is that the problem with controlled-loss is equivalent to the corresponding stochastic target problem up to a martingale and this observation will result in an augmentation of the state and control space by adding a martingale component to the original dynamics.

Theorem 1: Let assumptions (H1) and (H2) be satisfied. Then

$$v(t,x) = \inf\Big\{ z \in \mathbb{R} : \exists (u,\alpha) \in \mathcal{U}\times\mathcal{A} \text{ s.t. } \Big(Z^{\alpha}_{t,z}(T) \geq \psi(X^u_{t,x}(T)), X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \Big) \text{a.s.} \Big\} \quad (4)$$

where $\mathcal{A}$ denotes the set square-integrable $\mathbb{R}^p$-valued predictable processes and

$$Z^{\alpha}_{t,x}(\cdot) = z + \int_t^{\cdot} \alpha_s^T d\mathcal{B}_s.$$

The right side of (4) is a stochastic target problem as defined in [8]. In the unconstrained case this kind of problems has been widely studied in [9] and [8], where a geometric dynamic programming principle is stated for the function v leading to its characterization as a discontinuous viscosity solution of a non-linear second order partial differential equation, the analogous of the HJB equation for this problem. However a numerical approximation of v based this characterization seems not easy at all to obtain. Adding state-constraints to this framework would further complicate the analysis and no results are available at the moment.
We will propose a different way for solving the target problem, based on a level set approach, that has the advantage to lead to a computation of the function v, managing the eventual state-constraints by an exact penalization technique without any further controllability assumption (see [10] for the same idea applied to the deterministic case).
Let be $\mathcal{T} \in \mathbb{R}^{d+1}$ the target set defined by

$$\mathcal{T} := \Big\{ (x,z) \in \mathbb{R}^{d+1} : z \geq \psi(x) \Big\} \equiv \text{Epigraph}(\psi).$$

By Theorem 1 follows immediately that defined the backward reachable set

$$\mathcal{R}^{\mathcal{T},\mathcal{K}}_t := \Big\{ (x,z) \in \mathbb{R}^{d+1} : \exists (u,\alpha) \in \mathcal{U}\times\mathcal{A} \text{ s.t. } \Big((X^u_{t,x}(T), Z^{\alpha}_{t,z}(T)) \in \mathcal{T}, X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \Big) \text{a.s.} \Big\} \quad (5)$$

one has

$$v(t,x) = \inf\Big\{ z \in \mathbb{R} : (x,z) \in \mathcal{R}^{\mathcal{T},\mathcal{K}}_t \Big\}. \quad (6)$$

It is clear by the equality above that the characterization of the backward reachable set $\mathcal{R}^{\mathcal{T},\mathcal{K}}_t$ can be the starting point for the computation of v. The level-set approach we will apply, is based on the idea of Osher and Sethian [11] to characterize a set, the backward reachable set $\mathcal{R}^{\mathcal{T},\mathcal{K}}_t$ for us, as the level-set of a suitable continuous function. In our case this function will be the value function associated to an unconstrained auxiliary optimal control problem.

In particular considered two functions $f_{\mathcal{T}}$ and $g_{\mathcal{K}}$ such that

(H3) $f_{\mathcal{T}} : \mathbb{R}^{d+1} \to \mathbb{R}$, $g_{\mathcal{K}} : \mathbb{R}^d \to \mathbb{R}$ are positive and Lipschitz continuous functions such that

$$f_{\mathcal{T}}(x,z) = 0 \Leftrightarrow (x,z) \in \mathcal{T} \text{ and } g_{\mathcal{K}}(x) = 0 \Leftrightarrow x \in \mathcal{K}$$

and defined the following *unconstrained* optimal control problem

$$\vartheta(t,x,z) := \inf_{\mathcal{U}\times\mathcal{A}} \mathbb{E}\Big[f_{\mathcal{T}}(X^u_{t,x}(T), Z^{\alpha}_{t,z}(T)) + \textstyle\int_t^T g_{\mathcal{K}}(X^u_{t,x}(s))ds \Big] \quad (7)$$

we will prove that under suitable convexity assumptions the backward reachable set $\mathcal{R}^{\mathcal{T},\mathcal{K}}_t$ is the 0-level set of the function ϑ, that is

$$\mathcal{R}^{\mathcal{T},\mathcal{K}}_t = \Big\{ (x,z) \in \mathbb{R}^{d+1} : \vartheta(t,x,z) = 0 \Big\},$$

and thanks to this result, the following characterization is obtained for v:

$$v(t,x) = \inf\Big\{ z \in \mathbb{R} : \vartheta(t,x,z) = 0 \Big\}.$$

In virtue of this equality the remaining part of the work is devoted to the study of the HJB equation associated to the unconstrained optimal control problem (7), where some further difficulties arise from the unboundness of the controls in $\mathcal{A}$.

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