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Robust Open-Loop Nash Equilibria in the Non-cooperative LQ Game Revisited.

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Abstract

In this paper we reconsider the existence of worst-case Nash equilibria in a noncooperative multi-player setting, where the dynamics are described by a linear differential equation and players' preferences are modeled by quadratic cost functions. We consider an open-loop information structure. We show that these equilibria can be obtained by determining the open-loop Nash equilibria of an associated linear quadratic differential game with an additional initial state constraint. We derive both necessary and sufficient conditions for solvability of the finite planning horizon problem. In particular we demonstrate that unlike in the standard linear quadratic differential game setting uniqueness of equilibria may fail to hold. A both necessary and sufficient condition under which there is a unique equilibrium is provided.

Keywords robust control, noncooperative differential games, linear optimal control, Riccati equations.
AMS 91A10, 49N70, 93D09.

1. Introduction

In the last decades, there is an increased interest in studying diverse problems in economics, optimal control theory and engineering using dynamic games. Particularly the framework of linear quadratic differential games is often used to analyse problems due to its analytic tractability. In environmental economics, marketing and macroeconomic policy coordination, policy coordination problems are frequently modeled as dynamic games (see, e.g., the books and references in [8], [21], [12], [29], [26] and [19]). In optimal control theory the derivation of robust control strategies (in particular the H^∞ control problem) can be approached using the theory of (linear quadratic zero-sum) dynamic games (see the seminal work of [5]). In the area of military operations, pursuit-evasion problems and, more recently, problems of defending assets can also be approached using linear quadratic modeling techniques (see, e.g., [17], [24], [25]). Furthermore, this modeling paradigm has been used in the area of robot formation and communication networks (see, e.g., [16], [3]).

In this note we consider the open-loop linear quadratic differential game. This open-loop Nash strategy is often used as one of the benchmarks to evaluate outcomes of the game. Another benchmark that is often used is the state feedback strategy. Recently, [7] compares both strategies to see what the loss in performance of players may be using either one of these strategies. For the scalar game (see the paper for precise details on the game) they find that if there is a large number of players involved in the game, the ratio of losses for an individual player under a feedback and open-loop information structure ranges between $\frac{\sqrt{2}}{2}$ and $\sqrt{2}$. This indicates, that the difference in performance using either one of the information structures is not dramatic.

The linear quadratic differential game problem with an open-loop information structure has been considered by many authors and dates back to the seminal work of Starr and Ho in [30] (see, e.g., [27], [28], [9], [18], [15], [2], [10], [11], [6], [22] and [12]). In [13] the (regular indefinite) infinite-planning horizon case for affine systems under the assumption that every player is capable to stabilize the system by his own was studied, which result was generalized in [14] where it is just assumed that the system as a whole is stabilizable.

[23], [20] and [4] considered for a finite planning horizon the corresponding differential game problem for an open-loop information structure if the system is corrupted by deterministic noise. They introduced the notion of Nash/worst-case equilibrium to model noncooperative behavior in an uncertain environment. Under some assumptions, concerning in particular the existence of certain Riccati differential equations, they showed the problem has a solution.
In this note we will not make these assumptions upfront and reconsider the problem from scratch. Following the standard analysis as presented, e.g., in [12, Chapter 7] we will derive here both necessary and sufficient conditions for existence of a solution for this problem. In

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*Corresponding Author: J.C. Engwerda, Tilburg University, Dept. of Econometrics and O.R. P.O. Box: 90153, 5000 LE Tilburg, The Netherlands; e-mail: engwerda@uvt.nl particular we will show that for this problem also situations occur where there exist an infinite number of Nash/worst-case equilibria. Necessary and sufficient conditions are given under which a unique equilibrium occurs.

The outline of the paper is as follows. In section 2 we show that the problem can be reformulated as a linear quadratic differential game with an additional initial state constraint. This result is then used in section 3 to present existence conditions. Furthermore we present an example where an infinite number of equilibria exist. In the concluding remarks we discuss a number of issues left for future research. Finally, in the appendix we present the proof of the main theorem from section 3.

2. An equivalence result

In this paper we consider the problem to find Nash equilibria for a linear quadratic differential game that is subject to deterministic noise. With $u(t) := [u_1^T(t) \cdots u_N^T]^T$ this game is defined by the cost functions $J_i(t_f, x_0, u, w) :=$

$$\int_0^{t_f} [x^T(t), u^T(t)] M_i [x^T(t),\ u^T(t)]^T - w^T(t) R_{wi} w(t) dt,$$

$$\text{where } M_i = \begin{bmatrix} Q_i & V_{i11} & \cdots & \cdots & V_{i1N} \\ V_{i11}^T & R_{i1} & V_{i22} & \cdots & V_{i2N} \\ & & \ddots & & \\ V_{i1N}^T & V_{i2N}^T & \cdots & \cdots & R_{iN} \end{bmatrix}. \quad (1)$$

M_i is assumed to be symmetric, R_{ii} and R_{wi} positive definite (> 0), $i \in \overline{N}$[1], and $x(t) \in \mathbb{R}^n$ is the solution of the linear differential equation

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t) + Dw(t),\ x(0) = x_0. \quad (2)$$

The function $w(.) \in \mathcal{W} := L_2^k[0, t_f]$ is an unknown disturbance. The controls $u_i(t) \in \mathbb{R}^{m_i}$ considered by player i, $i \in \overline{N}$, are assumed to be such that u belong to $\mathcal{U} := L_2^m[0, t_f]$. Notice we make no definiteness assumptions w.r.t. matrix Q_i.

We assume that in this uncertain environment every player wants to minimize his individual cost function J_i by choosing u_i appropriately. Since all players interact, it is obvious that without making any further specifications the outcome of the game cannot be predicted as every player will preferably base his action on the actions taken by the other players in the game and his expectations concerning the disturbance that will occur. Therefore, depending on the information players have on the game it is to be expected that in the end a set of actions will be chosen from which no individual player has an incentive to deviate. That is, a so-called set of Nash equilibrium actions will be played. We will analyze this problem here under the assumption that the game is played under an open-loop information structure.

From [20] we recall the next definition of Nash/worst-case equilibrium, which we assume will be played by the players.

Definition 2.1 *We define the Nash/worst-case equilibrium in two stages. Consider $u \in \mathcal{U}$, then*

1. $\hat{w}_i(u) \in \mathcal{W}$ *is the worst-case disturbance from the point of view of the i^{th} player if*

   $$J_i(u, \hat{w}_i(u)) \geq J_i(u, w)$$

   holds for each $w \in \mathcal{W}$, $i \in \overline{N}$.

2. *The controls $(u_1^*, \cdots, u_N^*) \in \mathcal{U}$ form a Nash/worst-case equilibrium if for all $i \in \overline{N}$*
   - (a) *For all $u \in \mathcal{U}$, there exists a worst-case disturbance from the point of view of player i and*
   - (b) $J_i(u^*, \hat{w}_i(u^*)) \leq J_i((u_i, u_{-i}^*), \hat{w}_i(u_i, u_{-i}^*))$, *holds for each admissible control function (u_i, u_{-i}^*)[2] and corresponding worst-case disturbance $\hat{w}_i(u_i, u_{-i}^*)$.* ☐

The above definition reflects the idea that every player wants to secure against a for him worst-case realization of the disturbance. Matrix R_{wi} models his expectation about the disturbance and can be interpreted as a risk aversion parameter. In case he expects that only a small disturbance $Dw(.)$ might disrupt the system, he can express this by choosing R_{wi} large. A Nash/worst-case equilibrium models then a situation where every player has no incentive to change his policy given his worst-case expectations concerning the disturbance and the actions of his opponents. Clearly, in a situation where players can observe the realization of the disturbance and they can adapt their actions during the game other solution concepts like the soft-constrained feedback Nash equilibrium (see, e.g., [12, Chapter 9]) are more appropriate.

The next lemma 2.2 shows that Nash/worst-case equilibria can be determined as the OLN equilibria of an associated extended linear quadratic differential game.

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[1] $\overline{N} := \{1, \cdots, N\}$

[2] (v, u_{-i}) equals u where entry u_i is replaced by v. To simplify notation sometimes the brackets are dropped.

In this lemma we use the next notation. $\hat{E}_{Nn}$ will denote the block-column matrix containing N blocks of $n\times n$ identity matrices and $\overline{E}_{i,j}$ the block-column matrix containing i blocks of $n\times n$ zero matrices with block number matrix j replaced by the identity matrix. Furthermore, I and 0 (where sometimes we use an index to indicate the size of these matrices) will denote the identity matrix and zero matrix of appropriate size, respectively. $\mathrm{diag}(A)_N$ will denote the $N\times N$ block diagonal matrix with diagonal entries matrix A.

Lemma 2.2 *$u^*\in\mathscr{U}$ with corresponding worst-case disturbances $\hat{w}_i(u^*)$ is a Nash/worst-case equilibrium for (1,2) if and only if $(u^*,\hat{w}(u^*))$ is an open-loop Nash equilibrium of the $2N$-player differential game*

$$\dot{\overline{x}}(t)=\overline{A}\overline{x}(t)+\sum_{i=1}^{N}\overline{B}_iu_i(t)+\sum_{i=1}^{N}\overline{D}_iw_i(t),\ \overline{x}(0)=\overline{x}_0,$$

where player i likes to minimize his cost function $\overline{J}_i:=$

$$\int_0^T[\overline{x}^T(t),\ u^T(t)]\overline{M}_i[\overline{x}^T(t),\ u^T(t)]^T-w_i^T(t)R_{wi}w_i(t)dt,$$

w.r.t. u_i, $i\in\overline{N}$, and player i, $i=N+1,\cdots,2N$, likes to maximize $\overline{J}_i$ w.r.t. w_i.
Here $\overline{A}=\mathrm{diag}(A)_N$, $\overline{B}_i=\hat{E}_{Nn}B_i$, $\overline{D}_i=\overline{E}_{Ni}D$, $\overline{c}(t)=\hat{E}_{Nn}c(t)$,
$$\overline{M}_i=\begin{bmatrix}\overline{E}_{Ni}^T & 0\\ 0 & I_m\end{bmatrix}^T M_i\begin{bmatrix}\overline{E}_{Ni}^T & 0\\ 0 & I_m\end{bmatrix}\ \textit{and}\ \overline{x}_0=\hat{E}_{Nn}x_0.$$
Moreover, $J_i(u^*,\hat{w}(u^*))=\overline{J}_i(u^*,\hat{w}(u^*))$, $i\in\overline{N}$.

Proof:
$\Rightarrow$ Assume u^* is a Nash/worst-case equilibrium. Then, by definition, $(u_i^*,\hat{w}_i(u^*))$ constitutes a saddle-point solution for player i if the other players, j, play u_j^*. Consequently (see e.g. [12, Theorem 3.26]),

$$\begin{aligned}J_i(u^*,\hat{w}_i(u^*)) &= \max_w\min_{u_i}J_i((u_i,u_{-i}^*),w)\\ &= \min_{u_i}\max_w J_i((u_i,u_{-i}^*),w). \quad (3)\end{aligned}$$

Next, consider the minimization of $\overline{J}_i(u_i,u_{-i}^*,\hat{w}(u^*))$ w.r.t. u_i. Some elementary rewriting shows that this is equivalent with the minimization of $\tilde{J}_i:=$

$$\int_0^T\{[x_i^T(t),\ (u_i^T(t),\ u_{-i}^{*^T}(t))]M_i[x_i^T(t),\ (u_i^T(t),\ u_{-i}^{*^T}(t))]^T -\hat{w}_i^T(u^*)(t)R_{wi}\hat{w}_i(u^*)(t)\}dt$$

subject to the system

$$\dot{x}_i(t)=Ax_i(t)+B_iu_i(t)+\sum_{j\neq i}^{N}B_ju_j^*(t)+D\hat{w}_i(u^*)(t),$$

with $x_i(0)=x_0$. By (3) this minimum is attained at $u_i=u_i^*$.

Similarly, we have that the maximization of $\overline{J}_i(u^*,(w_i,\hat{w}_{-i}(u^*)))$ w.r.t. w_i is equivalent with the maximization of $\tilde{J}_i=$

$$\int_0^T[x_i^T(t),\ u^{*^T}(t)]M_i[x_i^T(t),\ u^{*^T}(t)]^T-w_i^T(t)R_{wi}w_i(t)dt$$

subject to the system

$$\dot{x}_i(t)=Ax_i(t)+\sum_{i=1}^{N}B_iu_i^*(t)+Dw_i(t),\ x_i(0)=x_0.$$

From (3) again, it follows that this maximum is attained at $w_i=\hat{w}_i(u^*)$. So, $(u^*,\hat{w}(u^*))$ is an OLN equilibrium for the $2N$ player differential game.
$\Leftarrow$ Let (u^*,w^*) be an OLN equilibrium for the $2N$ player differential game. Then

$$\overline{J}_i(u^*,w^*)\leq\overline{J}_i((u_i,u_{-i}^*),w^*)\ \text{for all admissible}\ u_i \quad (4)$$
$$\overline{J}_i(u^*,w^*)\geq\overline{J}_i(u^*,(w_i,w_{-i}^*))\text{for all admissible}\ w_i. \quad (5)$$

Now, consider the maximization of $J_i(u^*,w)$ w.r.t. w subject to the system

$$\dot{x}(t)=Ax(t)+\sum_{i=1}^{N}B_iu_i^*(t)+Dw(t),\ x(0)=x_0.$$

A simple elaboration of (5) shows that this maximum is attained at $w=w_i^*$. So, $\hat{w}_i(u^*)=w_i^*$.
Now, let (u_i,u_{-i}^*) be an arbitrary admissible control and $\hat{w}(u_i,u_{-i}^*))$ a corresponding worst-case control. Then, by definition of worst-case control, $J_i((u_i,u_{-i}^*),\hat{w}(u_i,u_{-i}^*))\geq J_i((u_i,u_{-i}^*),\hat{w}(u^*))$. Therefore, in particular

$$\begin{aligned}J_i((u_i,u_{-i}^*),\hat{w}(u_i,u_{-i}^*)) &\geq J_i((u_i,u_{-i}^*),\hat{w}(u^*))\\ &\geq \min_{u_i}J_i((u_i,u_{-i}^*),\hat{w}(u^*)).\end{aligned}$$

From (4) it follows that this lastmentioned minimum exists and is attained at u_i^*. That is, (u^*,w_i^*) satisfies also condition 2.(b) of Definition 2.1.
Finally, that $J_i(u^*,\hat{w}(u^*))=\overline{J}_i(u^*,\hat{w}(u^*))$, $i\in\overline{N}$, follows by a direct comparison of both functions. ☐

3. Necessary and sufficient conditions for Nash/worst-case equilibria

For presentation purposes we will consider here just the simplified two-player linear-quadratic differential game, where the dynamics of the game is described by the linear differential equation

$$\dot{x}(t)=Ax(t)+B_1u_1(t)+B_2u_2(t)+Dw(t), \quad (6)$$

with $x(0)=x_0$, and both players have a quadratic cost functional $J_i(u_1,u_2,w)$ given by:

$$\frac{1}{2}\int_0^T x^T(t)Q_ix(t)+\sum_{j=1}^{2}u_j^T(t)R_{ij}u_j(t)-w^T(t)R_{wi}w(t)dt$$

$$+\frac{1}{2}x^T(T)Q_{iT}x(T),\ i=1,2, \tag{7}$$

in which all matrices are symmetric, and R_{ii}, R_{wi} are positive definite.

Using the shorthand notation $S_i := B_iR_{ii}^{-1}B_i^T$, $S_{D_i} := DR_{wi}^{-1}D^T$, $Q_T = \text{diag}(Q_{iT})$, $\breve{Q}_T^T = [I_{2N}\ Q_T\ -Q_T]$ and H^+ for the Moore-Penrose inverse (see e.g. [1]) of matrix H the following theorem is proved in the Appendix.

Theorem 3.1 *Consider matrix* $M :=$

$$\begin{bmatrix} A & 0 & -S_1 & -S_2 & -S_{D_1} & 0 \\ 0 & A & -S_1 & -S_2 & 0 & -S_{D_2} \\ -Q_1 & 0 & -A^T & 0 & 0 & 0 \\ 0 & -Q_2 & 0 & -A^T & 0 & 0 \\ Q_1 & 0 & 0 & 0 & -A^T & 0 \\ 0 & Q_2 & 0 & 0 & 0 & -A^T \end{bmatrix}. \tag{8}$$

Assume that the four Riccati differential equations,

$$\dot{K}_i(t) = -A^TK_i(t)-K_i(t)A+K_i(t)S_iK_i(t)-Q_i,$$
$$K_i(T)=Q_{iT},\ i=1,2, \tag{9}$$
$$\dot{L}_i(t) = -A^TL_i(t)-L_i(t)A+L_i(t)S_{D_i}L_i(t)+Q_i,$$
$$L_i(T)=-Q_{iT},\ i=1,2, \tag{10}$$

have a symmetric solution $K_i(.)$, $L_i(.)$, *respectively, on* $[0,T]$.

Then, the two-player linear quadratic differential game (6,7) has a Nash/worst-case equilibrium for every initial state x_0 *if and only if with matrix* $H(T) :=$

$$[I_{2N}\ 0_{2N\times 4N}]e^{-MT}\breve{Q}_T,$$

$$H(T)H^+(T)\begin{bmatrix} I \\ I \end{bmatrix} = \begin{bmatrix} I \\ I \end{bmatrix}. \tag{11}$$

Moreover, if the above condition (11) applies, with $v(t) := e^{M(t-T)}\breve{Q}_Tz_1$, *where* $z_1 := H^+\begin{bmatrix} x_0 \\ x_0 \end{bmatrix} + (I - H^+H)q$, $q \in \mathbb{R}^{2n}$ *is an arbitrary vector, the set of equilibrium actions/worst-case disturbances are given by*

$$\begin{aligned} u_i(t) &= -R_{ii}^{-1}B_i^T\bar{E}_{6,i+2}^Tv(t),\ i=1,2, \\ w_i(t) &= -R_{wi}^{-1}D^T\bar{E}_{6,i+4}^Tv(t),\ i=1,2. \end{aligned}$$

The two-player linear quadratic differential game (6,7) has a unique Nash/worst-case equilibrium for every initial state x_0 *if and only if matrix* $H(T)$ *is invertible. The unique equilibrium actions can be calculated either from the above equations or from the linear two-point boundary value problem* $\dot{y}(t)=My(t)$, *with*

$$Py(0)+Qy(T) = [x_0^T\ x_0^T\ 0\ 0\ 0\ 0]^T. \tag{12}$$

Here

$$P = \begin{bmatrix} I\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 0 & 0 \\ -Q_T & I & 0 \\ Q_T & 0 & I \end{bmatrix}.$$

Denoting $[y_0^T(t),\ v_1^T(t),\ v_2^T(t),\ v_3^T(t),\ v_4^T(t)]^T := y(t)$, *with* $y_0 \in \mathbb{R}^{2n}$, *and* $v_i \in \mathbb{R}^n$, $i=1,\cdots,4$, *the equilibrium actions are*

$$u_i(t) = -R_{ii}^{-1}B_i^Tv_i(t) \text{ and } w_i(t) = -R_{wi}^{-1}D^Tv_{i+2}(t),$$

$i=1,2$, *respectively.* ☐

Remark 3.2 In case matrix $H(T)$ introduced in (11) is not invertible either the game has for some initial states no equilibrium, or, for every initial state there exists an infinite number of equilibrium actions. ☐

Example 3.3 In this example we present a game which has for every initial state an infinite number of Nash/worst-case equilibria.

Consider the scalar game, with $A=-1$, $B_i=D=Q_i=1$, $R_{11}=R_{w1}=4$, $R_{22}=R_{w2}=5.6689$, $Q_{1T}=-0.9398$, $Q_{2T}=-1.497$, $T=1$ and all other parameters zero.

The corresponding Riccati differential equations (9,10) are

$$\begin{aligned} \dot{k}_1(t) &= 2k_1(t)+\frac{1}{4}k_1^2(t)-1,\ k_1(1)=-0.9398; \\ \dot{k}_2(t) &= 2k_2(t)+0.1764k_2^2(t)-1,\ k_2(1)=-1.497; \\ \dot{l}_1(t) &= 2l_1(t)+\frac{1}{4}l_1^2(t)+1,\ l_1(1)=0.9398; \\ \dot{l}_2(t) &= 2l_2(t)+0.1764l_2^2(t)+1,\ l_2(1)=1.497. \end{aligned}$$

Some elementary analysis shows that all four differential equations have a solution on the time interval $[0,1]$. Matrix $H(T)$, introduced in (11), equals $\begin{bmatrix} 0.3771 & 0.3817 \\ 0.3771 & 0.3817 \end{bmatrix}$. Consequently, $H^+ = \begin{bmatrix} 0.6549 & 0.6629 \\ 0.6549 & 0.6629 \end{bmatrix}$. It is easily verified that $H(T)H^+(T)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So, by Theorem 3.1, for every x_0 this game has an infinite number of

Nash/worst-case equilibria.

With $z_1 := H^+ \begin{bmatrix} x_0 \\ x_0 \end{bmatrix} + (I - H^+H)q = \begin{bmatrix} 1.3179 \\ 1.3179 \end{bmatrix} x_0 + \begin{bmatrix} 0.503 \\ -0.497 \end{bmatrix} q$, $q \in I\!R$, and

$$v(t) = e^{M(t-T)} \breve{Q}_T z_1 = 10^{-1} *$$

$$\begin{bmatrix} -5.3 & -5.15 & 0.5 & 0.45 & 0 & 0 \\ -6.35 & 6.1 & -0.6 & 0.54 & 0 & 0 \\ -2.54 & -2.7 & 4.5 & -4.5 & 5 & 0 \\ -3.0 & 3.24 & -5.4 & -5.4 & 0 & 6.3 \\ 2.54 & 2.7 & -4.5 & 4.5 & -5 & -4.46 \\ 3.0 & -3.24 & 5.4 & 5.4 & -7.1 & -6.3 \end{bmatrix} *$$

$$\begin{bmatrix} -1.91e^{-1.1(t-1)} & -0.09e^{-1.1(t-1)} \\ -0.17e^{-\frac{8}{9}(t-1)} & -1.08e^{-\frac{8}{9}(t-1)} \\ 0.36e^{\frac{8}{9}(t-1)} & -1.86e^{\frac{8}{9}(t-1)} \\ 4.26e^{1.1(t-1)} & -0.117e^{1.1(t-1)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ q \end{bmatrix},$$

the equilibrium/worst-case actions are

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \\ w_1^*(t) \\ w_2^*(t) \end{bmatrix} = \begin{bmatrix} -0.254 & -0.27 & 0.45 & -0.45 \\ -0.30 & 0.324 & -0.54 & -0.54 \\ 0.254 & 0.27 & -0.45 & 0.45 \\ 0.30 & -0.324 & 0.54 & 0.54 \end{bmatrix}$$

$$* \begin{bmatrix} -1.91e^{-1.1(t-1)} & -0.09e^{-1.1(t-1)} \\ -0.17e^{-\frac{8}{9}(t-1)} & -1.08e^{-\frac{8}{9}(t-1)} \\ 0.36e^{\frac{8}{9}(t-1)} & -1.86e^{\frac{8}{9}(t-1)} \\ 4.26e^{1.1(t-1)} & -0.117e^{1.1(t-1)} \end{bmatrix} \begin{bmatrix} x_0 \\ q \end{bmatrix}. \square$$

Remark 3.4 It is easily seen that for the N-player case Theorem 3.1 applies with matrix $M = \begin{bmatrix} \mathrm{diag}(A)_N & -\hat{E}_{Nn}[S_1 \cdots S_N] & -\mathrm{diag}(S_{D_i}) \\ -\mathrm{diag}(Q_i) & -\mathrm{diag}(A^T)_N & 0 \\ \mathrm{diag}(Q_i) & 0 & -\mathrm{diag}(A^T)_N \end{bmatrix}$ and

$$H(T) = [I_{Nn}\ 0_{Nn \times 2Nn}] e^{-MT} \begin{bmatrix} I_{Nn} \\ \mathrm{diag}(Q_{iT}) \\ -\mathrm{diag}(Q_{iT}) \end{bmatrix}. \qquad \square$$

4. Concluding remarks

In this paper we reconsidered the finite planning horizon open-loop linear quadratic differential game that is disrupted by deterministic noise. We derived both necessary and sufficient conditions under which this game has an open-loop Nash/worst-case equilibrium. We showed that Nash/worst-case equilibria can be calculated from a "virtual" linear quadratic differential game where with every player a "nature" player is introduced that tries to maximize the performance criterium w.r.t. to the disturbance. Based on this equivalence result we derived both necessary and sufficient conditions for Nash/worst-case equilibria along the lines these conditions are obtained for the noise-free linear quadratic differential game. The only difficulty arises in an extra condition that is imposed on the initial state of the extended system. In an example we showed that, different from the noise free case, multiple equilibria may occur. As a result of the analysis we could also easily establish both necessary and sufficient conditions for existence of a unique equilibrium.

Open issues that remain to be settled are how these results can be used to arrive at both necessary and sufficient conditions for existence of Nash/worst-case equilibria for an arbitrary planning horizon $[0, t_f]$, where t_f ranges between 0 and T. In the noise-free case necessary and sufficient conditions for solvability of this problem can be formulated in terms of existence to a set of coupled Riccati differential equations. Unfortunately, due to the initial state restrictions for the extended system, such a generalization for the disturbed game is less obvious. Clearly, by e.g. ignoring this restriction and assuming that the corresponding "noise-free" Riccati differential equations have a solution one easily obtains a set of sufficient conditions for the existence to the problem. However, the question remains how far these conditions are necessary too. Such existence conditions may also help to solve the corresponding infinite planning horizon problem.

Appendix

Proof Theorem 3.1

By Lemma 2.2 the two-player linear quadratic differential game (6-7) has a Nash/worst-case equilibrium for every initial state x_0 if and only if with $\overline{x}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$; $\overline{A} := \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$; $\overline{B}_i := \begin{bmatrix} B_i \\ B_i \end{bmatrix}$; $\overline{D}_1 := \begin{bmatrix} D \\ 0 \end{bmatrix}$; $\overline{D}_2 := \begin{bmatrix} 0 \\ D \end{bmatrix}$; $\overline{Q}_1 := \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$; $\overline{Q}_2 := \begin{bmatrix} 0 & 0 \\ 0 & Q_2 \end{bmatrix}$ $\overline{Q}_{1T} := \begin{bmatrix} Q_{1T} & 0 \\ 0 & 0 \end{bmatrix}$; and $\overline{Q}_{2T} := \begin{bmatrix} 0 & 0 \\ 0 & Q_{2T} \end{bmatrix}$, the next four-player game has a Nash equilibrium for every initial state x_0.

$\dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + \overline{B}_1 u_1(t) + \overline{B}_2 u_2(t) + \overline{D}_1 w_1(t) + \overline{D}_2 w_2(t)$, with $\overline{x}(0) = \overline{x}_0 := \begin{bmatrix} x_0 \\ x_0 \end{bmatrix}$ and cost functional for the players given by $J_i(u_1,u_2,w_i)=$

$$\frac{1}{2}\int_0^T\{\overline{x}^T(t)\overline{Q}_i\overline{x}(t)+\sum_{j=1}^{2}u_j^T(t)R_{ij}u_j(t)-$$

$$w_i^T(t)R_{wi}w_i(t)\}dt+\frac{1}{2}\overline{x}^T(T)\overline{Q}_{iT}\overline{x}(T),$$

and $J_{i+2}(u_1,u_2,w_i)=-J_i(u_1,u_2,w_i),\ i=1,2.$
Unfortunately, the initial state of this extended system can not be arbitrarily chosen. Therefore, we cannot use directly existing results on open-loop LQ games to derive both necessary and sufficient existence conditions for a Nash equilibrium. However, we can follow the lines of the proof for the standard case (see, e.g., proof of [12, Theorem 7.1]) to obtain these conditions.
Suppose that $(u_i^*(.),w_i^*(.))$ is a Nash equilibrium. Then, by the maximum principle, the Hamiltonian $H_i=$

$$\frac{1}{2}(\overline{x}^T\overline{Q}_i\overline{x}+u_1^TR_{i1}u_1+u_2^TR_{i2}u_2-w_i^T(t)R_{wi}w_i(t))+$$

$$\psi_i^T(\overline{A}\overline{x}+\overline{B}_1u_1+\overline{B}_2u_2+\overline{D}_1w_1+\overline{D}_2w_2),$$

is minimized by player i w.r.t. u_i, $i=1,2$, and $H_{i+2}=$

$$\frac{1}{2}(-\overline{x}^T\overline{Q}_i\overline{x}-u_1^TR_{i1}u_1-u_2^TR_{i2}u_2+w_i^T(t)R_{wi}w_i(t))+$$

$$\psi_{i+2}^T(\overline{A}\overline{x}+\overline{B}_1u_1+\overline{B}_2u_2+\overline{D}_1w_1+\overline{D}_2w_2),$$

is minimized by player $i+2$ w.r.t. w_i, $i=1,2$. This yields the necessary conditions

$$u_i^*(t)=-R_{ii}^{-1}\overline{B}_i^T\psi_i(t)\text{ and }w_i^*(t)=-R_{wi}^{-1}\overline{D}_i^T\psi_{i+2}(t),$$

$i=1,2$, where the $2n$-dimensional vectors $\psi_i(t)$ satisfy

$$\dot{\psi}_i(t)=-\overline{Q}_i\overline{x}(t)-\overline{A}^T\psi_i(t),\psi_i(T)=\overline{Q}_{iT}\overline{x}(T),$$

$$\dot{\psi}_{i+2}(t)=\overline{Q}_i\overline{x}(t)-\overline{A}^T\psi_{i+2}(t),\psi_{i+2}(T)=-\overline{Q}_{iT}\overline{x}(T),$$

$i=1,2$, and

$$\dot{\overline{x}}(t)=\overline{A}\overline{x}(t)-\overline{S}_1^e\psi_1(t)-\overline{S}_2^e\psi_2(t)-\overline{S}_{D_1}^e\psi_3(t)-\overline{S}_{D_2}^e\psi_4(t);$$

with $\overline{x}^T(0)=[x_0^T\ x_0^T]$. Here $\overline{S}_i^e:=\begin{bmatrix}S_i & S_i\\ S_i & S_i\end{bmatrix}$, $\overline{S}_{D_1}^e:=\begin{bmatrix}S_{D_1} & 0\\ 0 & 0\end{bmatrix}$ and $\overline{S}_{D_2}^e:=\begin{bmatrix}0 & 0\\ 0 & S_{D_2}\end{bmatrix}$. In other words, if the problem has an open-loop Nash equilibrium then, with $\tilde{y}(t):=[\overline{x}^T(t),\psi_1^T(t),\cdots,\psi_4^T(t)]^T$, the differential equation

$$\dot{\tilde{y}}(t)=\begin{bmatrix}\overline{A} & -\overline{S}_1^e & -\overline{S}_2^e & -\overline{S}_{D_1}^e & -\overline{S}_{D_2}^e\\ -\overline{Q}_1 & -\overline{A}^T & 0 & 0 & 0\\ -\overline{Q}_2 & 0 & -\overline{A}^T & 0 & 0\\ \overline{Q}_1 & 0 & 0 & -\overline{A}^T & 0\\ \overline{Q}_2 & 0 & 0 & 0 & -\overline{A}^T\end{bmatrix}\tilde{y}(t) \tag{13}$$

with boundary conditions $\overline{x}^T(0)=[x_0^T\ x_0^T]$, $\psi_1(T)-\overline{Q}_{1T}\overline{x}(T)=0$, $\psi_2(T)-\overline{Q}_{2T}\overline{x}(T)=0$, $\psi_3(T)+\overline{Q}_{1T}\overline{x}(T)=0$ and $\psi_4(T)+\overline{Q}_{2T}\overline{x}(T)=0$, has a solution. Next, split $\psi_i^T=:[\psi_{i1}^T\ \psi_{i2}^T]$ and introduce $\tilde{\psi}_1^T:=[\psi_{11}^T\ \psi_{22}^T]$, $\tilde{\psi}_2^T:=[\psi_{31}^T\ \psi_{42}^T]$, $\tilde{\psi}_3^T:=[\psi_{12}^T\ \psi_{21}^T]$ and $\tilde{\psi}_4^T:=[\psi_{32}^T\ \psi_{41}^T]$. Then, with $y(t):=[x^T(t),\tilde{\psi}_1^T(t),\cdots,\tilde{\psi}_4^T(t)]^T$, $S^e:=\begin{bmatrix}S_1 & S_2\\ S_1 & S_2\end{bmatrix}$, $S_D^e:=\text{diag}(S_{D_i})$ and $Q^e:=\text{diag}(Q_i)$ the above reasoning shows that, if there is a Nash equilibrium, then for every x_0 the next linear two-point boundary value problem has a solution.

$$\dot{y}(t)=M^ey(t),\text{ with }Py(0)+Qy(T)=[\overline{x}_0^T\ 0\ 0\ 0\ 0]^T. \tag{14}$$

$$\text{Here }M^e=\begin{bmatrix}\overline{A} & -S^e & -S_D^e & -S^e & 0\\ -Q^e & -\overline{A}^T & 0 & 0 & 0\\ Q^e & 0 & -\overline{A}^T & 0 & 0\\ 0 & 0 & 0 & -\overline{A}^T & 0\\ 0 & 0 & 0 & 0 & -\overline{A}^T\end{bmatrix};$$

$$P=\begin{bmatrix}I\,0\,0\,0\,0\\ 0\,0\,0\,0\,0\\ 0\,0\,0\,0\,0\\ 0\,0\,0\,0\,0\\ 0\,0\,0\,0\,0\end{bmatrix}\text{ and }Q=\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ -Q_T & I & 0 & 0 & 0\\ Q_T & 0 & I & 0 & 0\\ 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & I\end{bmatrix}.$$

Some elementary rewriting shows that the above two-point boundary value problem (14) has a solution for every initial state x_0 if and only if

$$(P+Qe^{M^eT})y(0)=[\overline{x}_0^T\ 0\ 0\ 0\ 0]^T,$$

or, equivalently,

$$(Pe^{-M^eT}+Q)e^{M^eT}y(0)=[x_0^T\ x_0^T\ 0\ 0\ 0\ 0]^T, \tag{15}$$

is solvable for every x_0.
Denoting $z:=e^{M^eT}y(0)$ and $[W_1\ W_2\ W_3\ W_4\ W_5]:=[I\ 0\ 0\ 0\ 0]e^{-M^eT}$, the question whether (15) is solvable for every x_0 is equivalent with the question whether

$$\begin{bmatrix}W_1 & W_2 & W_3 & W_4 & W_5\\ -Q_T & I & 0 & 0 & 0\\ Q_T & 0 & I & 0 & 0\\ 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & I\end{bmatrix}z=\begin{bmatrix}\overline{x}_0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \tag{16}$$

has a solution for every x_0. An elementary analysis shows that a solution exists for every x_0 if and only if

$$(W_1+W_2Q_T-W_3Q_T)z_1=\overline{x}_0 \tag{17}$$

has a solution for every x_0. Or, using the definition of W_i: $\bar{E}_{5,1}^Te^{-M^eT}\begin{bmatrix}I_{3N}\\ 0_{2N\times 3N}\end{bmatrix}\check{Q}_Tz_1=\overline{x}_0,$

or $[I\ 0\ 0\ 0\ 0]e^{-M^eT}[I\ Q_T\ \ -Q_T\ 0\ 0]^T z_1 = \overline{x}_0$ has a solution for every x_0. Using the structure of matrix M^e it is now easily verified that this equation has a solution if and only if with matrix $H(T) := [I\ 0\ 0]e^{-MT}\breve{Q}_T$ the equation $H(T) = \overline{x}_0$ has a solution for every x_0. This last problem is equivalent with the problem under which conditions the matrix equation $H(T)X = \begin{bmatrix} I \\ I \end{bmatrix}$ has a solution X. From this one directly obtains condition (11) (see e.g. [1, Exercise 10.49]).

If (11) holds, all solutions of equation (17) are

$$z_1 = H^+\overline{x}_0 + (I - H^+H)q,\ \text{where } q \text{ is an arbitrary vector.}$$

From (16) it follows that the set of points z yielding a consistent initial boundary value problem are

$$z^T = [z_1^T\ \ (Q_T z_1)^T,\ -(Q_T z_1)^T,\ 0,\ 0].$$

The with z corresponding consistent initial state of the boundary value problem is $y_0 = e^{-M^eT}z$. The solution of the initial boundary value problem is then

$$y(t) = e^{M^et}y_0 = e^{M^e(t-T)}z = \begin{bmatrix} e^{M(t-T)}\breve{Q}_T z_1 \\ 0 \\ 0 \end{bmatrix}.$$

Using the definition of $y(t)$ in terms of the state and co-state variables, yields then the corresponding equilibrium controls: $u_i(t) = -R_{ii}^{-1}[B_i^T\ \ B_i^T]\begin{bmatrix} \psi_{i1}(t) \\ \psi_{i2}(t) \end{bmatrix} = -R_{ii}^{-1}B_i^T\bar{E}_{6,i+2}^T e^{M(t-T)}\breve{Q}_T z_1$ and $w_i(t) = -R_{wi}^{-1}\bar{D}_i^T \begin{bmatrix} \psi_{i+21}(t) \\ \psi_{i+22}(t) \end{bmatrix} = -R_{wi}^{-1}D^T\bar{E}_{6,i+4}^T e^{M(t-T)}\breve{Q}_T z_1,\ i = 1,2.$

”$\Leftarrow$ part” By assumption the Riccati differential equations (9,10) have a solution on $[0,T]$. Since $H(T)$ satisfies (11), it is clear from the ”$\Rightarrow$ part” of the proof that the two-point boundary value problem (12) has for every x_0 a consistent initial value y_0 yielding a unique solution for the boundary value problem. Assume y_0 is a with x_0 consistent initial value of the boundary value problem. Denote the solution $y(t)$ of this two-point boundary value problem (12) by $[x_1^T(t),\ x_2^T(t), \psi_1(t), \cdots, \psi_4(t)]^T$, where the dimension of x_i and ψ_i is n. Now consider

$$m_i(t) := \psi_i(t) - K_i(t)x_i(t) \text{ and}$$
$$m_{i+2}(t) := \psi_{i+2}(t) - L_i(t)x_i(t),\ i = 1,2.$$

Then, $m_i(T) = 0$. Furthermore, differentiation of $m_1(t)$ gives $\dot{m}_1(t) = \dot{\psi}_1(t) - \dot{K}_1(t)x_1(t) - K_1(t)\dot{x}_1(t)$ $= -Q_1x_1(t) - A^T\psi_1(t) - [-A^TK_1(t) - K_1(t)A + K_1(t)S_1K_1(t) - Q_1]x_1(t) - K_1(t)[Ax_1(t) - S_1\psi_1(t) - S_2\psi_2(t) - S_{D_1}\psi_3(t)] = -A^T[m_1(t) + K_1(t)x_1(t)] + A^TK_1(t)x_1(t) - K_1(t)S_1K_1(t)x_1(t) + K_1(t)S_1[m_1(t) + K_1(t)x_1(t)] + K_1(t)S_2[m_2(t) + K_2(t)x_2(t)] + K_1(t)S_{D_1}[m_3(t) + L_1(t)x_1(t)] = -A^Tm_1(t) + K_1(t)[S_1m_1(t) + S_2m_2(t) + S_{D_1}m_3(t)] + K_1(t)[S_2K_2(t)x_2(t) + S_{D_1}L_1(t)x_1(t)]$. Next, consider

$$u_i^*(t) = -R_{ii}^{-1}B_i^T(K_i(t)x_i(t) + m_i(t)) \text{ and}$$
$$w_i^*(t) = -R_{wi}^{-1}D^T(L_i(t)x_i(t) + m_{i+2}(t))\ i = 1,2.$$

By [12, Theorem 5.11] the minimization w.r.t. u_1 of $J_1(u_1, u_2^*, w_1^*, w_2^*) :=$

$$\int_0^T \{\overline{x}^T(t)\overline{Q}_i\overline{x}(t) + u_1^T(t)R_{11}u_1(t) + u_2^{*^T}(t)R_{12}u_2^*(t) - w_1^{*^T}(t)R_{w1}w_1^*(t)\}dt + \overline{x}^T(T)\overline{Q}_{iT}\overline{x}(T),$$

where $\dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + \overline{B}_1u_1(t) + \overline{B}_2u_2^*(t) + \sum_{i=1}^2 \overline{D}_iw_i^*(t)$, with $\overline{x}(0) = \overline{x}_0 = [x_0^T,\ x_0^T]^T$, has a unique solution. This solution is $\overline{u}_1(t) = -R_{11}^{-1}\overline{B}_1^T(\overline{K}_1(t)\overline{x}(t) + \overline{m}_1(t))$, where $\overline{K}_1(t)$ solves the Riccati differential equation, with $\overline{K}_1(T) = \overline{Q}_{1T}$,

$$\dot{\overline{K}}_1(t) = -\overline{A}^T\overline{K}_1(t) - \overline{K}_1(t)\overline{A} + \overline{K}_1(t)\overline{S}_1^e\overline{K}_1(t) - \overline{Q}_1,$$

and $\overline{m}_1(t)$ solves the linear differential equation

$$\dot{\overline{m}}_1(t) = (\overline{K}_1(t)\overline{S}_1^e - \overline{A}^T)\overline{m}_1(t) - \overline{K}_1(t)(\overline{B}_2u_2^*(t) + \sum_{i=1}^2 \overline{D}_iw_i^*(t)),\ \overline{m}_1(T) = 0. \tag{18}$$

It is easily verified that $\overline{K}_1(t) := \begin{bmatrix} K_1(t) & 0 \\ 0 & 0 \end{bmatrix}$ solves the Riccati differential equation, where $K_1(t)$ solves (9). Using this in (18) shows that $\overline{m}_1(t) = \begin{bmatrix} \tilde{m}_1(t) \\ 0 \end{bmatrix}$, where $\tilde{m}_1(t)$ solves the differential equation

$$\dot{\tilde{m}}_1(t) = (K_1(t)S_1 - A^T)\tilde{m}_1(t) - K_1(t)(B_2u_2^*(t) + Dw_1^*(t)),\ \tilde{m}_1(T) = 0. \tag{19}$$

Consequently, $\overline{u}_1(t) = -R_{11}^{-1}B_1^T(K_1(t)\overline{x}_1(t) + \tilde{m}_1(t))$, where $\overline{x}_1(t)$ is the through this optimal control implied solution of the differential equation, with $\overline{x}_1(0) = x_0$,

$$\dot{\overline{x}}_1(t) = (A - S_1K_1)\overline{x}_1(t) - S_1\tilde{m}_1(t) + B_2u_2^*(t) + Dw_1^*(t). \tag{20}$$

Substitution of $u_2^*(t)$ and $w_1^*(t)$ into (19,20) shows, that the variables $\tilde{m}_1$ and $\tilde{x}$ satisfy

$$\dot{\tilde{m}}_1(t) = (K_1(t)S_1 - A^T)\tilde{m}_1(t) + K_1(t)(S_2(K_2(t)x_2(t) + m_2(t)) + S_{D_1}(L_1(t)x_1(t) + m_3(t))),\ \tilde{m}_1(T) = 0, \text{ and}$$

$$\dot{\overline{x}}_1(t) = (A - S_1K_1(t))\overline{x}_1(t) - S_2(K_2(t)x_2(t) + m_2(t)) - S_{D_1}(L_1(t)x_1(t) + m_3(t)) - S_1\tilde{m}_1(t),\ \overline{x}_1(0) = x_0.$$

It is easily verified that a solution of this set of differential equations is given by $\tilde{m}_1(t) = m_1(t)$ and $\bar{x}_1(t) = x_1(t)$. Since its solution is unique this implies that $\bar{u}_1(t) = u_1^*(t)$. Or, stated differently,

$$J_1(u_1^*, u_2^*, w_1^*, w_2^*) \leq J_1(u_1, u_2^*, w_1^*, w_2^*), \text{ for all } u_1.$$

Similarly it can be shown that the corresponding inequalities for the cost functions for the other players apply. Which shows that $(u_1^*, u_2^*, w_1^*, w_2^*)$ is a Nash equilibrium for the extended game. So, by Lemma 2.2, these strategies yield a Nash/worst-case equilibrium.

Finally, the uniqueness result follows since z_1 is uniquely determined if and only if $H(T)$ is invertible.

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