21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Convolution behaviors and topological algebra

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***Abstract*— We investigate one-dimensional convolution behaviors that comprise differential and delay-differential behaviors in particular. We thus continue work of, for instance, Fliess, Glüsing-Lürssen, Mounier, Rocha, Vettori, Willems, Yamamoto, Zampieri of the last twenty-five years. The signal space of these behaviors is the space $\mathcal{E}$ of complex-valued smooth functions on the real line with its standard Fréchet topology on which the dual space $\mathcal{E}'$ with the strong topology of distributions with compact support acts by a variant of the convolution product. Also via convolution $\mathcal{E}'$ is a commutative domain and topological algebra and ${}_{\mathcal{E}'}\mathcal{E}$ is a module. By definition *generalized behaviors* (gen. beh.) are closed, translation-invariant subspaces or, equivalently, closed $\mathcal{E}'$-submodules $\mathcal{B}$ of some $\mathcal{E}^\ell$, $\ell \geq 0$, that, by duality or orthogonality, are in one-one correspondence with closed submodules U of $\mathcal{E}'^{1\times\ell}$ or with their finitely generated (f.g.) Hausdorff factor modules $M := \mathcal{E}'^{1\times\ell}/U$. A gen. beh. is called a *behavior* if it can be described by *finitely* many convolution equations as usual. The torsion elements of $\mathcal{E}$, i.e., the functions which are annihilated by some nonzero distribution in $\mathcal{E}'$, are called *mean-periodic functions* and were studied by analysts like Delsarte (1935), Schwartz (1947), Kahane (1953), Ehrenpreis (1955, 1960) and Berenstein and Taylor (1980). Their results are significant for the study of convolution behaviors. Typical new results are the following: (i) The $\mathcal{E}'$-submodule $\mathcal{PE}$ of $\mathcal{E}$ of polynomial-exponential functions is injective and thus permits (Willems') elimination procedure for the polynomial-exponential part of gen. beh. whereas ${}_{\mathcal{E}'}\mathcal{E}$ is not injective. (ii) We construct all *autonomous* gen. beh.. (iii) We show that many gen. beh., in particular all autonomous ones, are indeed behaviors, but cannot presently decide whether even all gen. beh. are behaviors. Schwartz' seminal result (1947) in this context was that every closed, translation-invariant subspace of $\mathcal{E}$ is a behavior and indeed described by two equations. (iv) We construct and investigate input/output decompositions of gen. beh.. The study of convolution behaviors requires methods of *topological algebra* since the operator domain $\mathcal{E}'$ is not noetherian and therefore the usual purely algebraic methods of one-dimensional systems theory (polynomial matrix or differential operator approach) do not work.**

I. Introduction

This paper is a survey of a larger journal paper in preparation on the subject of the title and gives indications of some proofs only. The abstract describes the intention and scope of our work. One-dimensional convolution equations have already a longer history in mathematics and were especially studied in the analytic theory of mean-periodic functions [6], [24], [18], [17], [7], [1], [2]. The paper [2] gives an excellent survey of the theory and its history. The study of such equations and their solution spaces from Willems' behavioral point of view is of more recent origin and was pursued, for instance, by Fliess and Mounier [10], [20], Glüsing-Lürssen [11], [12], [13], van Eijndhoven and Habets [14], [9], Rocha and Willems [23], Vettori and Zampieri [26], [27], [28], Yamamoto [29], Yamamoto and Willems [30]. In connection with mean-periodic functions only generalized subbehaviors of $\mathcal{E}$ are considered. The main results of this paper are Theorems 4.1, 4.2, 5.2, 5.4, 6.2, 6.5.

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II. Basic data

We give the basic data in more detail. We quote our papers [4], [5] for several results from the literature that we especially collected there and need here again. Let $\mathcal{E} := \mathrm{C}^\infty(\mathbb{R},\mathbb{C})$ be the space of smooth complex-valued functions $w(t)$ of the real variable t, interpreted as time. According to [25, pp.88-90] this space has the Fréchet topology of compact convergence of the functions and their derivatives. Its topological dual $\mathcal{E}'$ of continuous linear functions $T : \mathcal{E} \to \mathbb{C}$ is the space of distributions with compact support. The canonical bilinear form

$$< -,- >: \mathcal{E}' \times \mathcal{E} \to \mathbb{C},\ < T, w >:= T(w), \tag{1}$$

is non-degenerate, i.e., $< T,- >,\ T \in \mathcal{E}'$, and $< -,w >$, $w \in \mathcal{E}$, are injective. The weak locally convex Hausdorff topologies on $\mathcal{E}'$ and $\mathcal{E}$ are the least ones for which the latter linear functions are continuous and then each of the two spaces is the weak dual of the other [3, §II.6]. The strong topology [25, p. 89] on $\mathcal{E}'$ of uniform convergence on bounded sets of $\mathcal{E}$ makes $\mathcal{E}$ reflexive, i.e., the Gelfand map

$$\mathcal{E} \to \mathcal{E}'', w \mapsto (T \mapsto < T, w >)\,, \tag{2}$$

is a topological isomorphism. The closed subspaces with respect to the weak and strong topologies coincide [3, §II.6.3, Cor. 3]. The forms and topologies are canonically extended to finite powers $\mathcal{E}'^{1\times\ell}$ (rows), $\ell \geq 0$, and $\mathcal{E}^\ell$ (columns). If $U \subseteq \mathcal{E}'^{1\times\ell}$ resp. $\mathcal{B} \subseteq \mathcal{E}^\ell$ are subspaces their closures are denoted by $\mathrm{cl}_E(U)$ resp. $\mathrm{cl}_E(\mathcal{B})$. Their *polar subspaces* are

$$\begin{gathered} U^o := \left\{w \in \mathcal{E}^\ell;\ < U, w >= 0\right\},\\ \mathcal{B}^o := \left\{\xi \in \mathcal{E}'^{1\times\ell};\ < \xi, \mathcal{B} >= 0\right\},\ \text{hence}\\ U^o = U^{ooo},\ \mathcal{B}^o = \mathcal{B}^{ooo}. \end{gathered} \tag{3}$$

The important *bipolar theorem* [3, Thm. II.6.1] implies

$$\mathrm{cl}_E(U) = U^{oo} \text{ and } \mathrm{cl}_E(\mathcal{B}) = \mathcal{B}^{oo}. \tag{4}$$

For $S, T \in \mathcal{E}'$ and $w \in \mathcal{E}$ the *convolution product* on $\mathcal{E}'$ is defined by

$$ST := S * T,\ < S * T, w >=< S_s, < T_t, w(s + t) >> . \tag{5}$$

ISBN: 978-90-367-6321-9

This product makes $\mathcal{E}'$ a commutative (integral) domain and a topological algebra with respect to the strong topology [25, Ch. VI, Thms. IV, VII, XIV]. In deviation from the standard stronger terminology we call a $\mathbb{C}$-space M a *topological $\mathcal{E}'$-module* if it has the structure of an $\mathcal{E}'$-module and of a locally convex Hausdorff space and if all multiplications

$$\mathcal{E}' \to M,\ T \mapsto Tx,\ x \in M, \tag{6}$$

are continuous, for instance $\mathcal{E}'$. Then $\mathcal{E}$ is a topological $\mathcal{E}'$-module [25, Thm. VI.V] via the action $T \circ w$, defined by

$$\begin{aligned} &< T_1, T\circ w >:=< T_1T, w >,\ T_1, T \in \mathcal{E}', w \in \mathcal{E}. \text{ Then} \\ &T \circ w = \check{T} * w \text{ with } < \check{T}_t, w(t) >:=< T_t, w(-t) > . \end{aligned} \tag{7}$$

This module structure is important since it comprises many special cases of engineering significance, for instance

$$\begin{aligned} &(\delta_h \circ w)(t) = w(t+h),\ h \in \mathbb{R}, < \delta_h, w >:= w(h), \\ &((\delta_1 - \delta_0)\circ w)(t) = w(t+1) - w(t), \\ &\delta := \delta_0,\ -\delta' \circ w = \delta' * w = w', \\ &(T \circ w)(t) = (\check{T} * w)(t) = \int_{-\infty}^{\infty} T(x-t)w(x)dx \text{ for } T \in \\ &\mathcal{L}_0^1(\mathbb{R},\mathbb{C}) := \{\text{integrable functions with compact support}\} . \end{aligned} \tag{8}$$

Thus convolution equations comprise differential, delay-differential and integral equations. We identify

$$s := \frac{d}{dt} = -\delta',\ \mathbb{C}[s] = \mathbb{C}[\frac{d}{dt}] = \mathbb{C}[-\delta'] = \oplus_{k=0}^{\infty}\mathbb{C}\delta^{(k)} \subset \mathcal{E}', \tag{9}$$

and thus differential operators with constant coefficients belong to $\mathcal{E}'$. As usual the scalar multiplication $T \circ w$ is extended to matrices via

$$R \circ w := \left(\sum_{j=1}^{\ell} R_{ij} \circ w_j\right)_{1\le i\le k},\ R \in \mathcal{E}'^{k\times\ell},\ w \in \mathcal{E}^{\ell}. \tag{10}$$

Convolution behaviors are defined with $\mathcal{E}'$ as ring of operators and ${}_{\mathcal{E}'}\mathcal{E}$ as signal module. For $\mathcal{E}'$-submodules $U \subseteq \mathcal{E}'^{1\times\ell}$ resp. $\mathcal{B} \subseteq \mathcal{E}^{\ell}$ we define the orthogonal submodules [28, §2]

$$\begin{aligned} &U^{\perp} := \left\{w \in \mathcal{E}^{\ell};\ U \circ w = 0\right\}, \\ &\mathcal{B}^{\perp} := \left\{\xi \in \mathcal{E}'^{1\times\ell};\ \xi \circ \mathcal{B} = 0\right\} \text{ and obtain [4, Thm. 2.23]} \\ &U^{\perp} = U^{o},\ \mathcal{B}^{\perp} = \mathcal{B}^{o}, \\ &\mathrm{cl}_E(U) = U^{oo} = U^{\perp\perp},\ \mathrm{cl}_E(\mathcal{B}) = \mathcal{B}^{oo} = \mathcal{B}^{\perp\perp}, \\ &\text{especially } U^{\perp} = 0 \iff \mathrm{cl}_E(U) = \mathcal{E}'^{1\times\ell}. \end{aligned} \tag{11}$$

A closed subspace $\mathcal{B} \subseteq \mathcal{E}^{\ell}$ is called a *generalized convolution or ${}_{\mathcal{E}'}\mathcal{E}$-behavior* (gen. beh.) if it is translation-invariant, i.e., if $\delta_h \circ \mathcal{B} = \mathcal{B}$ for all $h \in \mathbb{R}$. According to [25, (VI,3;16)] this implies that $T \circ \mathcal{B} = \check{T} * \mathcal{B} \subseteq \mathcal{B}$ is invariant under all $T \in \mathcal{E}'$, i.e., is an $\mathcal{E}'$-submodule. Equation (11) shows that gen. beh. are of the form $\mathcal{B} = U^{\perp},\ U := \mathcal{B}^{\perp}$, and thus in one-one-correspondence with closed submodules of $\mathcal{E}'^{1\times\ell}$. The gen. beh. $\mathcal{B}$ is called a *behavior* if $\mathcal{B}^{\perp}$ has a dense and finitely generated (f.g.) submodule $U_1 = \mathcal{E}'^{1\times k}R,\ R \in \mathcal{E}'^{k\times\ell}$. This signifies that

$$\begin{aligned} &\mathcal{B}^{\perp} = \mathrm{cl}_E(U_1) = U_1^{\perp\perp} \text{ or} \\ &\mathcal{B} = \mathcal{B}^{\perp\perp} = U_1^{\perp} = \left\{w \in \mathcal{E}^{\ell};\ R \circ w = 0\right\}. \end{aligned} \tag{12}$$

This is Willems' *kernel representation* with the generalization that $R \circ w = 0$ is a linear system of convolution equations instead of the customary differential equations. These behaviors were already studied in the papers [27], [28], [13] by Glüsing-Lürssen, Vettori and Zampieri and [9] by Eijndhoven and Habets. A main result [24, Thm. 13 on p. 914] of Schwartz' seminal paper shows that every gen. beh. $\mathcal{B} \subseteq \mathcal{E}$ is a behavior and that indeed the closed ideal $\mathcal{B}^{\perp} \subseteq \mathcal{E}'$ contains a dense ideal with *two* generators. Below we extend this result to all autonomous gen. beh. and to large classes of nonautonomous ones. Schwartz' result also shows that a convolution behavior cannot in general be described by a matrix of maximal row rank. For any submodule $U \subseteq \mathcal{E}'^{1\times\ell}$ with f.g. topological factor module $M := \mathcal{E}'^{1\times\ell}/U$ (with the identification topology) the canonical Malgrange isomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{E}'}(\mathcal{E}'^{1\times\ell}/U, \mathcal{E}) = \mathrm{Hom}_{\mathcal{E}'}(\mathcal{E}'^{1\times\ell}/\mathrm{cl}_E(U), \mathcal{E}) \cong \\ &\mathcal{B} := U^{\perp} = \mathrm{cl}_E(U)^{\perp},\ \varphi \leftrightarrow w,\ \varphi(\xi + U) = \xi \circ w, \end{aligned} \tag{13}$$

holds. The left exact functor $\mathrm{Hom}_{\mathcal{E}'}(-,\mathcal{E})$ on f.g. $\mathcal{E}'$-modules thus is an equally important tool for convolution behaviors as it is for differential behaviors.

The subspace $\mathcal{PE} \subseteq \mathcal{E}$ of polynomial-exponential functions is the torsion submodule of $\mathcal{E}$ as $\mathbb{C}[s] = \mathbb{C}[\frac{d}{dt}]$-module, i.e.,

$$\begin{aligned} &\mathcal{PE} := \{w \in \mathcal{E};\ \exists 0 \neq T \in \mathbb{C}[s] \text{ with } T \circ w = 0\} = \\ &\bigoplus_{z\in\mathbb{C}} \mathcal{PE}(z),\ \mathcal{PE}(z) := \bigcup_{k\in\mathbb{N}} \mathrm{ann}_{\mathcal{E}}((s-z)^k) \\ &\mathrm{ann}_{\mathcal{E}}((s-z)^k) := \left\{w \in \mathcal{E};\ (s-z)^k \circ w = 0\right\} = \\ &\mathbb{C}[t]_{<k}e^{zt} = \bigoplus_{i=0}^{k-1} \mathbb{C}e_{z,i},\ \mathbb{C}[t]_{<k} := \bigoplus_{i=0}^{k-1} \mathbb{C}t^i, \\ &e_{z,i} := \frac{t^i}{i!}e^{zt},\ (s-z)^j \circ e_{z,i} = \begin{cases} e_{z,i-j} & \text{if } j \le i \\ 0 & \text{if } j > i \end{cases} \end{aligned} \tag{14}$$

(cf. [9, (7)]). Since the ring $\mathcal{E}'$ is commutative the annihilators $\mathrm{ann}_{\mathcal{E}}((s-z)^k)$ are $\mathbb{C}$-finite-dimensional (f.d.) $\mathcal{E}'$-submodules of $\mathcal{E}$ and indeed behaviors and therefore also the $\mathcal{PE}(z) = \mathbb{C}[t]e^{zt}$ and $\mathcal{PE} = \oplus_{z\in\mathbb{C}}\mathcal{PE}(z)$ are $\mathcal{E}'$-submodules of $\mathcal{E}$, but not closed. For any $U \subseteq \mathcal{E}'^{1\times\ell}$ and gen. beh. $\mathcal{B} := U^{\perp} \subseteq \mathcal{E}^{\ell}$ the isomorphism (13) induces the isomorphism

$$\mathrm{Hom}_{\mathcal{E}'}(\mathcal{E}'^{1\times\ell}/U, \mathcal{PE}) \cong \mathcal{B} \bigcap \mathcal{PE}^{\ell} \tag{15}$$

for the polynomial-exponential part of $\mathcal{B}$. The following essential result was the fundamental result Thm. 6 of [24] for $\mathcal{B} \subseteq \mathcal{E}$ and was extended to arbitrary gen. beh. in [13, (3.3)] by means of [22].

Result 2.1: The polynomial-exponential part $\mathcal{B} \bigcap \mathcal{PE}^{\ell}$ of a gen. beh. $\mathcal{B} \subseteq \mathcal{E}^{\ell}$ is dense in $\mathcal{B}$, hence

$$\mathcal{B}^{\perp} = \mathcal{B}^{o} = \left(\mathcal{B} \bigcap \mathcal{PE}^{\ell}\right)^{o} = \left(\mathcal{B} \bigcap \mathcal{PE}^{\ell}\right)^{\perp}.$$

This is false in higher dimensions and therefore the theory of this paper cannot be extended to higher dimensions.

III. The use of complex variables

The following considerations were used for multidimensional analytic behaviors over Stein algebras in [5, §3, §5]. Let $\mathcal{O} := \mathcal{O}(\mathbb{C})$ denote the $\mathbb{C}$-algebra of entire functions (everywhere convergent power series) in the complex variable $s \in \mathbb{C}$. It is a Fréchet algebra with the topology of compact convergence. The *Laplace transform* [15, Thm. 7.1.14]

$$\mathcal{L} : \mathcal{E}' \to \mathcal{O},\ T \mapsto \widehat{T},\ \widehat{T}(s) := < T_t, e^{ts} >, \tag{16}$$

is an injective algebra homomorphism where $\mathcal{E}'$ resp. $\mathcal{O}$ are furnished with the convolution resp. with the pointwise multiplication. For $a > 0$, $p \in \mathbb{N}$ and $F \in \mathcal{O}$ define

$$\begin{gathered}\|F\|_{a,p} := \sup_{z\in\mathbb{C}} |F(z)|(1+|z|)^{-p} e^{-a|\Re(z)|},\\ \mathcal{O}_{a,p} := \{F \in \mathcal{O};\ \|F\|_{a,p} < \infty\} = \\ \left\{F \in \mathcal{O};\ \exists C > 0 \forall z \in \mathbb{C}:\ |F(z)| \le C(1+|z|)^p e^{a|\Re(z)|}\right\},\\ PWS := \bigcup_{a>0, p\in\mathbb{N}} \mathcal{O}_{a,p}.\end{gathered} \tag{17}$$

Then $\mathcal{O}_{a,p}$ is a Banach space with the norm $\| - \|_{a,p}$ and a convergent sequence in $\mathcal{O}_{a,p}$ is compactly convergent in particular. The Paley-Wiener-Schwartz theorem [15, Thm. 7.3.1] implies the algebra isomorphism

$$\begin{gathered}\mathcal{L} : \mathcal{E}' \cong PWS,\ T \mapsto \widehat{T},\\ \mathrm{supp}(T) \subseteq [-a,a] := \{t \in \mathbb{R};\ |t| \le a\} \iff \widehat{T} \in \bigcup_{p\in\mathbb{N}} \mathcal{O}_{a,p}\end{gathered} \tag{18}$$

where $\mathrm{supp}(T)$ is the support of T. This theorem was an important tool already in [27] and [13].
A sequence $(T_n)_{n\in\mathbb{N}} \in \mathcal{E}'^{\mathbb{N}}$ converges weakly (and then also strongly) to $T \in \mathcal{E}'$ if and only if $\lim_{n\to\infty} < T_n, w > = < T, w >$ for all $w \in \mathcal{E}$. According to [8, Lemma 5.17, p.155], [2, p.211] this is equivalent to the existence of

$$\begin{gathered}a > 0, p \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}:\ \widehat{T_n}, \widehat{T} \in \mathcal{O}_{a,p} \text{ and}\\ \lim_{n\to\infty} \|\widehat{T_n} - \widehat{T}\|_{a,p} = 0.\end{gathered} \tag{19}$$

This enables the construction of distributions with support in $[-a,a]$ by limit processes in the Banach space $\mathcal{O}_{a,p}$ and is very important for the present paper. In the sequel we identify

$$\begin{gathered}\mathcal{E}' = PWS \subset \mathcal{O}, T = \widehat{T},\ T(s) := \widehat{T}(s) = < T_t, e^{ts} >,\\ \widehat{\delta_h} = e^{hs}, h \in \mathbb{R}, -\delta' = \widehat{-\delta'} = s.\end{gathered} \tag{20}$$

This identification coincides with that in (9), i.e., s is identified with $\frac{d}{dt} = -\delta' \in \mathcal{E}'$ and with $s = \widehat{-\delta'}(s) \in \mathcal{O}$. Hörmander uses the Fourier-Laplace transform $T \mapsto < T_t, e^{-its} > = \widehat{T}(-is)$ instead of $\mathcal{L}$. This has the consequence only that various constants have to be changed and that $s = i\frac{d}{dt}$.
The algebra $\mathcal{O}$ is a *Stein algebra* [5, §6] and has many valuable algebraic properties. An ideal $\mathfrak{b}$ of $\mathcal{O}$ is closed if and only if it is f.g. and then even a principal ideal, hence $\mathcal{O}$ is a Bézout domain and therefore also coherent [5, Def. and Cor. 2.3]. More generally, submodules $U \subseteq \mathcal{O}^{1\times\ell}$ are closed if and only if they are f.g. and then indeed free. This implies that the f.g. *Stein modules* $\mathcal{O}^{1\times\ell}/U$, U closed, [5, Result 6.1, Thm. 6.2] are precisely the finitely presented or coherent ones. The principal ideals $\mathcal{E}'f,\ 0 \ne f \in \mathcal{E}'$, are not closed in general. Indeed, the following equivalences hold [16, Thms. 16.3.10, 16.5.7, Def. 16.3.12]:

$$\mathcal{E}'f \text{ is closed } \iff \mathcal{E}'f = \mathcal{E}' \bigcap \mathcal{O}f \iff f \circ \mathcal{E} = \mathcal{E}. \tag{21}$$

Then f is called *invertible*. Smooth functions with compact support, considered as distributions, are never invertible whereas all differential operators $f \in \mathbb{C}[s] \subset \mathcal{E}'$ are invertible due to the standard result $f \circ \mathcal{E} = \mathcal{E}$. This applies especially to the prime powers $(s-z)^k = (-\delta' - z)^k$ and was widely exploited in the seminal paper [24]. For every $z \in \mathbb{C}$ the ring $\mathcal{O}_z := \mathbb{C} < s - z >$ is the ring of *locally convergent* power series at z. It is a discrete valuation ring (DVR), i.e., a principal ideal domain with the unique (up to association) prime element $s - z$ and unique maximal ideal $\mathfrak{m}_z = \mathcal{O}_z(s-z)$. The identity theorem implies the inclusion $\mathcal{O} \subset \mathcal{O}_z,\ f = f_z := \sum_{n=0}^{\infty} f^{(n)}(z)(n!)^{-1}(s-z)^n$. The closure of an $\mathcal{O}$-submodule $V \subseteq \mathcal{O}^{1\times\ell}$ is

$$\mathrm{cl}_O(V) = \left\{w \in \mathcal{O}^{1\times\ell};\ \forall z \in \mathbb{C}:\ w_z \in \mathcal{O}_z V \subseteq \mathcal{O}_z^{1\times\ell}\right\} \tag{22}$$

[5, Thm. 6.1,(8)]. Further maximal ideals are

$$\begin{gathered}\mathfrak{m}_z \supset \mathfrak{m}_O(z) := \mathcal{O} \bigcap \mathfrak{m}_z = \{f \in \mathcal{O};\ f(z) = 0\} = \\ \mathcal{O}(s-z) \supset \mathfrak{m}_E(z) := \mathcal{E}' \bigcap \mathfrak{m}_O(z) = \mathcal{E}'(s-z) \supset \\ \mathfrak{m}_P(z) := \mathbb{C}[s] \bigcap \mathfrak{m}_O(z) = \mathbb{C}[s](s-z).\end{gathered} \tag{23}$$

The equation $\mathfrak{m}_E(z) = \mathcal{E}'(s-z)$ follows from the invertibility of $s - z$. The maximal ideals induce local rings

$$\mathcal{O}_z \supset \mathcal{O}_{\mathfrak{m}_O(z)} \supset \mathcal{E}'_{\mathfrak{m}_E(z)} \supset \mathbb{C}[s]_{\mathfrak{m}_P(z)}, \tag{24}$$

all of which are DVRs with the unique prime $s - z$, up to units. For $\mathcal{E}'_{\mathfrak{m}_E(z)}$ this is again a consequence of the invertibility of $s - z$. Equations (23) and (24), in turn, imply the identifications or canonical isomorphims ($k \ge 1$)

$$\begin{gathered}\mathcal{O}_z/\mathcal{O}_z(s-z)^k = \mathcal{O}/\mathcal{O}(s-z)^k = \\ \mathcal{E}'_{\mathfrak{m}_E(z)}/\mathfrak{m}_E(z)^k = \mathcal{E}'/\mathcal{E}'(s-z)^k = \mathbb{C}[s]/\mathbb{C}[s](s-z)^k = \\ \oplus_{i=0}^{k-1} \mathbb{C}\overline{(s-z)^k} \cong \mathbb{C}[s-z]_{<k}\end{gathered} \tag{25}$$

where $\overline{f}$ denotes the residue class. These factor algebras have $\mathbb{C}$-dimension k. Since $\mathcal{E}'(s-z)^k$ is closed the algebra $\mathcal{E}'/\mathcal{E}'(s-z)^k$ is Hausdorff with the coinduced factor topology and this coincides with the topology as f.d. $\mathbb{C}$-space. The identifications (25) finally induce the identification of the completions of these DRVs [5, (94)], viz.

$$\widehat{\mathcal{O}_z} = \widehat{\mathcal{O}_{\mathfrak{m}_O(z)}} = \widehat{\mathcal{E}'_{\mathfrak{m}_E(z)}} = \widehat{\mathbb{C}[s]_{\mathfrak{m}_P(z)}} = \mathbb{C}[[s-z]] \tag{26}$$

where $\mathbb{C}[[s-z]]$ is the DVR of *formal* power series in $s - z$. Since the inclusion $A \subset \widehat{A}$ of a local noetherian ring into its completion is faithfully flat, i.e., the functor $M \mapsto \widehat{A} \otimes_A$

M preserves and reflects exact sequences of A-modules, the preceding considerations imply that all inclusions of DVRs

$$\mathbb{C}[[s-z]] \supset \mathcal{O}_z \supset \mathcal{O}_{\mathfrak{m}_O(z)} \supset \mathcal{E}'_{\mathfrak{m}_E(z)} \supset \mathbb{C}[s]_{\mathfrak{m}_P(z)}, \quad (27)$$

are faithfully flat.

Let $\mathfrak{b} = \mathcal{O}f$ be any nonzero closed ideal of $\mathcal{O}$. Its associated analytic variety is the the countable discrete set of zeros of f or $\mathfrak{b}$, viz.

$$V_{\mathbb{C}}(f) := V_{\mathbb{C}}(\mathfrak{b}) := \{z \in \mathbb{C};\ \forall g \in \mathfrak{b}:\ g(z) = 0\}. \quad (28)$$

Recall that $\mathfrak{m}_O(z)^k = \mathcal{O}(s-z)^k \subset \mathfrak{m}_z^k = \mathcal{O}_z(s-z)^k$. The *multiplicity* $k(z) := \mathrm{mult}(f,z)$ of f at z is defined by the equivalent conditions

$$\begin{gathered} f \in \mathfrak{m}_O(z)^{k(z)} \setminus \mathfrak{m}_O(z)^{k(z)+1} \iff f \in \mathfrak{m}_z^{k(z)} \setminus \mathfrak{m}_z^{k(z)+1} \\ \iff \exists g \in \mathcal{O} \text{ with } f = g(s-z)^{k(z)} \text{ and } g(z) \neq 0 \iff \\ f^{(i)}(z) = 0 \text{ for } i = 0, \cdots, k(z)-1,\ f^{(k(z))}(z) \neq 0, \end{gathered} \quad (29)$$

hence $V_{\mathbb{C}}(f) = \{z \in \mathbb{C};\ \mathrm{mult}(f,z) \geq 1\}$. The family $\big(V_{\mathbb{C}}(f), (\mathrm{mult}(f,z))_{z \in V_{\mathbb{C}}(f)}\big)$ is called the *cospectrum* [24] or *multiplicity variety* [1, p.117] of $\mathfrak{b}$. The Weierstraß and Mittag-Leffler theorems imply the algebra isomorphism

$$\begin{gathered} \Delta : \mathcal{O}/\mathcal{O}f \cong \prod_{z \in V_{\mathbb{C}}(f)} \mathcal{O}/\mathcal{O}(s-z)^{k(z)},\ k(z) := \mathrm{mult}(f,z), \\ g + \mathcal{O}f \mapsto \Big(g + \mathcal{O}(s-z)^{k(z)}\Big)_{z \in V_{\mathbb{C}}(f)} \\ \text{with } \ker(\Delta) = \mathfrak{b} = \bigcap_{z \in V_{\mathbb{C}}(f)} \mathcal{O}(s-z)^{k(z)} \text{ and} \\ \mathcal{O}/\mathcal{O}(s-z)^{k(z)} \underset{(25)}{=} \mathbb{C}[s]/\mathbb{C}[s](s-z)^{k(z)} = \\ \oplus_{i=0}^{k(z)-1} \mathbb{C}\overline{(s-z)^i} \ni \overline{g} := g + \mathcal{O}f = \sum_{i=0}^{k(z)-1} \frac{g^{(i)}(z)}{i!} \overline{(s-z)^i}. \end{gathered} \quad (30)$$

This implies in particular that the cospectra or multiplicity varieties are in one-one correspondence with the nonzero closed ideals of $\mathcal{O}$. Notice that for finite $V_{\mathbb{C}}(f)$ the isomorphism Δ is a consequence of the *Chinese Remainder Theorem*. There is no simple characterization of the closed ideals $\mathcal{O}f$ or of the multiplicity varieties of distributions $f \in \mathcal{E}'$. Already Schwartz [24, pp. 882-] used the principal parts $\mathrm{pp}_z(f^{-1}) = \sum_{i=1}^{k(z)} a_{z,i}(s-z)^{-i}$ for $z \in V_{\mathbb{C}}(f)$ with $a_{z,i} \in \mathbb{C}$ and $f^{-1} - \mathrm{pp}_z(f^{-1}) \in \mathcal{O}_z$ to construct

$$\begin{gathered} \epsilon_z := f\,\mathrm{pp}_z(f^{-1}) = f(s-z)^{-k(z)} \sum_{i=1}^{k(z)} a_{z,i}(s-z)^{k(z)-i} \in \mathcal{O} \\ \text{with } \epsilon_z \equiv \delta_{z,z'} \mod \mathcal{O}(s-z')^{k(z')} \text{ for } z, z' \in V_{\mathbb{C}}(f) \text{ and} \\ \overline{\epsilon_z} \cdot \overline{\epsilon_{z'}} = \delta_{z,z'}\overline{\epsilon_z} \text{ in } \mathcal{O}/\mathcal{O}f. \text{ If} \\ f \in \mathcal{O}_{a,p} \subset \mathcal{E}' \text{ then also } \epsilon_z = \sum_{i=1}^{k(z)} a_{z,i}(s-z)^{-i} f \in \mathcal{O}_{a,p} \end{gathered} \quad (31)$$

and for all $w \in \mathcal{E}$ the action $\epsilon_z \circ w \in \mathcal{E}$ is defined.

IV. Injectivity, elimination and closure

Elimination in Willems' sense for gen. beh. occurs as follows: Assume

$$\begin{gathered} P \in \mathcal{E}'^{\ell_2 \times \ell_1},\ U_1 \subseteq \mathcal{E}'^{1 \times \ell_1} \text{ and} \\ U_2 := (\circ P)^{-1}(U_1) = \big\{\eta \in \mathcal{E}'^{1 \times \ell_2};\ \eta P \in U_1\big\}. \end{gathered} \quad (32)$$

The U_i induce f.g. modules $M_i := \mathcal{E}'^{1 \times \ell_i}/U_i$, gen. beh.

$$\begin{gathered} \mathcal{B}_i := U_i^{\perp} = \mathrm{cl}_E(U_i)^{\perp} \underset{\text{ident.}}{=} \mathrm{Hom}_{\mathcal{E}'}(M_i, \mathcal{E}) \subseteq \mathcal{E}^{\ell_i} \\ \text{and maps } (\circ P)_{\mathrm{ind}} : \mathcal{E}'^{1 \times \ell_2}/U_2 \to \mathcal{E}'^{1 \times \ell_1}/U_1, \\ P\circ = \mathrm{Hom}((\circ P)_{\mathrm{ind}}, \mathcal{E}) : \mathcal{B}_1 = \mathrm{Hom}_{\mathcal{E}'}(M_1, \mathcal{E}) \to \mathcal{B}_2. \end{gathered} \quad (33)$$

Notice that the U_i are not necessarily closed. The map $(\circ P)_{\mathrm{ind}}$ is, of course, injective. In Thm. 4.1,(2), below we show that $\mathrm{cl}_E(P \circ \mathcal{B}_1) = \mathcal{B}_2$. If $P \circ \mathcal{B}_1$ is a gen. beh. it is obtained by *elimination* in Willems' terminology. Even if U_1 is f.g. U_2 is not necessarily so and therefore the treatment of gen. beh. instead of behaviors only is mandatory.

A module ${}_{\mathcal{E}'}\mathcal{F}$ is injective if and only if the left exact functor $\mathrm{Hom}_{\mathcal{E}'}(-, \mathcal{F})$ is exact or, equivalently, transforms injections into surjections, and an *injective cogenerator* if in addition this functor also reflects exactness or if $\mathrm{Hom}_{\mathcal{E}'}(M, \mathcal{F}) = 0$ implies $M = 0$. If $0 \neq f \in \mathcal{E}' = \mathcal{E}'^{1 \times 1}$ is not invertible the map $\circ f : \mathcal{E}' \to \mathcal{E}'$ is injective, but $f\circ = \mathrm{Hom}(\circ f, \mathcal{E}) : \mathcal{E} \to \mathcal{E}$ is not surjective and therefore $f \circ \mathcal{E}$ is not a gen. beh. and ${}_{\mathcal{E}'}\mathcal{E}$ is not injective and does not admit elimination. But we are now going to show that $\mathcal{PE}$ admits elimination. By [21, Thms. 1.14, 6.6] the module $\mathcal{PE}(z) = \mathbb{C}[t]e^{zt}$ is the minimal injective cogenerator over the local ring $\mathbb{C}[s]_{\mathfrak{m}_P(z)}$ and therefore also over its completion $\widehat{\mathbb{C}[s]_{\mathfrak{m}_P(z)}} = \mathbb{C}[[s-z]]$ [19, Thm. 18.6,(iii)]. Here the action of $\mathbb{C}[s]$ on $\mathcal{PE}(z)$ by differentiation can be canonically extended to $\mathbb{C}[[s-z]]$ since $(s-z)^j \circ \frac{t^i}{i!}e^{zt} = 0$ for $j > i$. This latter *nilpotence of the action* moreover implies that $\mathbb{C}[s]$- and $\mathbb{C}[[s-z]]$-submodules of $\mathcal{PE}(z)$ coincide. The flatness of the inclusion $\mathcal{E}' \subset \mathcal{E}'_{\mathfrak{m}_E(z)}$ and the faithful flatness of the inclusions (27) then furnish the following

Theorem 4.1: Let $\mathcal{B} \subseteq \mathcal{E}^{\ell}$ be any gen. beh..

1) The module $\mathcal{PE}(z) = \mathbb{C}[t]e^{zt}$ is the least injective cogenerator over $\mathcal{E}'_{\mathfrak{m}_E(z)}$. In particular, there is a matrix $R^z \in \mathcal{E}'^{k_z \times \ell}$ that is of maximal row rank k_z and unique up to row equivalence over $\mathcal{E}'_{\mathfrak{m}_E(z)}$ such that
$$\mathcal{B} \bigcap \mathcal{PE}(z)^{\ell} = \big\{w \in \mathcal{PE}(z)^{\ell};\ R^z \circ w = 0\big\}. \quad (34)$$
2) The module $\mathcal{PE} = \bigoplus_{z \in \mathbb{C}} \mathbb{C}[t]e^{zt}$ is injective for f.g. $\mathcal{E}'$-modules, i.e., $\mathrm{Hom}_{\mathcal{E}'}(-, \mathcal{PE})$ preserves exactness of sequences of f.g. modules. For the data from 1) resp. (33) this implies
$$\mathcal{B} \bigcap \mathcal{PE}^{\ell} = \bigoplus_{z \in \mathbb{C}} \big\{w \in \mathcal{PE}(z)^{\ell};\ R^z \circ w = 0\big\} \text{ resp.} \quad (35)$$
$$P \circ (\mathcal{B}_1 \bigcap \mathcal{PE}^{\ell}) = \mathcal{B}_2 \bigcap \mathcal{PE}^{\ell},\ \mathrm{cl}_E(P \circ \mathcal{B}_1) = \mathcal{B}_2. \quad (36)$$
in particular ${}_{\mathcal{E}'}\mathcal{PE}$ admits elimination.

Notice that for arbitrarily chosen matrices $R^z \in \mathcal{E}'^{k_z \times \ell}$ the gen. beh. $\mathcal{B} := \mathrm{cl}_E\left(\bigoplus_{z\in\mathbb{C}}\left\{w \in \mathcal{PE}(z)^\ell;\ R^z \circ w = 0\right\}\right)$ does not satisfy (35). This is in contrast to Thm. 5.2 below.

Result 2.1, Thm. 4.1,(1), and (22) imply the following important result on the connection between algebra and topology.

Theorem 4.2: Let $U \subseteq \mathcal{E}'^{1\times\ell}$ be any $\mathcal{E}'$-submodule.

1) The closure of U in $\mathcal{E}'^{1\times\ell}$ is

$$\mathrm{cl}_E(U) = \mathcal{E}'^{1\times\ell} \bigcap \bigcap_{z\in\mathbb{C}} U_{\mathfrak{m}_E(z)},\ \text{hence also}$$
$$\mathrm{cl}_E(U)_{\mathfrak{m}_E(z)} = U_{\mathfrak{m}_E(z)} \text{ and } \left(\mathcal{E}'^{1\times\ell}/U\right)_{\mathfrak{m}_E(z)} = \left(\mathcal{E}'^{1\times\ell}/\,\mathrm{cl}_E(U)\right)_{\mathfrak{m}_E(z)} = \mathcal{E}'^{1\times\ell}_{\mathfrak{m}_E(z)}/U_{\mathfrak{m}_E(z)}. \tag{37}$$

2) The closures in $\mathcal{E}'^{1\times\ell}$ and in $\mathcal{O}^{1\times\ell}$ are related by

$$\mathrm{cl}_O(U) = \mathrm{cl}_O(\mathcal{O}U) \text{ and } \mathrm{cl}_E(U) = \mathcal{E}'^{1\times\ell}\bigcap \mathrm{cl}_O(U),$$
$$\text{hence } \mathcal{E}'^{1\times\ell}/\,\mathrm{cl}_E(U) \underset{\text{ident.}}{\subset} \mathcal{O}^{1\times\ell}/\,\mathrm{cl}_O(U) \tag{38}$$

hold. If $U = \sum_{i=1}^m \mathcal{E}'u_i$ is f.g. then $\mathrm{cl}_O(U) = \sum_{i=1}^m \mathcal{O}'u_i$, especially $\mathrm{cl}_O(\mathcal{E}'f) = \mathcal{O}f$ and $\mathrm{cl}_E(\mathcal{E}'f) = \mathcal{E}'\bigcap \mathcal{O}f$ for $f \in \mathcal{E}'$ (cf. (21)).

This theorem enables the study of $\mathcal{E}'^{1\times\ell}/\,\mathrm{cl}_E(U)$ by means of the f.g. modules $\mathcal{E}'^{1\times\ell}_{\mathfrak{m}(z)}/U_{\mathfrak{m}(z)}$ over the DVRs $\mathcal{E}'_{\mathfrak{m}(z)}$ with their well-known simple structure.

Example 4.3: We use (14) and illustrate Thm. 4.1,(1), by solving the equation $f \circ y = u =: e_{z,k} = t^k/(k!)^{-1}e^{zt}$ for $y \in \mathbb{C}[t]e^{zt}$ where $0 \neq f = (s-z)^j g \in \mathcal{E}'$ and $g(z) \neq 0$. Recall that over the principal ideal domains $\mathcal{E}'_{\mathfrak{m}_E(z)} \subset \mathbb{C}[[s-z]]$ injectivity and divisibility of a module coincide. The condition $g(z) \neq 0$ implies $g^{-1} = \sum_{i=0}^\infty a_i(s-z)^i \in \mathcal{O}_z = \mathbb{C} < s-z >$ with $a_i \in \mathbb{C}$ and $a_0 = g(z)^{-1}$. Define

$$y := g^{-1}\circ e_{z,k+j} = \sum_{i=0}^\infty a_i(s-z)^i \circ e_{z,k+j} = \sum_{i=0}^{k+j} a_i e_{z,k+j-i} \in \mathbb{C}[t]e^{zt} \Longrightarrow f\circ y = (s-z)^j \circ g\circ g^{-1}\circ e_{z,k+j} = (s-z)^j \circ e_{z,k+j} = e_{z,k} = u. \tag{39}$$

V. AUTONOMOUS BEHAVIORS

Let $\mathcal{D}' := \mathcal{D}'(\mathbb{R},\mathbb{C})$ denote the space of distributions and $\mathcal{D}'_+$ its subspace of distributions with left bounded support. The convolution product makes $\mathcal{D}'_+$ an integral domain [25, Thm. VI.XIV] and $\mathcal{E}'$ is a subalgebra of $\mathcal{D}'_+$. The subspace $\mathcal{E}_+ = \mathcal{D}'_+ \bigcap \mathcal{E}$ of smooth functions with left bounded support is an ideal of $\mathcal{D}'_+$ and therefore a torsionfree $\mathcal{D}'_+$-module. We conclude that $\mathcal{E}_+$ is a torsionfree $\mathcal{E}'$-submodule of ${}_{\mathcal{E}'}\mathcal{E}$.

Theorem and Definition 5.1: (cf. [13, Prop. 3.10] [28, Def. 2, Prop. 4]) For a not necessarily closed submodule $U \subseteq \mathcal{E}'^{1\times\ell}$ and the associated f.g. $\mathcal{E}'$-module $M := \mathcal{E}'^{1\times\ell}/U$ and gen. beh. $\mathcal{B} := U^\perp$ the following properties are equivalent:

1) M is a torsion module, i.e., there is a nonzero $f \in \mathcal{E}'$ with $fM = 0$ or $\mathcal{E}'^{1\times\ell}f \subseteq U$.
2) $\mathcal{B}\bigcap \mathcal{E}'^\ell_+ = 0$. i.e., the past of a trajectory of $\mathcal{B}$ determines its future.
3) There is a nonzero $g \in \mathcal{E}'$ with

$$g\circ\mathcal{B} = 0 \text{ or } \mathcal{B} \subseteq (\mathcal{E}'^{1\times\ell}g)^\perp = \left\{w \in \mathcal{E}^\ell;\ g\circ w = 0\right\}. \tag{40}$$

If U is closed one may choose $f = g$ in (1) and (3).

Under these conditions $\mathcal{B}$ is called *autonomous*.

There is no structure theorem for arbitrary closed $\mathcal{E}'$-submodules $U \subseteq \mathcal{E}'^{1\times\ell}$ or gen. beh. $\mathcal{B} = U^\perp$, but we *construct* all *autonomous* gen. beh. in the next theorem. So assume

$$U \subseteq \mathcal{E}'^{1\times\ell},\ M := \mathcal{E}'^{1\times\ell}/U,\ 0 \neq f \in \mathcal{E}',\quad fM = 0 \text{ or } \mathcal{E}'^{1\times\ell}f \subseteq U,\ \mathcal{B} := U^\perp \subseteq (\mathcal{E}'^{1\times\ell}f)^\perp. \tag{41}$$

We use data from (28), (29) and (31) for f. Recall from Thm. 4.1 that $\mathcal{PE}(z),\ z\in\mathbb{C}$, is the least injective cogenerator over the DVR $\mathcal{E}'_{\mathfrak{m}_E(z)}$ and that

$$\mathcal{B}\bigcap \mathcal{PE}^\ell = \bigoplus_{z\in\mathbb{C}}\left(\mathcal{B}\bigcap\mathcal{PE}(z)^\ell\right),\ \mathcal{PE}(z) = \mathbb{C}[t]e^{zt},$$
$$\mathcal{B}\bigcap \mathcal{PE}(z)^\ell \underset{\text{ident.}}{=} \mathrm{Hom}_{\mathcal{E}'}(M,\mathcal{PE}(z)) \underset{\text{ident.}}{=} \mathrm{Hom}_{\mathcal{E}'_{\mathfrak{m}_E(z)}}(M_{\mathfrak{m}_E(z)},\mathcal{PE}(z)) \text{ where}$$
$$M_{\mathfrak{m}_E(z)} = \mathcal{E}'^{1\times\ell}_{\mathfrak{m}_E(z)}/U_{\mathfrak{m}_E(z)},\ f\mathcal{E}'^{1\times\ell}_{\mathfrak{m}_E(z)} \subseteq U_{\mathfrak{m}_E(z)}. \tag{42}$$

Recall

$$k(z) := \mathrm{mult}(f,z),\ f = (s-z)^{k(z)}f^z,\ f^z \in \mathcal{E}',\quad f^z(z) \neq 0, z \in V_{\mathbb{C}}(f) \iff k(z) > 0. \tag{43}$$

The condition $f^z(z) \neq 0$ signifies that f^z is a unit in $\mathcal{E}'_{\mathfrak{m}_E(z)}$. If $f(z) \neq 0$ then $f = f^z$ itself is a unit in $\mathcal{E}'_{\mathfrak{m}_E(z)}$ and hence

$$\mathcal{E}'^{1\times\ell}_{\mathfrak{m}_E(z)} = f\mathcal{E}'^{1\times\ell}_{\mathfrak{m}_E(z)} = U_{\mathfrak{m}_E(z)},\ M_{\mathfrak{m}_E(z)} = 0,\quad \mathcal{B}\bigcap\mathcal{PE}(z)^\ell = 0\ (f(z)\neq 0,\ z \notin V_{\mathbb{C}}(f)). \tag{44}$$

For $z \in V_{\mathbb{C}}(f)$ recall (25) and define

$$U(z) := U + \mathcal{E}'^{1\times\ell}(s-z)^{k(z)} \Longrightarrow U(z)/\mathcal{E}'^{1\times\ell}(s-z)^{k(z)} \subseteq \mathcal{E}'^{1\times\ell}/\mathcal{E}'^{1\times\ell}(s-z)^{k(z)} = \underset{(25)}{=} \mathbb{C}[s]^{1\times\ell}/\mathbb{C}[s]^{1\times\ell}(s-z)^{k(z)}. \tag{45}$$

This implies that $U(z)$ is closed and that there is a matrix $R^z \in \mathbb{C}[s]^{\ell\times\ell}$ such that

$$\deg_s(R^z) < k(z),\ U(z) = \mathcal{E}'^{1\times\ell}R^z + \mathcal{E}'^{1\times\ell}(s-z)^{k(z)},$$
$$U(z)^\perp = \left\{w \in \mathbb{C}[t]^\ell_{<k(z)}e^{zt};\ R^z\circ w = 0\right\} \subseteq \left(\mathcal{E}'^{1\times\ell}(s-z)^{k(z)}\right)^\perp = \mathbb{C}[t]^\ell_{<k(z)}e^{zt}. \tag{46}$$

Localization w.r.t. $\mathfrak{m}_E(z)$ furnishes

$$\mathcal{E}'^{1\times\ell}_{\mathfrak{m}_E(z)}f = \mathcal{E}'^{1\times\ell}_{\mathfrak{m}_E(z)}(s-z)^{k(z)} \subseteq U_{\mathfrak{m}_E(z)},\ \text{hence}$$
$$U_{\mathfrak{m}_E(z)} = U_{\mathfrak{m}_E(z)} + \mathcal{E}'^{1\times\ell}_{\mathfrak{m}_E(z)}(s-z)^{k(z)} = U(z)_{\mathfrak{m}_E(z)} \text{ and}$$
$$U(z)^\perp = U(z)^\perp\bigcap\mathcal{PE}(z)^\ell = \mathcal{B}\bigcap\mathcal{PE}(z)^\ell. \tag{47}$$

Theorem 5.2: Let $f \in \mathcal{E}' \subset \mathcal{O}$ be nonzero with the data from (28), (29) and (31).

1) If $\mathcal{B} \subseteq \mathcal{E}^\ell$ is any autonomous gen. beh. with $f \circ \mathcal{B} = 0$ then there are polynomial matrices $R^z \in \mathbb{C}[s]^{\ell\times\ell}$, $z \in V_\mathbb{C}(f)$, of degree $\deg_s(R^z) < k(z)$ such that
$$\begin{gathered}\mathcal{B}\bigcap\mathcal{PE}^\ell = \oplus_{z\in V_\mathbb{C}(f)} U(z)^\perp \text{ where}\\ U(z) := \mathcal{E}'^{1\times\ell}R^z + \mathcal{E}'^{1\times\ell}(s-z)^{k(z)},\\ \mathcal{B}\bigcap\mathcal{PE}(z)^\ell = U(z)^\perp = \\ \left\{w \in \mathbb{C}[t]^\ell_{<k(z)}e^{zt};\ R^z \circ w = 0\right\}.\end{gathered} \tag{48}$$

2) Conversely, choose arbitrary polynomial matrices R^z, $z \in V_\mathbb{C}(f)$, as in item 1) and define
$$\begin{gathered}U(z) := \mathcal{E}'^{1\times\ell}R^z + \mathcal{E}'^{1\times\ell}(s-z)^{k(z)} = \\ \mathcal{E}'^{1\times 2\ell}\begin{pmatrix} R^z \\ (s-z)^{k(z)}\operatorname{id}_\ell\end{pmatrix} \subseteq \mathcal{E}'^{1\times\ell},\\ \text{hence } U(z)^\perp \subset \mathbb{C}[t]^\ell_{<k(z)}e^{zt} \subset \mathcal{PE}(z)^\ell,\ z \in V_\mathbb{C}(f),\end{gathered} \tag{49}$$
where the $U(z)$ are closed and the $U(z)^\perp$ are $\mathbb{C}$-finite-dimensional *differential* behaviors. Then $\oplus_{z\in V_\mathbb{C}(f)}U(z)^\perp$ is an $\mathcal{E}'$-submodule of $\mathcal{PE}^\ell$ and
$$\mathcal{B} := \operatorname{cl}_E\left(\oplus_{z\in V_\mathbb{C}(f)}U(z)^\perp\right) \tag{50}$$
is the unique autonomous gen. beh. $\mathcal{B}$ with
$$\mathcal{B}\bigcap\mathcal{PE}(z)^\ell = \begin{cases} U(z)^\perp & \text{if } z \in V_\mathbb{C}(f)\\ 0 & \text{if } z \in \mathbb{C}\setminus V_\mathbb{C}(f)\end{cases} \tag{51}$$
and moreover $f \circ \mathcal{B} = 0$.

3) With the $\epsilon_z \in \mathcal{E}'$ from (31) one obtains:
$$\begin{gathered}\forall z \in V_\mathbb{C}(f):\ \epsilon_z \circ \mathcal{B} = U(z)^\perp,\\ \forall w \in \mathcal{B}\bigcap\mathcal{PE}^\ell:\ w = \sum_{z\in V_\mathbb{C}(f)} \epsilon_z \circ w, \text{ and}\\ \mathcal{B} \to \prod_{z\in V_\mathbb{C}(f)} U(z)^\perp,\ w \mapsto (\epsilon_z \circ w)_{z\in V_\mathbb{C}(f)},\end{gathered} \tag{52}$$
is injective.

Remark 5.3: Consider the special case $\ell = 1$ and $\mathcal{B} \subsetneq \mathcal{E}$ in Thm. 5.2. In this case the injectivity of the map in (52) was established in Schwartz' fundamental theorem [24, Thm. 9] from which he inferred that $\mathcal{B}\bigcap\mathcal{PE}$ is dense in $\mathcal{B}$ whereas we simply *used* Result 2.1. The quoted analysts proved a stronger version of Thm. 5.2,(3). Indeed they showed that for *all* $w \in \mathcal{B}$ the sum representation $w = \sum_{z\in V_\mathbb{C}(f)} \epsilon_z \circ w$ holds where a suitable notion of convergence has to be used [24, Thm. 11], [2, Thms. 2,3, 4]. The sum representations $w = \sum_{z\in V_\mathbb{C}(f)} \epsilon_z \circ w$ generalize the *convergent Fourier series*. Notice that
$$\begin{gathered}\epsilon_z \circ w = \sum_{i=0}^{k(z)-1} b_{z,i}e_{z,i} \in \epsilon_z \circ \mathcal{B} = U(z)^\perp \subseteq \mathbb{C}[t]^\ell_{<k(z)}e^{zt}\\ \text{where } b_{z,i} \in \mathbb{C}^\ell,\ e_{z,i} = \frac{t^i}{i!}e^{zt}, \text{ hence}\\ w \underset{[2]}{=} \sum_{z\in V_\mathbb{C}(f)} \epsilon_z \circ w = \sum_{z\in V_\mathbb{C}(f), 0\le i<k(z)} b_{z,i}\frac{t^i}{i!}e^{zt}.\end{gathered} \tag{53}$$

Theorem 5.4: Every autonomous gen. beh. $\mathcal{B} \subseteq \mathcal{E}^\ell$ is indeed a behavior. If $0 \neq f \in \mathcal{E}'$ with $\operatorname{supp}(f) \subseteq [-a,a]$, $a > 0$, annihilates $\mathcal{B}$, i.e., if $f \circ \mathcal{B} = 0$, then there is a matrix $R \in \mathcal{E}'^{\ell\times\ell}$ with $\operatorname{supp}(R) \subseteq [-a,a]$ such that
$$\begin{gathered}\mathcal{B} = \left\{w \in \mathcal{E}^\ell;\ R\circ w = 0,\ f \circ w = 0\right\}, \text{ hence}\\ \mathcal{B}^\perp = \operatorname{cl}_E\left(\mathcal{E}'^{1\times 2\ell}\begin{pmatrix} R \\ f\operatorname{id}_\ell\end{pmatrix}\right).\end{gathered} \tag{54}$$

Proof: For $\ell = 1$ we give an indication of the proof that furnishes [24, Thm. 13 on p.914], but differs from the proof of Schwartz. Let $\mathfrak{b}$ be a nonzero closed ideal of $\mathcal{E}' = PWS$, $0 \neq f \in \mathfrak{b}\bigcap\mathcal{O}_{a,p}$ and $\mathcal{B} := \mathfrak{b}^\perp$. Consider the data from (28) and (29) for f, especially $V := V_\mathbb{C}(f)$. Then
$$\begin{gathered}\operatorname{cl}_O(\mathfrak{b}) = \operatorname{cl}_O(\mathcal{O}\mathfrak{b}) = \bigcap_{z\in V}\mathcal{O}(s-z)^{m(z)},\ m(z) \le k(z),\\ g_z := \frac{f(s)}{(s-z)^{k(z)-m(z)}} \in \mathcal{O}_{a,p},\ \operatorname{mult}(g_z, z) = m(z).\end{gathered} \tag{55}$$
We choose $a(z) > 0$ for $z \in V_\mathbb{C}(f)$ such that $\sum_{z\in V} a(z)\|g_z\|_{a,p} < \infty$. Equation (17) implies $g := \sum_{z\in V} a(z)g_z \in \mathcal{O}_{a,p} \subset \mathcal{E}' = PWS$. Let $h := \gcd_\mathcal{O}(f,g)$. Then
$$\begin{gathered}\forall z \in V:\ \operatorname{mult}(g,z) = m(z) \Longrightarrow \forall z \in V:\ \operatorname{mult}(h,z) = \\ \min(\operatorname{mult}(f,z), \operatorname{mult}(g,z)) = \min(k(z), m(z)) = m(z),\\ \text{also } \forall z \in \mathbb{C}\setminus V:\ \operatorname{mult}(h,z) = 1 \Longrightarrow \operatorname{cl}_O(\mathfrak{b}) = \mathcal{O}h \Longrightarrow\\ \operatorname{cl}_E(\mathcal{E}'f + \mathcal{E}'g) = \mathcal{E}'\bigcap(\mathcal{O}f + \mathcal{O}g) = \mathcal{E}'\bigcap\mathcal{O}h = \\ \mathcal{E}'\bigcap\operatorname{cl}_O(\mathfrak{b}) = \operatorname{cl}_E(\mathfrak{b}) = \mathfrak{b} \Longrightarrow\\ \mathcal{B} = \mathfrak{b}^\perp = \{w \in \mathcal{E};\ f\circ w = g \circ w = 0\}.\end{gathered} \tag{56}$$
■

Corollary 5.5: If $\mathcal{B}_1 \subset \mathcal{E}^{\ell_1}$ is an autonomous behavior and $P \in \mathcal{E}'^{\ell_2\times\ell_1}$ then also $\mathcal{B}_2 := \operatorname{cl}_E(P\circ\mathcal{B}_1)$ is a behavior.

The emphasis of [27] is on *delay-differential* behaviors. To discuss these we need the algebras
$$\mathcal{R} := \bigoplus_{h\in\mathbb{R}}\mathbb{C}[\delta']\delta_h = \bigoplus_{h\in\mathbb{R}}\mathbb{C}[s]e^{hs} \subset \mathcal{H} := \operatorname{quot}(\mathcal{R})\bigcap\mathcal{O} \tag{57}$$
where $\mathcal{R}$ is the ring of delay-differential operators and $\operatorname{quot}(\mathcal{R}) \subset \mathcal{M} = \operatorname{quot}(\mathcal{O})$ is its quotient field inside the field $\mathcal{M}$ of meromorphic functions. Each $h \in \mathcal{H}$ has the simpler form $h = fg^{-1}$ with $f \in \mathcal{R}$ and $0 \neq g \in \mathbb{C}[s]$ (cf. [13, Thm. 2.2], [5, (85)]) and hence $\mathcal{H} \subset \mathcal{E}'$ since g is invertible. A behavior of the form $\mathcal{B} = \left\{w \in \mathcal{E}^\ell;\ R\circ w = 0\right\}$ with $R \in \mathcal{H}^{k\times\ell}$ is called *delay-differential* [13, Def. 3.1].

Remark 5.6: If $h \in \mathcal{H}$ annihilates a gen. beh. $\mathcal{B}$ this is autonomous and hence a behavior, but not necessarily delay-differential. The closed image of an autonomous delay-differential behavior is a behavior, but again not necessarily delay-differential.

VI. Characteristic variety and input/output structures

Let $\mathcal{K} := \operatorname{quot}(\mathcal{E}') \subset \mathcal{M}$ be the quotient field of $\mathcal{E}'$ inside the field $\mathcal{M}$ of meromorphic functions. Consider a f.g. $\mathcal{E}'$-module $M = \mathcal{E}'^{1\times\ell}/U$ with its factor module

$\overline{M} := \mathcal{E}'^{1\times\ell}/\operatorname{cl}_E(U)$ and associated gen. beh. $\mathcal{B} := U^\perp = \operatorname{cl}_E(U)^\perp$. The annihilator ideal

$$\mathfrak{a} := \operatorname{ann}_{\mathcal{E}'}(\overline{M}) = \operatorname{ann}_{\mathcal{E}'}(\mathcal{B}) = \{f \in \mathcal{E}';\ f \circ \mathcal{B} = 0\} \quad (58)$$

is closed. Thm. 4.2,(1), implies

$$\begin{gathered}\mathcal{K}U = \mathcal{K}\operatorname{cl}_E(U) = \mathcal{K}U_{\mathfrak{m}_E(z)},\ \text{hence } \mathcal{K}\otimes_{\mathcal{E}'} M = \\ \mathcal{K}^{1\times\ell}/\mathcal{K}U = \mathcal{K}\otimes_{\mathcal{E}'_{\mathfrak{m}_E(z)}} M_{\mathfrak{m}_E(z)},\ z \in \mathbb{C}.\end{gathered} \quad (59)$$

Since $\mathcal{E}'_{\mathfrak{m}_E(z)}$ is a DVR the module $M_{\mathfrak{m}_E(z)}$ is free if and only it is torsionfree. We define the *characteristic variety*

$$\operatorname{char}(\mathcal{B}) := \left\{z \in \mathbb{C};\ M_{\mathfrak{m}_E(z)} \text{ is not free}\right\}, \quad (60)$$

especially $\operatorname{char}(\mathcal{B}) = \{z \in \mathbb{C};\ \operatorname{rank}(R(z)) < \operatorname{rank}(R)\}$ if $\mathcal{B} = \left\{w \in \mathcal{E}^\ell;\ R\circ w = 0\right\}$ is a behavior. The module M is a *torsion module* or $\mathcal{B}$ is *autonomous* if and only if $\mathcal{K}\otimes_{\mathcal{E}'} M = \mathcal{K}^{1\times\ell}/\mathcal{K}U = 0$ or $\mathfrak{a} \neq 0$ and then

$$\begin{gathered}\operatorname{char}(\mathcal{B}) = V_{\mathbb{C}}(\mathfrak{a}) = \left\{z \in \mathbb{C};\ \mathcal{B}\bigcap\mathcal{PE}(z)^\ell \neq 0\right\} \text{ and} \\ \mathcal{B}\bigcap\mathcal{PE}^\ell = \bigoplus_{z\in\operatorname{char}(\mathcal{B})}\left(\mathcal{B}\bigcap\mathcal{PE}(z)^\ell\right).\end{gathered} \quad (61)$$

The direct sum decomposition in (61) generalizes the well-known and important *modal decomposition* of one-dimensional autonomous differential behaviors.

Theorem and Definition 6.1: (cf. [13, Thm. 3.12]) If U is closed the following properties are equivalent:

(i) $U = \mathcal{E}'^{1\times\ell}\bigcap\mathcal{K}U$.
(ii) $M \subset \mathcal{K}\otimes_{\mathcal{E}'} M$, i.e., M is torsionfree.
(iii) There is a matrix $P \in \mathcal{E}'^{\ell\times m}$ such that $\ker(\circ P) = \left\{\xi \in \mathcal{E}'^{1\times\ell};\ \xi P = 0\right\} = U$ or $(\circ P_{\text{ind}}) : M = \mathcal{E}'^{1\times\ell}/U \to \mathcal{E}'^{1\times m}$ is injective.
(iv) There is $P \in \mathcal{E}'^{\ell\times m}$ such that $\operatorname{cl}_E(P\circ\mathcal{E}^m) = \mathcal{B}$ or $P\circ\mathcal{PE}^m = \mathcal{B}\bigcap\mathcal{PE}^\ell$.
(v) $\operatorname{char}(\mathcal{B}) = 0$.
(vi) $\mathcal{B} = \operatorname{cl}_E(\mathcal{B}\bigcap\mathcal{D}^\ell)$.

Under these conditions $\mathcal{B}$ is called *weakly controllable* (cf. [13, p. 11]). The condition (v) is usually described as *spectral controllability* of $\mathcal{B}$.
(2) The gen. beh. $\mathcal{B}_{\text{cont}} := \operatorname{Hom}_{\mathcal{E}'}(M/\operatorname{tor}(M), \mathcal{E}) = \operatorname{cl}_E(\mathcal{B}\bigcap\mathcal{D}^\ell)$ is the largest weakly controllable gen. beh. contained in $\mathcal{B}$.
Here $\operatorname{tor}(M)$ is the torsion submodule of M and $\mathcal{D} := \mathcal{D}(\mathbb{R})$ the torsionfree $\mathcal{E}'$-submodule of $\mathcal{E}_+ \subset \mathcal{E}$ of smooth functions with compact support. Notice that conditions (i)-(iii) of Thm. 6.1 imply that U is closed.
We define the *output* resp. *input rank* of $\mathcal{B}$ as

$$\begin{gathered}p := \operatorname{rank}(U) := \dim_{\mathcal{K}}(\mathcal{K}U) \text{ resp.} \\ m := \operatorname{rank}(M) := \dim_{\mathcal{K}}(\mathcal{K}\otimes_{\mathcal{E}'} M),\ \text{hence } p+m=\ell.\end{gathered} \quad (62)$$

In general there are different subsets $I \subseteq [\ell] := \{1,\cdots,\ell\}$ with p elements such that the projection $\mathcal{K}^{1\times\ell} \to \mathcal{K}^I$ induces an isomorphism $\mathcal{K}U \cong \mathcal{K}^I$. Such an I is called an *input/output (IO) structure* of U, M or $\mathcal{B}$ and $\mathcal{B}$ with this structure is called an IO behavior. After the usual permutation of the components of $\mathcal{E}'^{1\times\ell}$ one assumes $I = [p]$ and gets $\mathcal{K}U = \mathcal{K}^{1\times p}(\mathrm{id}_p, -H),\ H \in \mathcal{K}^{p\times m}$. The unique matrix H is called the *transfer matrix* of the IO behavior. Equation (59) implies that IO structures and transfer matrices of U, $\operatorname{cl}_E(U)$ and $U_{\mathfrak{m}_E(z)}$ coincide. An IO structure induces the usual exact sequence

$$\begin{gathered}0 \to \mathcal{E}'^{1\times m} \overset{(\circ(0,\mathrm{id}_m))_{\text{ind}}}{\longrightarrow} M \overset{\left(\circ\left(\begin{smallmatrix}\mathrm{id}_p\\0\end{smallmatrix}\right)\right)_{\text{ind}}}{\longrightarrow} M^0 \to 0 \text{ where} \\ M = \mathcal{E}'^{1\times(p+m)}/U,\ M^0 = \mathcal{E}'^{1\times p}/U^0,\ U^0 := U\left(\begin{smallmatrix}\mathrm{id}_p\\0\end{smallmatrix}\right).\end{gathered} \quad (63)$$

The module M^0 is a torsion module and thus $\mathcal{B}^0 := (U^0)^\perp \subset \mathcal{E}^p$ is autonomous and therefore a behavior by Thm. 5.4. Application of $\operatorname{Hom}_{\mathcal{E}'}(-,\mathcal{E})$ to this sequence furnishes the exact gen. beh. sequence

$$\begin{gathered}\begin{matrix}0\to & \mathcal{B}^0 & \overset{\text{inj}}{\longrightarrow} & \mathcal{B} & \overset{\text{proj}}{\longrightarrow} & \mathcal{E}^m \\ & y & \mapsto & \left(\begin{smallmatrix}y\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}y\\u\end{smallmatrix}\right) & \mapsto & u\end{matrix} \quad \text{and} \\ \operatorname{cl}_E(\operatorname{proj}(\mathcal{B})) = \mathcal{E}^m,\ \operatorname{proj}\left(\mathcal{B}\bigcap\mathcal{PE}^{p+m}\right) = \mathcal{PE}^m.\end{gathered} \quad (64)$$

In particular, every polynomial-exponential input u gives rise to a trajectory $(y,u)^\top \in \mathcal{B}$. Even if U is closed U^0 is not necessarily so, but $\operatorname{cl}_E(U)\left(\begin{smallmatrix}\mathrm{id}_P\\0\end{smallmatrix}\right) \subseteq \operatorname{cl}_E\left(U\left(\begin{smallmatrix}\mathrm{id}_P\\0\end{smallmatrix}\right)\right)$ implies

$$\begin{gathered}\operatorname{cl}_E(U^0) = \operatorname{cl}_E(\operatorname{cl}_E(U)^0),\ \text{hence} \\ \mathcal{B}^0 = (U^0)^\perp = (\operatorname{cl}_E(U)^0)^\perp,\end{gathered} \quad (65)$$

i.e., $\mathcal{B}^0$ does not depend on the choice of U with $U^\perp = \mathcal{B}$. If in (64) the module $\mathcal{B}^\perp = \operatorname{cl}_E(U)$ has a dense f.g. submodule $\mathcal{E}'^{1\times k}(P,-Q)$ with $(P,-Q) \in \mathcal{E}'^{k\times(p+m)}$ then the gen. beh. are behaviors and as usual (cf. [13, Thm. 3.9])

$$\begin{gathered}\operatorname{rank}(P,-Q) = \operatorname{rank}(P) = p,\ PH = Q, \\ \mathcal{B} = \left\{(y,u)^\top \in \mathcal{E}^{p+m};\ P\circ y = Q\circ u\right\}, \\ \mathcal{B}^0 = \{y \in \mathcal{E}^p;\ P\circ y = 0\}.\end{gathered} \quad (66)$$

A partial converse of (66) is the following

Theorem 6.2: Assume an IO structure $\mathcal{K}U = \mathcal{K}^{1\times p}(\mathrm{id}_p, -H)$ of $\mathcal{B}$ as in (64) and

$$\mathcal{B}^0 = \{y \in \mathcal{E}^p;\ P\circ y = 0\},\ P \in \mathcal{E}'^{k\times p},\ \operatorname{rank}(P) = p.$$

The matrix P exists since the autonomous gen. beh. $\mathcal{B}^0$ is a behavior. Define $Q := PH \in \mathcal{K}^{k\times m}$. Then

1) $Q \in \mathcal{H}_E^{k\times m}$ where $\mathcal{H}_E := \mathcal{K}\bigcap\mathcal{O} = \bigcap_{z\in\mathbb{C}}\mathcal{E}'_{\mathfrak{m}_E(z)}$.
2) If $Q \in \mathcal{E}'^{k\times m}$ then also $\mathcal{B}$ is a behavior, viz.
$$\mathcal{B} = \left\{(y,u)^\top \in \mathcal{E}^{p+m};\ P\circ y = Q\circ u\right\}.$$
3) If H has an *invertible* common denominator $f \neq 0$ ($fH \in \mathcal{E}'^{p\times m}$), for instance if $H \in \operatorname{quot}(\mathcal{R})^{p\times m}$, then $Q \in \mathcal{E}'^{k\times m}$ and hence $\mathcal{B}$ is a behavior by item 2). Then also the controllable part $\mathcal{B}_{\text{cont}}$ of $\mathcal{B}$ is a behavior ((cf. [13, lines before Prop. 3.13])) and the projection $\operatorname{proj} : \mathcal{B} \to \mathcal{E}^m,\ (y,u)^\top \mapsto u$, is surjective. (cf. [26, Thm. 3.8]).

The preceding theorem shows that also many *nonautonomous* gen. beh. are indeed behaviors.

If $\mathcal{K}U = \mathcal{K}^{1\times p}(\mathrm{id}_p, -H)$ is any IO structure then by Thm. and Def. 6.1 $\mathcal{B} := U^\perp$ is weakly controllable if and only if

$$U = \left\{(\xi,\eta) \in \mathcal{E}'^{1\times(p+m)};\ \xi H + \eta = 0\right\}. \quad (67)$$

Assume conversely that any $H \in \mathcal{K}^{p\times m}$ is given. It can, of course, be written in the fractional form

$$H = ND^{-1},\ D \in \mathcal{E}'^{m\times m},\ N \in \mathcal{E}'^{p\times m},\ \det(D) \neq 0.$$

$$\text{Then } U := \left\{(\xi,\eta) \in \mathcal{E}'^{1\times(p+m)};\ \xi H + \eta = 0\right\} = \left\{(\xi,\eta) \in \mathcal{E}'^{1\times(p+m)};\ \xi N + \eta D = 0\right\} \tag{68}$$

is closed, $\mathcal{K}U = \mathcal{K}(\mathrm{id}_p, -H)$ is an IO structure with transfer matrix H and the associated gen. beh. $\mathcal{B} := U^\perp$ is weakly controllable due to (67). As for differential behaviors this $\mathcal{B}$ is the *unique weakly controllable realization* of H.

Corollary 6.3: Assume that $\mathcal{B} \subseteq \mathcal{E}^{p+m}$ is the unique weakly controllable realization of the transfer matrix $H \in \mathcal{K}^{p\times m}$. Then $\mathcal{B}$ is a behavior if either (i) $m = 1$ or (ii) there is a matrix $Y = \left(\begin{smallmatrix} N \\ D \end{smallmatrix}\right) \in \mathcal{E}'^{(p+m)\times q}$ such that

$$N = HD,\ \mathrm{rank}(D) = m,\ \forall z \in \mathbb{C}:\ \mathrm{rank}(Y(z)) = m. \tag{69}$$

The case (i) is proven like Thm. 5.4 and (ii) follows from [26, Thm. 2.12].

Lemma 6.4: (Rosenbrock equations) Let $\mathcal{B}_1 \subseteq \mathcal{E}^{p_1+m}$ be an IO gen. beh. with transfer matrix H_1 and autonomous part $\mathcal{B}_1^0$, $P = \left(\begin{smallmatrix} Y & U \\ 0 & \mathrm{id}_m \end{smallmatrix}\right) \in \mathcal{E}'^{(p_2+m)\times(p_1+m)}$ and (cf. (33))

$$\mathcal{B}_2 := \mathrm{cl}_E(P \circ \mathcal{B}_1) = \mathrm{cl}_E\left\{\left(\begin{smallmatrix} Y\circ y_1 + U\circ u \\ u \end{smallmatrix}\right);\ \left(\begin{smallmatrix} y_1 \\ u \end{smallmatrix}\right) \in \mathcal{B}_1\right\}. \tag{70}$$

Then also $\mathcal{B}_2$ is an IO gen. beh. with transfer matrix $H_2 = U + YH_1$ and autonomous part $\mathcal{B}_2^0 = \mathrm{cl}_E(Y \circ \mathcal{B}_1^0)$.

Theorem 6.5: If in the situation of Lemma 6.4 the transfer matrix H_1 of $\mathcal{B}_1$ has an invertible common denominator, for instance if $H_1 \in \mathrm{quot}(\mathcal{R})^{p_1\times m}$ or if $\mathcal{B}_1$ is a delay-differential behavior, then also H_2 has an invertible common denominator and therefore both $\mathcal{B}_1$ and $\mathcal{B}_2$ are behaviors.

REFERENCES

[1] C.A. Berenstein, B.A. Taylor, 'A New Look at Interpolation Theory for Entire Functions of One Variable', *Advances in Mathematics* 33(1979), 109-143

[2] C.A. Berenstein, B.A. Taylor, 'Mean-Peridic Functions', *Internat. J. Math. and Math. Sci.* 3(1980), 199-235

[3] N. Bourbaki, *Topological Vector Spaces*, Springer, New York, 1987

[4] H. Bourlès, U. Oberst, 'Duality for Differential-Difference Systems over Lie Groups', *SIAM J. Control Optim.* 48(2009), 2051-2084

[5] H. Bourlès, U. Oberst, 'Elimination, fundamental principle and duality for analytic linear systems of partial differential-difference equations with constant coefficients', *Math. of Control, Signals, Systems* 24(2012), 351-402

[6] J. Delsarte, 'Les fonctions moyenne-périodiques', *J. Math. Pures et Appl.* 14(1935), 403-453

[7] L. Ehrenpreis, 'Solutions of some problems of division IV', *Amer. J. Math.* 82(1960), 522-588

[8] L. Ehrenpreis, *Fourier Analysis In Several Complex Variables*, Wiley-Interscience, New York, 1970

[9] S.J.L. van Eijndhoven, L.C.G.J.M. Habets, 'Equivalence of Convolution Systems in a Behavioral Framework', *Math. Control Signals Systems* 16(2003), 175-206

[10] M. Fliess, H. Mounier, 'Interpretation and comparison of various types of delay system controllabilities', pp. 330-335 in *Proceedings IFAC Conference System Structure and Control, Nantes, France*, July 1995

[11] H. Glüsing-Lürssen, 'A behavioral approach to delay-differential systems', *SIAM J. Control Optim.* 35(1997), 480-499

[12] H. Glüsing-Lürssen, *Linear Delay-Differential Systems with Commensurate Delays: An Algebraic Approach*, Lecture Notes in Mathematics 1770, Springer, Berlin, 2002

[13] H. Glüsing-Lürssen, P. Vettori, S. Zampieri, 'The algebraic structure of DD systems: a behavioral perspective', *Kybernetica* 37(2001), 397-426

[14] L.C. Habets, 'System equivalence for AR-systems over rings-With an application to delay-differential systems', *Math. Contr. Sign. Syst.* 12(1999), 219-244

[15] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, Berlin, 1983

[16] L. Hörmander, *The Analysis of Linear Partial Differential Operators II*, Springer, Berlin, 1983

[17] J.-P. Kahane, 'Sur quelques problèmes d'unicité et de prolongement, relatifs aux fonctions approchables par de sommes d'exponentielles', *Ann. Inst. Fourier* 5(1953/54), 39-130

[18] A.F. Leont'ev, 'Differential-difference equations', *Mat. Sb.* 24(1949), 347-374

[19] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986

[20] H. Mounier, 'Algebraic interpretations of the spectral controllability of a linear delay system', *Forum Math.* 10(1998), 39-58

[21] U. Oberst, 'Variations on the Fundamental Principle for Linear Systems of Partial Differential and Difference Equations with Constant Coefficients', *Applicable Algebra in Engineering, Communication and Computing* 6(1995), 211-243

[22] F. Parreau, Y. Weit, 'Schwartz's theorem on mean-periodic vector-valued functions', *Bull. Soc. Math. France* 117(1989), 319-325

[23] P.M. Rocha and J.C. Willems, 'Behavioral controllability of delay-differential systems', *SIAM J. Control Optim.* 35(1997), 254-264

[24] L. Schwartz, 'Théorie générale des fonctions moyenne-périodiques', *Ann. Math.* 48(1947), 857-929

[25] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966

[26] P. Vettori, S. Zampieri, 'Controllability of Systems Described by Convolutional or Delay-Differential Equations', *SIAM J. Control Optim.* 39(2000), 728-756.

[27] P. Vettori, S. Zampieri, 'Some Results on Systems Described by Convolution Equations', *IEEE Trans. Autom. Control* 46(2001), 793-797

[28] P. Vettori, S. Zampieri, 'Stability and stabilizability of delay-differential systems', pp. 307-322 in B. Hanzon, M. Hazewinkel (Eds.), *Constructive Algebra and Systems Theory*, Royal Netherlands Academy of Arts and Sciences, 2006

[29] Y. Yamamoto, 'Pseudo-rational input/output maps and their realizations: A fractional representation approach to infinite-dimensional systems', *SIAM J. Control Optim.* 26(1988), 1415-1430

[30] Y. Yamamoto, J.C. Willems, 'Behavioral Controllability and Coprimeness for a Class of Infinite-Dimensional Systems', *Proceedings of the 47th Conference on Decision and Control, Cancun, 2008*, 1513-1518