

Consensus under Persistent Interconnections in a Ring Topology: a Small Gain Approach

Nohemi Alvarez¹ and Antonio Loria²

Abstract— We study consensus problem over a network of dynamic agents with time-dependent communication links which may disconnect for long intervals of time. We assume that the nodes are interconnected in a ring topology. The originality of our results lays, in part, in our method of proof: we leave behind graph theory for linear time-invariant systems and use instead, stability theory. In particular, we employ the small-gain theorem to establish simple checkable conditions on the network interconnections, to guarantee the achievement of consensus. Simulations on an academic example are provided to demonstrate the effectiveness of the theoretical results.

I. INTRODUCTION

In spite of the considerable bulk of literature in the past years, the consensus problem is still of great interest for researchers of several disciplines due to the multiple applications related to networked multi-agent systems. Satellite formation flying [1], [2], coupled oscillators [3], formation tracking control for mobile robots [4], coupled air traffic control [5] just to mention a few. These applications justify the design of appropriate consensus protocols to drive all dynamic agents to a common value. The consensus problem consists in establishing conditions under which the differences between any two motions among a group of dynamic systems, converge to zero asymptotically.

Although the case of constant fixed topologies with permanent all-to-all interconnections is completely solved by various means –see [6], the problem of consensus under intermittent interconnections is still of actual interest. It is clearly motivated, for instance, by scenarios in which communication failures appear. In the pioneer paper [7] the author presents a coordination problem for a network of agents with single integrator dynamics. The network coupling is allowed to be time-dependent and non-bidirectional. It is represented by a linear time-varying system in continuous time with a Metzler matrix. Stability analysis shows that all state components converge to a common value as time grows unbounded using a Lyapunov function.

In [8] the authors provide a theoretical explanation for the observed behavior of Vicsek model [9]. Explicitly takes into account possible changes of each agent’s nearest neighbors over time, can be thought of as a consensus

problem. Using a classical convergence result show that all agents converge to a common steady state provided all agents are “linked together” with sufficient frequency as the system evolves.

The approaches in [7] and [8] are based on undirected graphs. When the information exchange is unidirectional, that is, consensus may to be achieved in the presence of limited information exchange. In [10] the authors present both continuous and discrete time consensus problem for switching directed graphs. Considering that the Laplacian matrix is piecewise constant it is shown that consensus is achieved if and only if the topology has a directed spanning tree. In [11] a class of nonlinear consensus protocol for networks of dynamic agents whose dynamics are simple integrators is prescribed. A Lyapunov function is introduced and quantifies the total disagreement among the agents, guaranteeing that consensus problem is globally asymptotically achieved.

In this paper, we solve the consensus problem for a group of agents modeled by single integrators, interconnected in a ring topology. We assume that the communication links may fail therefore, the communication may be interrupted by arbitrarily long but uniformly bounded intervals of time. More precisely we establish that persistency of excitation of each interconnection between any pair of agents, is necessary condition to achieve consensus. This condition is also sufficient provided that the intensity of *one* of the interconnections is relatively small. The results are established using the small-gain theorem.

The paper is organized as follows. In section II we establish the principal protocol and consensus problem. Section III contains the main results. In section IV we present some illustrative simulation results and Section V offers our concluding remarks.

II. PROBLEM STATEMENT

Consider N dynamic agents

$$\Psi_i : \dot{x}_i = u_i, \quad i \in \{1, 2, \dots, N\} \quad (1)$$

where u_i represents a protocol of interconnection. The most common continuous consensus protocol under an all-to-all communication assumption, has been studied

N. Alvarez is with Univ. Paris Sud. A. Loria is with CNRS. Address: LSS-SUPELEC, 91192 Gif-sur-Yvette, France. E-mail: nohemi.alvarez@lss.supelec.fr, antonio.loria@lss.supelec.fr

in [12], [13] and is given by

$$u_i = - \sum_{j=1}^N a_{ij}(t)(x_i - x_j), \quad \forall i \in \{1, \dots, N\} \quad (2)$$

where $a_{ij}(t)$ is the (i, j) entry of the adjacency matrix and x_i is the information state of the i -th agent. In this paper, we study consensus under the assumption of a ring-communication topology, that is, for each i , we define

$$u_i = \begin{cases} -a_{ii+1}(t)(x_i - x_{i+1}), & \forall i \in \{1, \dots, N-1\} \\ -a_{i1}(t)(x_i - x_1), & i = N \end{cases} \quad (3)$$

where $a_{ii+1} \geq 0$, it is strictly positive whenever information flows from the $(i+1)$ th node to the i th node. Under (3), the system (1) has the form

$$\dot{x} = -L(t)x \quad (4)$$

where $L(t)$ is the following Laplacian matrix

$$L(t) := \begin{bmatrix} a_{12}(t) & -a_{12}(t) & 0 & 0 & 0 \\ 0 & a_{23}(t) & -a_{23}(t) & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{N-1N}(t) & -a_{N-1N}(t) \\ -a_{N1}(t) & 0 & \cdots & \cdots & a_{N1}(t) \end{bmatrix} \quad (5)$$

and $x := (x_1, \dots, x_N)^\top$ is the vector containing the state of each agent. The Laplacian $L(t)$ has associated a graphic representation which is shown in Figure 1.

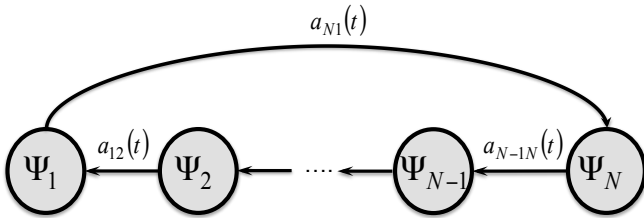


Fig. 1. Ring topology with time dependent communication links.

The system (4) reaches consensus if for any initial condition all the states reaches a common value as t tends to infinity. The consensus problem has been thoroughly studied both for the case of constant and time-varying interconnections, mostly under the assumption of an all-to-all communication topology. Typically, graph theory is used to establish that consensus is reached if there exists a directed spanning tree (any node may be reached from any node). In the case that the interconnections are time-varying, a similar result was established in [7].

In this paper, we take a different approach to the analysis of consensus: stability theory. Indeed, our analysis builds upon the elementary observation that consensus is equivalent to the asymptotic stability of the origin of

$$\dot{z} = A(t)z \quad (6)$$

where we defined the error states

$$z_i = x_i - x_{i+1} \quad \forall i \in \{1, \dots, N-1\}, \quad (7)$$

the vector containing all the errors z_i is defined as $z = (z_1, \dots, z_{N-1})^\top$ and

$$A(t) := \begin{bmatrix} -a_{12}(t) & a_{23}(t) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & -a_{N-2N-1}(t) & a_{N-1N}(t) \\ -a_{N1}(t) & \cdots & -a_{N1}(t) & -(a_{N1} + a_{N-1N})(t) \end{bmatrix} \quad (8)$$

The latter follows from evaluating the time derivative of (7) to obtain, substituting the dynamics of the agents i ,

$$\dot{z}_i = \begin{cases} -a_{ii+1}(t)z_i + a_{i+1i+2}(t)z_{i+1} & i \in \{1, \dots, N-2\} \\ -a_{N-1N}(t)z_{N-1} + a_{N1}(t)z_N & i = N-1. \end{cases} \quad (9)$$

Our main result, which is presented next, establishes sufficient and necessary conditions for the origin of (6) to be uniformly globally exponentially stable therefore, for the system (4) to reach consensus uniformly and exponentially fast.

III. MAIN RESULTS

We assume that the functions a_{ii+1} may be equal to zero for intervals of time whose length is uniformly bounded. That is, we assume that each coefficient a_{ii+1} is *persistently exciting*. The latter property, which is well-known in the literature of adaptive control systems –see [14], is defined as follows.

Definition 1: Persistence of Excitation. A locally integrable function $a : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is said to be persistently exciting (PE) if there exist positive numbers T and μ such that

$$\int_t^{t+T} a(\tau)d\tau \geq \mu, \quad \forall t \geq 0 \quad (10)$$

It is an elementary but important fact (see [15]) that if $t \mapsto a(t)$ is persistently exciting then

$$e^{-\int_{t_1}^t a(\tau)d\tau} \leq \bar{k}e^{-k(t-t_1)}, \quad \forall t \geq t_1 \geq 0 \quad (11)$$

where $\bar{k} = e^\mu$ and $k = \mu/T$.

Example 1: Consider a PE signal a with $T = 3$ and $\mu = 1$ wich is shown in the Figure (2). The signal a satisfies the property (11) with $\bar{k} = 2.78$ and $k = 0.33$, see Figure (3).

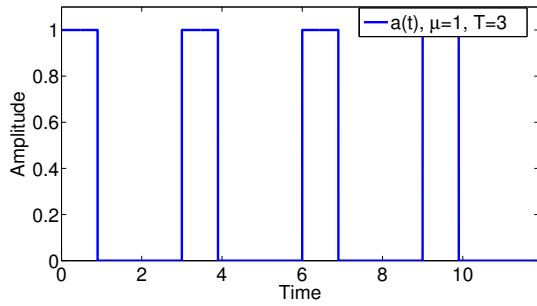


Fig. 2. Persistently exciting signal a with $T = 3$ and $\mu = 1$.

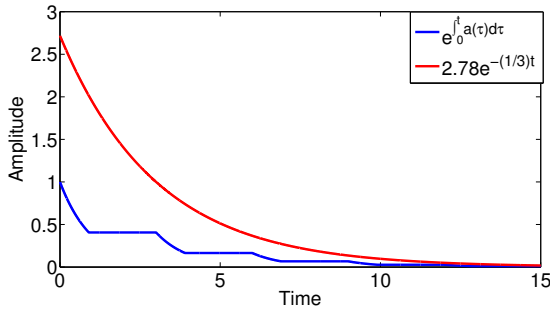


Fig. 3. Property of a persistently exciting signal.

We are ready to present our main result.

Theorem 1: If

$$|a_{N1}|_{\infty} := \sup_{t \geq 0} |a_{N1}(t)|$$

is sufficiently small, the system (4) reaches consensus uniformly in the initial times if and only if a_{ii} is persistently exciting.

A quantification of “sufficiently small” is given in Lemma 2.

To prove the theorem we start by observing that the matrix $A(t)$ in (6) may be partitioned as

$$A(t) = A_1(t) + A_2(t) \quad (12)$$

with

$$A_1(t) := \begin{bmatrix} -a_{12}(t) & a_{23}(t) & 0 & \cdots & 0 \\ 0 & -a_{23}(t) & a_{34}(t) & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -a_{N-2N-1}(t) & a_{N-1N}(t) \\ 0 & 0 & 0 & \cdots & -a_{N-1N}(t) \end{bmatrix} \quad (13)$$

and

$$A_2(t) := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -a_{N1}(t) & -a_{N1}(t) & -a_{N1}(t) & \cdots & -a_{N1}(t) \end{bmatrix} \quad (14)$$

Therefore, (6) is equivalent to the feedback-interconnected system

$$\Sigma : \begin{cases} \Sigma_1 : \begin{cases} \dot{z} = A_1(t)z + y_2 \\ y_1 = z \end{cases} \\ \Sigma_2 : y_2 = A_2(t)y_1 \end{cases} \quad (15)$$

—see the illustration in Figure 4, below.

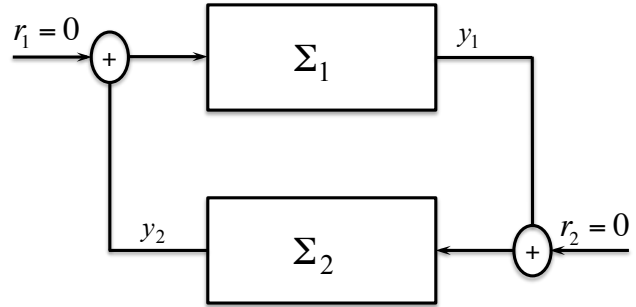


Fig. 4. Interconnected system

The interest of this observation is that the stability conditions for (15), hence for (6), may be derived by computing the norms of the systems Σ_1 and Σ_2 and invoking the *small gain theorem* —see e.g. [16]. Indeed, the system Σ is a particular case of that covered by the small gain theorem with the inputs $r_1(t) = 0$ and $r_2(t) = 0$ for every nonnegative t —see Figure 4.

In other words, in view of (12), the system (6) may be studied as a perturbed system with nominal dynamics

$$\Sigma_1 : \dot{z} = A_1(t)z \quad (16)$$

and perturbation (output injection) $A_2(t)y_1$. Moreover, note that the perturbation only depends on the interconnection function $a_{N1}(t)$ hence, we shall establish exponential stability under the condition that the $|\cdot|_{\infty}$ norm of $a_{N1}(t)$ is sufficiently small and the origin of (16) is exponentially stable. The stability of system (16) is given by the following result.

Lemma 1: Let

$$\dot{\Phi}(t, t_o) = A_1(t)\Phi(t, t_o), \quad \Phi(t_o, t_o) = I_{N-1} \quad \forall t \geq t_o > 0 \quad (17)$$

where

$$A_1(t) := \begin{bmatrix} -a_{12}(t) & a_{23}(t) & 0 & \cdots & 0 \\ 0 & -a_{23}(t) & a_{34}(t) & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -a_{N-2N-1}(t) & a_{N-1N}(t) \\ 0 & 0 & 0 & \cdots & -a_{N-1N}(t) \end{bmatrix} \quad (18)$$

Assume that, for every $i = 1, \dots, N - 1$, $a_{ii+1}(t)$ is a bounded persistently exciting signal. Then, there exist $\bar{\alpha} > 0$, $\alpha > 0$ such that

$$\|\Phi(t, t_o)\| \leq \bar{\alpha}e^{-\alpha(t-t_o)} \quad \forall t \geq t_o \geq 0. \quad (19)$$

Proof: Note that the solution of the differential equation (17) is given by $\Phi(t, t_o) = [\phi_{ij}(t, t_o)]$ where

$$\phi_{ij}(t, t_o) = \begin{cases} 0 & i > j \\ e^{-\int_{t_o}^t a_{ii+1}(s)ds} & i = j \\ \int_{t_o}^t \phi_{ii}(t, s) a_{i+1, i+2}(s) \phi_{i+1, j}(s, t_o) ds & i < j \end{cases} \quad (20)$$

We show that every element of $\Phi(t, t_o)$ is bounded by an exponential function. Taking the absolute value of (20) and using the property (11) for its diagonal entries, we have

$$\begin{aligned} |\phi_{ij}(t, t_o)| &= 0 & i > j \\ |\phi_{ii}(t, t_o)| &\leq \bar{k}_i e^{-k_i(t-t_o)} & i = j \\ |\phi_{ij}(t, t_o)| &\leq \int_{t_o}^t |\phi_{ii}(t; s)| |a_{i+1, i+2}(s)| |\phi_{i+1, j}(s; t_o)| ds & i < j \end{aligned} \quad (21)$$

For each $j = i + 1$ such that $i < N - 1$ the integral (20) depends on ϕ_{ii} and $\phi_{i+1, i+1}$ which are bounded by $\bar{k}_i e^{-k_i(t-t_o)}$ and $\bar{k}_{i+1} e^{-k_{i+1}(t-t_o)}$, respectively. Consequently,

$$|\phi_{ij}(t, t_o)| \leq \bar{k}_i \bar{k}_j |a_{i+1, i+2}|_\infty \left[\frac{1}{|k_i - k_j|} e^{-\min\{k_j, k_i\}(t-t_o)} \right]$$

where by assumption, $|a_{i+1, i+2}|_\infty$ is bounded. Thus, all elements of $\Phi(t, t_o)$ are bounded in norm by a decaying exponential. ■

Now, we can compute the input-output gain of the system Σ_1 as follows. The norm of the output of Σ_1 is given by

$$\|y_1(t)\| \leq \|\Phi(t, t_o)\| \|z(t_o)\| + \int_{t_o}^t \|\Phi(t, \tau)\| \|y_2(\tau)\| d\tau.$$

By Lemma 1 we have $\|\Phi(t, t_o)\| \leq \bar{\alpha} e^{-\alpha(t-t_o)}$, then

$$\|y_1(t)\| \leq \bar{\alpha} e^{-\alpha(t-t_o)} \|z(t_o)\| + \int_{t_o}^t \bar{\alpha} e^{-\alpha(t-t_o)} \|y_2(\tau)\| d\tau.$$

Solving the integral and taking the initial condition for z equals to zero, we obtain

$$\|y_1(t)\| \leq \frac{\bar{\alpha}}{\alpha} (1 - e^{-\alpha(t-t_o)}) \|y_2\|_\infty.$$

Thus,

$$\|\Sigma_1\|_\infty \leq \frac{\bar{\alpha}}{\alpha}. \quad (22)$$

On the other hand, the norm of the output of Σ_2 is

$$\|y_2\|_\infty \leq \|A_2(t)\|_\infty \|y_1\|_\infty.$$

Hence,

$$\|\Sigma_2\|_\infty \leq \|A_2(t)\|_\infty \leq (N-1) |a_{N1}|_\infty. \quad (23)$$

Finally, the proof of the theorem is completed by the following lemma, which establishes the stability of the

feedback interconnected system (4) based on the computed input/output gains.

Lemma 2: Consider the interconnected system

$$\Sigma : \begin{cases} \Sigma_1 & : \begin{cases} \dot{z} &= A_1(t)z + y_2 \\ y_1 &= z \end{cases} \\ \Sigma_2 & : y_2 = A_2(t)y_1 \end{cases} \quad (24)$$

Under the assumptions of Lemma 1 the interconnected system (24) is stable if $|a_{N1}|_\infty \leq \frac{\alpha}{(N-1)\bar{\alpha}}$. In particular, all components of $x(t)$ converge to a common value as $t \rightarrow \infty$.

Proof: We apply the small gain theorem to the interconnected system (24) using the upper bounds (22) and (23) of $\|\Sigma_1\|$ and $\|\Sigma_2\|$, respectively, this is

$$\|\Sigma_1\| \|\Sigma_2\| \leq (N-1) |a_{N1}|_\infty \frac{\bar{\alpha}}{\alpha} < 1. \quad (25)$$

It follows that

$$|a_{N1}|_\infty < \frac{\alpha}{\bar{\alpha}} \left(\frac{1}{N-1} \right) \quad (26)$$

is a sufficient condition for the asymptotically stability of (24). ■

IV. EXAMPLE AND SIMULATION RESULTS

For the sake of illustration, let us study a system with four interconnected agents (see. Figure 5).

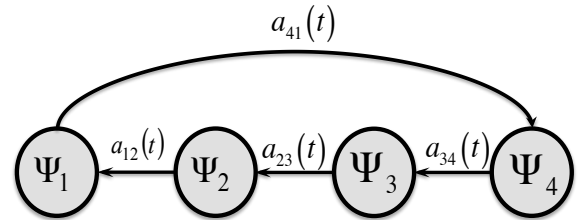


Fig. 5. Ring topology with four interconnected agents

The information exchange among agents is represented by the persistently exciting signals a_{12} , a_{23} and a_{34} shown in Figure 6 with parameters in Table 1.

TABLE I
PARAMETERS T_i AND μ_i CORRESPONDING TO THE SIGNAL a_{ii+1} ,
 $i = 1, 2, 3$.

	T_i	μ_i
$a_{12}(t)$	2	1
$a_{23}(t)$	3	0.3
$a_{34}(t)$	2.5	1.25

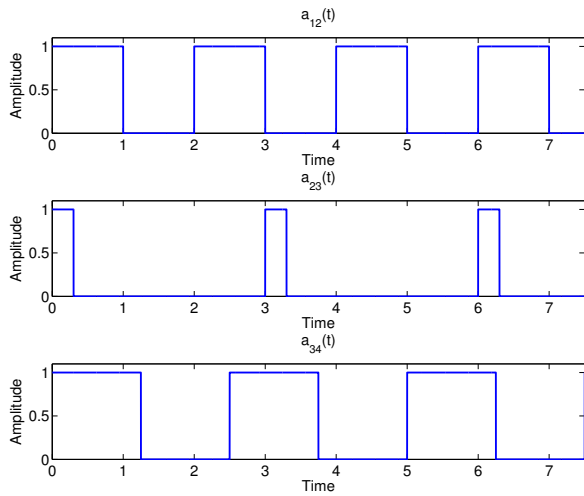


Fig. 6. Persistently exciting signals $a_{12}(t)$ (top), $a_{23}(t)$ (middle) and $a_{34}(t)$ (bottom).

For this example, the system $\dot{z} = A(t)z(t)$ is partitioned into two matrices A_1 and A_2 defined as

$$A_1(t) = \begin{bmatrix} -a_{12}(t) & a_{23}(t) & 0 \\ 0 & -a_{23}(t) & a_{34}(t) \\ 0 & 0 & -a_{34}(t) \end{bmatrix}$$

and

$$A_2(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_{41}(t) & -a_{41}(t) & -a_{41}(t) \end{bmatrix}$$

To apply Theorem 1, is necessary to compute $\bar{\alpha}$ and α . By Lemma 1 we have

$$\begin{aligned} \bar{\alpha} &= \bar{k}_1 + |a_{34}|_{\infty} \frac{\bar{k}_2 \bar{k}_3}{|\bar{k}_2 - \bar{k}_3|} \\ &\quad + |a_{23}|_{\infty} |a_{34}|_{\infty} \frac{\bar{k}_1 \bar{k}_2 \bar{k}_3}{|\bar{k}_2 - \bar{k}_3| |k_1 - \min(k_2, k_3)|}, \\ \alpha &= \min(k_1, k_2, k_3) \end{aligned}$$

where the values of \bar{k}_i and k_i are shown in Table II.

TABLE II

PARAMETERS \bar{k}_i AND k_i CORRESPONDING TO THE SIGNAL a_{ii+1} , $i = 1, 2, 3$.

	\bar{k}_i	k_i
$a_{12}(t)$	2.71	0.5
$a_{23}(t)$	1.34	0.1
$a_{34}(t)$	3.49	0.5

Therefore, we have

$$\alpha = 13.1285 \quad \text{and} \quad \bar{\alpha} = 47.1147.$$

By applying Theorem 1, we know that the origin of $\dot{z}(t) = A(t)z(t)$ is exponentially stable if

$$|a_{41}(t)| \leq 0.092. \quad (27)$$

To illustrate the feasibility of the results obtained we performed simulations using SIMULINK of MATLAB.

The initial conditions of the agents are $x_1(0) = -1$, $x_2(0) = 3$, $x_3(0) = -2$ and $x_4(0) = -0.5$. The persistently exciting signal a_{41} , satisfying (27), is shown in Figure 7.

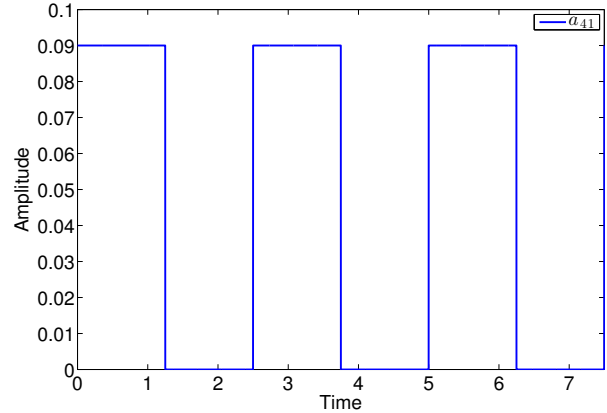


Fig. 7. Persistently exciting signals $a_{41}(t)$.

Figure 8 depicts that all trajectories converge to a common value.

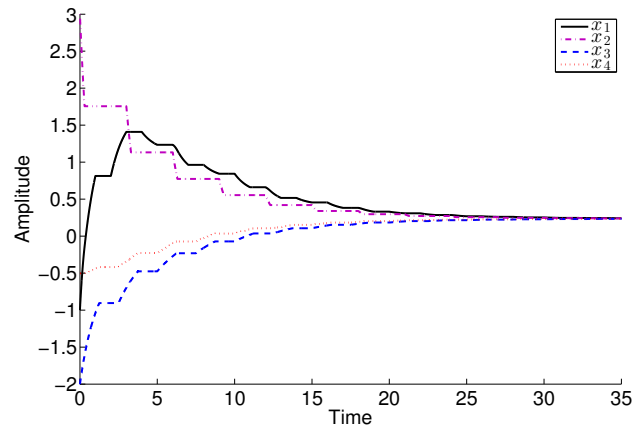


Fig. 8. Trajectories of states $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$.

In [10], it is shown that having a spanning tree topology is a sufficient condition to achieve consensus under the assumption of piecewise constant communication links among the agents. Therefore, by using this result we can also conclude that the foregoing example reaches consensus due to the persistently exciting signals a_{12} , a_{23} , a_{34} and a_{41} satisfy this condition. However, we want to stress that our result is based on the assumptions that the time-varying communication links among the agents are represented by persistently exciting signals and consequently our result is different with respect to the one presented in [10].

V. CONCLUSIONS

In this paper, we introduced a consensus problem for a network of dynamic agents with a ring topology, under the assumption that each interconnection between any pair of agents is represented by bounded and persistently exciting signals. By using the small-gain theorem we obtained that the system reached consensus if the intensity of *one* of the interconnections is relatively small. Simulation results were presented and are consistent with our theoretical results.

ACKNOWLEDGMENT

N. Alvarez benefits from a scholarship by CONACyT, México.

REFERENCES

- [1] J. Carpenter, "Decentralized control of satellite formations," *Int. J. Robust Nonlinear Control*, vol. 12, no. 2, pp. 141–161, 2002.
- [2] A. Sarlette, R. Sepulchre, and N. Leonard, "Cooperative attitude synchronization in satellite swarms: a consensus approach," in *Proc. 17th IFAC Symp. Autom. Control Aerosp.*, pp. 223–228, 2007.
- [3] L. Conteville and E. Panteley, "Linear reformulation of the kuramoto model: Asymptotic mapping and stability properties," *2nd Conference on Communications, Computing and Control Applications (CCCA)*, pp. 1–6, 2012.
- [4] J. Dasdemir and A. Loria, "A simple formation-tracking controller of mobile robots based on a "spanning-tree" communication," *51st IEEE Conf. Decision and Control*, pp. 1796–1801, 2012.
- [5] C. Tomlin, G. Pappas, and S. Sastry, "Conflict resolution for air traffic management: A study in multiagent hybrid systems communication," *IEEE Trans. Autom. Contr.*, vol. 43, no. 4, pp. 509–521, 1998.
- [6] W. Ren and R. W. Beard, *Distributed consensus in multivehicle cooperative control*. Springer verlag, 2005.
- [7] L. Moreau, "Stability of continuous-time distributed consensus algorithms," *43rd IEEE Conf. Decision and Control*, vol. 4, pp. 3998–4003, 2004.
- [8] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Automat. Contr.*, vol. 48, pp. 988–1001, 2003.
- [9] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, and O. Schochet, "Novel type of phase transitions in a system of self-driven particles," *Phys. Rev. Lett.*, vol. 75, pp. 1226–1229, 1995.
- [10] W. Ren and R. W. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Trans. on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.
- [11] R. Olfati-Saber and R. M. Murray, "Consensus protocols for networks of dynamic agents," in *Proc. 2003 Am. Control Conf. pp. 951–956.*, pp. 951–956, 2003.
- [12] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. on Automatic Control*, vol. 49, pp. 1520–1533, 2004.
- [13] F. Xiao and L. Wang, "State consensus for multi-agent systems with switching topologies and time-variant delays," *International Journal of Control*, vol. 79, no. 10, pp. 1277–1284, 2006.
- [14] B. Anderson, "Exponential stability of linear equations arising in adaptive identification," *IEEE Trans. Autom. Control*, vol. 22, pp. 83–88, 1977.
- [15] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Prentice Hall, 1995.
- [16] H. Khalil, *Nonlinear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 2 ed., 1996.