21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

The i/s/o resolvent set and the i/s/o resolvent matrix of an i/s/o system in continuous time

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***Abstract*— Let $\Sigma = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ be an i/s/o system in continuous time. Here the state space $\mathcal{X}$, the input space $\mathcal{U}$, and the output space $\mathcal{Y}$ are Hilbert spaces, and the generator S of Σ is a closed operator $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$. A continuously differentiable $\mathcal{X}$-valued function x is a classical trajectory of Σ on $[0,\infty)$ with input function u and output function y (both of which are assumed to be continuous) if $\left[\begin{smallmatrix}x(t)\\ u(t)\end{smallmatrix}\right] \in \mathrm{dom}\,(S)$ and $\left[\begin{smallmatrix}\dot{x}(t)\\ y(t)\end{smallmatrix}\right] = S\left[\begin{smallmatrix}x(t)\\ u(t)\end{smallmatrix}\right]$ for all $t \geq 0$. For such a system Σ we define the notions of the resolvent set $\rho(\Sigma)$ and the i/s/o resolvent matrix $\widehat{\mathfrak{S}}$. The i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ of Σ is an analytic 2×2 block operator matrix defined on $\rho(\Sigma)$, and intuitively it maps the initial state x_0 at time zero and the restriction to $\rho(\Sigma)$ of the formal Laplace transform $\hat{u}$ of the input u into the restriction to $\rho(\Sigma)$ of the formal Laplace transforms $\hat{x}$ and $\hat{y}$ of the state x and the output y. The i/s/o resolvent matrix is a fundamental tool in the frequency domain analysis of Σ, and it makes it possible to give natural extensions of many significant notions in the theory of well-posed i/s/o systems to the class of possibly non-well-posed i/s/o systems with a nonempty resolvent set. Examples of such notions which can be extended are controllability, observability, minimality, restrictions, projections, compressions, intertwinements, similarities, and pseudo-similarities.**

***MSC 2010*— 47A10, 47A48, 93C25**

***Index Terms*— I/s/o resolvent set, i/s/o resolvent matrix, frequency domain trajectories, restriction, projection, compression, dilation, intertwinement**

I. The Resolvent Set and the Resolvent of an Operator

If A is a closed linear operator in a Hilbert (or Banach) space $\mathcal{X}$, then the resolvent set of A consists of those points $\lambda \in C$ for which $(\lambda - A)$ has a bounded, everywhere defined inverse, and this inverse $(\lambda - A)^{-1}$ is called the resolvent of A. One way to motivate this definition is the following: Consider the linear stationary dynamical system

$$\Sigma\colon \begin{cases} x(t) \in \mathrm{dom}\,(A)\,, \\ \dot{x}(t) = Ax(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \tag{1}$$

with no input and no output. We call x a classical trajectory of Σ if $x \in C^1(\mathbb{R}^+;\mathcal{X})$ and x satisfies (1). If x is Laplace transformable, then by taking Laplace transforms in (1) we get

$$\lambda\hat{x}(\lambda) - x_0 = A\hat{x}(\lambda), \tag{2}$$

for all $\lambda \in \mathbb{C}$ for which the Laplace transform converges. Clearly $\lambda \in \rho(A)$ if and only if for every $x_0 \in \mathcal{X}$ the equation (2) has a unique solution $\hat{x}(\lambda)$ which depends continuously on $x(0)$, and if $\lambda \in \rho(A)$, then $\hat{x}(\lambda) = (\lambda - A)^{-1}x_0$.

The same argument can be used to define the resolvent set and the resolvent of a multi-valued closed linear operator A in $\mathcal{X}$ (also called a relation in $\mathcal{X}$). In this case we rewrite (1) into

$$\Sigma\colon \begin{cases} x(t) \in \mathrm{dom}\,(A)\,, \\ \dot{x}(t) \in Ax(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \tag{3}$$

and (2) is replaced by

$$\lambda\hat{x}(\lambda) - x_0 \in A\hat{x}(\lambda). \tag{4}$$

Also in this case we say that λ belongs to the resolvent set $\rho(A)$ of A if for every $x_0 \in \mathcal{X}$ the equation (4) has a unique solution $\hat{x}(\lambda)$ which depends continuously on x_0, and we define the resolvent of A evaluated at $\lambda \in \rho(A)$ to be the bounded linear operator which maps x_0 into $\hat{x}(\lambda)$. This resolvent is still denoted by $(\lambda - A)^{-1}$ also in the case where A is multi-valued (but of course $(\lambda - A)^{-1}$ is single-valued and bounded).

II. The I/S/O Resolvent Set and the I/S/O Resolvent Matrix of an I/S/O System

The same argument can be extended to the case of a linear stationary dynamical system with nontrivial inputs and outputs. This time the time domain dynamics is described by an equation of the type

$$\Sigma\colon \begin{cases} \left[\begin{smallmatrix}x(t)\\ u(t)\end{smallmatrix}\right] \in \mathrm{dom}\,(S)\,, \\ \begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} = S\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{5}$$

Here x is a continuously differentiable function on $\mathbb{R}^+$ with values in the Hilbert space $\mathcal{X}$ (the state space), u and y are continuous functions on $\mathbb{R}^+$ with values in the Hilbert spaces $\mathcal{U}$ (the input space) and $\mathcal{Y}$ (the output space), respectively, and S is assumed to be a closed operator $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ with dense domain. We call such a system an *i/s/o system*, and denote it by $\Sigma = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$. It is also possible to study the multi-valued case, where the operator S in (5) is replaced by a closed multi-valued operator S whose domain need not be dense in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$, and the equation (5) is replaced by the

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ISBN: 978-90-367-6321-9

inclusion

$$\Sigma\colon \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}\,(S)\,, \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{6}$$

We call a system of the this type an *i/s/o pseudo-system*.

For the purpose of the following discussion there is no significant difference between the i/s/o system (5) and the i/s/o pseudo-system (6), so we may as well start by discussing (6) instead of (5). By a classical trajectory of (6) we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ where $x \in C^1(\mathbb{R}^+;\mathcal{X})$, $u \in C(\mathbb{R}^+;\mathcal{U})$, and $y \in C(\mathbb{R}^+;\mathcal{Y})$ which satisfies (6). If x, u, and y are Laplace transformable, then it follows from (6) (since we assume S to be closed) that the Laplace transforms $\hat{x}$, $\hat{u}$, and $\hat{y}$ of x, u, and y satisfy

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} \in S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}. \tag{7}$$

Definition 1. 1) A point $\lambda \in \mathbb{C}$ belongs to the *i/s/o resolvent set* of S if for every $x_0 \in \mathcal{X}$ and for every $\hat{u}(\lambda) \in \mathcal{U}$ there is a unique pair of vectors $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ satisfying (6), and $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$ depends continuously on $\left[\begin{smallmatrix} x_0 \\ \hat{u}(\lambda) \end{smallmatrix}\right]$. This set is alternatively called the *resolvent set of* Σ (where Σ is the i/s/o system defined by (6)) and denoted by $\rho_{\mathrm{iso}}(S)$ or by $\rho(\Sigma)$.

2) For each $\lambda \in \rho(\Sigma)$ we define the *i/s/o resolvent matrix of* Σ *(or of* S*) at* λ to be the bounded linear operator which maps $\left[\begin{smallmatrix} x_0 \\ \hat{u}(\lambda) \end{smallmatrix}\right]$ into $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$.

The above definition is both natural and simple, and it may be surprising that in the case where S is single-valued and densely defined the above definition is equivalent to the condition that S is a so called "operator node" in the sense of [5].

Definition 2 ([5, Definition 4.7.2])**.** By an *operator node* on a triple of Hilbert spaces $(\mathcal{X},\mathcal{U},\mathcal{Y})$ we mean a linear operator $S\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ with the following properties. We let $P_{\mathcal{X}}$ be the coordinate map which maps $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ into x, denote $\mathrm{dom}\,(A) = \left\{x \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathrm{dom}\,(S)\right\}$, define the *main operator* $A\colon \mathrm{dom}\,(A) \to \mathcal{X}$ of S by $Ax = P_{\mathcal{X}} S \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$, and require the following conditions to hold:

1) S is closed.
2) $\mathrm{dom}\,(A)$ is dense in $\mathcal{X}$ and $\rho(A) \neq \emptyset$.
3) $P_{\mathcal{X}} S$ can be extended to a bounded linear operator $\begin{bmatrix} A_{-1} & B \end{bmatrix} : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \mathcal{X}_{-1}$, where $\mathcal{X}_{-1}$ is the so called *extrapolation space* induced by A (i.e., the completion of $\mathcal{X}$ with respect to the norm $\|x\|_{\mathcal{X}_{-1}} = \|(\alpha - A)^{-1}x\|_{\mathcal{X}}$ where α is some fixed point in $\rho(A)$).
4) $\mathrm{dom}\,(S) = \left\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right] \;\middle|\; A_{-1}x + Bu \in \mathcal{X}\right\}$.

Theorem 3. *An operator* $S\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ *is an operator node in the sense of Definition 2 if and only if* $\mathrm{dom}\,(S)$ *is dense in* $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ *and* $\rho_{\mathrm{iso}}(S) \neq \emptyset$*. Moreover, if* $\rho_{\mathrm{iso}}(S) \neq \emptyset$*, then* $\rho_{\mathrm{iso}}(S) = \rho(A)$ *where* A *is the main operator of* S*.*

The proof of this theorem (and the proofs of all other results that will be mentioned below) is found in [1].

Definition 4. 1) By a *regular i/s/o system* $\Sigma = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ we mean an i/s/o system of the type (5) where the (single-valued) generating operator S has dense domain in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ and a nonempty i/s/o resolvent set.

2) By a *regular i/s/o pseudo-system* $\Sigma = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ we mean an i/s/o pseudo-system of the type (6) where the (possibly multi-valued) generating operator S has a nonempty i/s/o resolvent set (but its domain need not be dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$).

For each $\lambda \in \rho_{\mathrm{iso}}(S)$ the i/s/o resolvent matrix $\widehat{\mathfrak{S}}(\lambda)$ of S is a bounded linear operator $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$, and hence $\widehat{\mathfrak{S}}(\lambda)$ has a block matrix representation $\widehat{\mathfrak{S}}(\lambda) = \left[\begin{smallmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{smallmatrix}\right]$.

Definition 5. The components $\widehat{\mathfrak{A}}$, $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{C}}$, and $\widehat{\mathfrak{D}}$ in the block matrix representation $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ of the i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ of Σ (all of which are analytic operator-valued functions defined on $\rho_{\mathrm{iso}}(S)$) are called as follows:

1) $\widehat{\mathfrak{A}}$ is the *s/s (state/state) resolvent function* of Σ,
2) $\widehat{\mathfrak{B}}$ is the *i/s (input/state) resolvent function* of Σ,
3) $\widehat{\mathfrak{C}}$ is the *s/o (state/output) resolvent function* of Σ,
4) $\widehat{\mathfrak{D}}$ is the *i/o (input/output) resolvent function* of Σ,

Out of these functions the s/s resolvent function $\widehat{\mathfrak{A}}$ is the (standard) resolvent of the main operator A of S (both in the single-valued case and the multi-valued case). In different mathematical communities the i/o resolvent function $\widehat{\mathfrak{D}}$ is know as the under different names, such as "transfer function", or "characteristic function", or "Weyl function", and in operator theory the i/s resolvent function $\widehat{\mathfrak{B}}$ is sometimes called the Γ-field.

The s/s resolvent function $\widehat{\mathfrak{A}}$ can be used to check if a regular i/s/o pseudo-system $\Sigma = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ is actually a regular i/s/o system.

Lemma 6. *Let* $\Sigma = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ *be a regular i/s/o pseudo-system with main operator* A *and s/s resolvent function* $\widehat{\mathfrak{A}}$*. Then*

1) *The following conditions are equivalent:*
   a) S *is single-valued;*
   b) A *is single-valued;*
   c) $\widehat{\mathfrak{A}}(\lambda)$ *is injective for some* $\lambda \in \rho_{\mathrm{iso}}(S)$ *(or equivalently, for all* $\lambda \in \rho_{\mathrm{iso}}(S)$*).*
2) *Also the following conditions are equivalent:*
   a) $\mathrm{dom}\,(S)$ *is dense in* $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$*;*
   b) $\mathrm{dom}\,(A)$ *is dense in* $\mathcal{X}$*;*
   c) $\widehat{\mathfrak{A}}(\lambda)$ *has dense range for some* $\lambda \in \rho_{\mathrm{iso}}(S)$ *(or equivalently, for all* $\lambda \in \rho_{\mathrm{iso}}(S)$*).*

In particular, Σ *is a regular i/s/o system if and only if* A *is single-valued and has dense domain, or equivalently, if and only if* $\widehat{\mathfrak{A}}(\lambda)$ *is injective and has dense range for some* $\lambda \in \rho_{\mathrm{iso}}(S)$ *(or equivalently, for all* $\lambda \in \rho_{\mathrm{iso}}(S)$*).*

III. Input/State/Output Pseudo-Resolvents

Lemma 7. *The i/s/o resolvent matrix* $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ *of a regular i/s/o pseudo-system* $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *satisfies the* i/s/o resolvent identity

$$\widehat{\mathfrak{S}}(\lambda) - \widehat{\mathfrak{S}}(\mu) = (\mu - \lambda) \begin{bmatrix} \widehat{\mathfrak{A}}(\mu) \\ \widehat{\mathfrak{C}}(\mu) \end{bmatrix} \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \end{bmatrix} \tag{8}$$

for all μ, $\lambda \in \rho_{\mathrm{iso}}(S)$.

Motivated by Lemma 7 we make the following definition.

Definition 8. Let Ω be an open subset of the complex plane $\mathbb{C}$. An analytic $\mathcal{L}(\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right])$-valued function $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ defined in Ω is called an *i/s/o pseudo-resolvent in* $(\mathcal{X}, \mathcal{U}, \mathcal{Y}; \Omega)$ if it satisfies the identity (8) for all μ, $\lambda \in \Omega$.

Thus, the i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ of a regular i/s/o pseudo-system $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is an i/s/o pseudo-resolvent in $\rho(\Sigma)$.

In [3] Mark Opmeer makes systematic use of the notion of an i/s/o pseudo-resolvent, but instead of calling $\widehat{\mathfrak{S}}$ an i/s/o pseudo-resolvent he calls $\widehat{\mathfrak{S}}$ a "resolvent linear system", and calls $\widehat{\mathfrak{A}}$ the "pseudo-resolvent", $\widehat{\mathfrak{B}}$ the "incoming wave function", $\widehat{\mathfrak{C}}$ the "outgoing wave function", and $\mathfrak{D}$ the "characteristic function" of the resolvent linear system $\widehat{\mathfrak{S}}$. In the same article he also investigates what can be said about time domain trajectories (in the distribution sense) of resolvent linear systems satisfying some additional conditions. One of these additional set of conditions is that Ω should contain some right-half plane and that $\widehat{\mathfrak{S}}$ should satisfy a polynomial growth bound in this right-half plane.

The converse of Lemma 7 is also true in the following form.

Theorem 9. *Let* Ω *be an open subset of the complex plane* $\mathbb{C}$. *Then every i/s/o pseudo-resolvent* $\widehat{\mathfrak{S}}$ *in* $(\mathcal{X}, \mathcal{U}, \mathcal{Y}; \Omega)$ *is the restriction to* Ω *of the i/s/o resolvent of some i/s/o pseudo-system* $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *satisfying* $\rho(\Sigma) \supset \Omega$. *The i/s/o pseudo-system* Σ *is determined uniquely by* $\widehat{\mathfrak{S}}$, *and* $\widehat{\mathfrak{S}}$ *has a unique extension to* $\rho(\Sigma)$. *This extension is maximal in the sense that* $\widehat{\mathfrak{S}}$ *cannot be extended to an i/s/o pseudo-resolvent on any larger open subset of* $\mathbb{C}$.

This result is well-known in the case where the system has no input and no output (so that S is equal to its main operator A), and where $\widehat{\mathfrak{A}}(\lambda)$ is injective and has dense range for some $\lambda \in \Omega$; see, e.g., [4, Theorem 9.3, p. 36]. A multi-valued version of this theorem, still with no input and output, is found in [2, Remark, pp. 148–149].

Theorem 9 can be used in the following way: If we start from some i/s/o system or pseudo-system Σ, and modify the i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ of Σ by, e.g., restricting it to some subspace or projecting it onto some other subspace, then as long as the resulting block matrix function remains an i/s/o pseudo-resolvent it follows from Theorem 9 that this new i/s/o pseudo-resolvent is the i/s/o resolvent matrix of some new i/s/o pseudo-system (possibly with different state, input, or output spaces). This can be used, e.g., in the study of frequency domain restrictions, projections, compressions, dilations, and intertwinements of regular i/s/o pseudo-systems, as will be explained in more detail below.

IV. Frequency Domain Trajectories of I/S/O pseudo-systems

Equations (5) and (6) describe the *time domain evolution* of an i/s/o system or pseudo-system Σ. From the time domain inclusion (6) we get the frequency domain inclusion (7) by taking (formal) Laplace transforms as explained above. It is possible to introduce the notion of a *frequency domain trajectory* by replacing (6) by (7), and at the same time replacing the time domain interval $\mathbb{R}^+$ by some open subset Ω of the complex (frequency domain) plane $\mathbb{C}$.

Definition 10. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o pseudo-system, and let Ω be an open subset of $\mathbb{C}$. By a (frequency domain) Ω-trajectory of Σ with initial state $x_0 \in \mathcal{X}$ we mean a triple of analytic functions $\left[\begin{smallmatrix} \hat{x} \\ \hat{u} \\ \hat{y} \end{smallmatrix}\right]$ defined in Ω with values in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$ which satisfy (7) for all $\lambda \in \Omega$.

It follows immediately from the above definition that if $\Omega' \subset \Omega$, then the restriction to Ω' of an Ω-trajectory of Σ with initial state x_0 is an Ω'-trajectory of Σ with the same initial state.

For a regular i/s/o pseudo-system the question of existence of Ω-trajectories has a natural answer: If Ω is an arbitrary open subset of $\rho(\Sigma)$, then for every $x_0 \in \mathcal{X}$ and for every analytic function $\hat{u}$ in Ω the i/s/o pseudo-system Σ has a unique Ω-trajectory $\left[\begin{smallmatrix} \hat{x} \\ \hat{u} \\ \hat{y} \end{smallmatrix}\right]$ with initial state x_0 and input function $\hat{u}$, and this trajectory is given by

$$\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \widehat{\mathfrak{S}}(\lambda) \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda)x_0 + \widehat{\mathfrak{B}}(\lambda)\hat{u}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda)x_0 + \widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda) \end{bmatrix} \tag{9}$$

for all $\lambda \in \Omega$. If $\rho(\Sigma)$ is connected, then it turns out that the notion of an Ω-trajectory of Σ is independent of the choice of $\Omega \subset \rho(\Sigma)$ in the sense that if Ω_1, $\Omega_2 \subset \rho(\Sigma)$, and if $\left[\begin{smallmatrix} \hat{x} \\ \hat{u} \\ \hat{y} \end{smallmatrix}\right]$ is analytic in $\Omega_1 \cup \Omega_2$, then the restriction of $\left[\begin{smallmatrix} \hat{x} \\ \hat{u} \\ \hat{y} \end{smallmatrix}\right]$ to Ω_1 is an Ω_1-trajectory of Σ with initial state x_0 if and only if the restriction of $\left[\begin{smallmatrix} \hat{x} \\ \hat{u} \\ \hat{y} \end{smallmatrix}\right]$ to Ω_2 is an Ω_2-trajectory of Σ with the same initial state.

V. Frequency Domain Restrictions, Projections, and Compressions of I/S/O Pseudo-Systems

There is a rich theory about restrictions, projections, compressions, and dilations for discrete time i/s/o systems of the form

$$\Sigma\colon \begin{cases} x(n+1) = Ax(n) + Bu(n), \\ \quad\;\; y(n) = Cx(n) + Du(n), \end{cases} \quad n \in \mathbb{Z}^+, \tag{10}$$

where $x(n) \in \mathcal{X}$, $u(n) \in \mathcal{U}$, and $y(n) \in \mathcal{Y}$, and A, B, C, and D are bounded linear operators. Analogous (but more technical) results also exist for linear stationary well-posed i/s/o systems in continuous time in the sense of [5]; see, e.g., [5] and [1] for details. However, the non-well-posed continuous time theory is significantly more difficult, and not many results are available for that case. In the non-well-posed case it is even far from obvious to what extent the notions mentioned above can be described in terms of either classical or generalised time domain trajectories of the system. One solution to this problem is to replace the time domain by the frequency domain, and to study restrictions, projections, compressions, and dilations in the frequency domain. There all these notions have natural interpretations in terms of frequency domain trajectories of the system.

Definition 11. Let $\Sigma_1 = (S_1, \mathcal{X}_1; \mathcal{U}, \mathcal{Y})$ and $\Sigma_2 = (S_2, \mathcal{X}_2, \mathcal{U}, \mathcal{Y})$ be two regular i/s/o pseudo-systems (with the same input and output spaces), and suppose that $\mathcal{X}_1$ is a closed subspace of $\mathcal{X}_2$ with a complement $\mathcal{Z}_1$ in $\mathcal{X}_2$, so that $\mathcal{X}_2 = \mathcal{X}_1 \dotplus \mathcal{Z}_1$. Let Ω be an open subset of $\rho(\Sigma_1) \cap \rho(\Sigma_2)$.

1) Σ_1 is the Ω-*restriction* of Σ_2 to $\mathcal{X}_1$ if every Ω-trajectory of Σ_1 with initial state $x_0 \in \mathcal{X}_1$ is also an Ω-trajectory of Σ_2 with the same initial state. In this case we also call Σ_2 an *extension* of Σ_1.
2) Σ_1 is the Ω-*projection* of Σ_2 onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ if $\begin{bmatrix} P_{\mathcal{X}_1}^{\mathcal{Z}_1}\hat{x} \\ \hat{u} \\ \hat{y} \end{bmatrix}$ is an Ω-trajectory of Σ_1 with initial state $P_{\mathcal{X}_1}^{\mathcal{Z}_1} x_0$ whenever $\begin{bmatrix} \hat{x} \\ \hat{u} \\ \hat{y} \end{bmatrix}$ is an Ω-trajectory of Σ_2 with initial state $x_0 \in \mathcal{X}_2$.
3) Σ_1 is the Ω-*compression* of Σ_2 onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ if the following condition holds: If $\begin{bmatrix} \hat{x} \\ \hat{u} \\ \hat{y} \end{bmatrix}$ is an Ω-trajectory of Σ_2 with initial state $x_0 \in \mathcal{X}_1$, then $\begin{bmatrix} P_{\mathcal{X}_1}^{\mathcal{Z}_1}\hat{x} \\ \hat{u} \\ \hat{y} \end{bmatrix}$ is an Ω-trajectory of Σ_1 with the same initial state. In this case we also call Σ_2 a *dilation* of Σ_1 along $\mathcal{Z}_1$.

Clearly, every Ω-restriction and every Ω-projection is also an Ω-compression.

The notions that we have defined above can be expressed by means of formulas involving the i/s/o resolvent matrices of Σ_1 and Σ_2. For example, Ω-compressions can be characterised as follows:

Lemma 12. *Let $\Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$ be two regular i/s/o pseudo-systems with i/s/o resolvent matrices $\begin{bmatrix} \widehat{\mathfrak{A}}_i & \widehat{\mathfrak{B}}_i \\ \widehat{\mathfrak{C}}_i & \widehat{\mathfrak{D}}_i \end{bmatrix}$ of Σ_i, $i = 1, 2$, and with $\mathcal{X}_2 = \mathcal{X}_1 \dotplus \mathcal{Z}_1$, and let Ω be an open subset of $\rho(\Sigma_1) \cap \rho(\Sigma_2)$. Then Σ_1 is the Ω-compression of Σ_2 onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ if and only if the following four conditions hold for all $\lambda \in \Omega$:*

$$\begin{aligned} \widehat{\mathfrak{A}}_1(\lambda) &= P_{\mathcal{X}_1}^{\mathcal{Z}_1} \widehat{\mathfrak{A}}_2(\lambda)|_{\mathcal{X}_1}, & \widehat{\mathfrak{B}}_1(\lambda) &= P_{\mathcal{X}_1}^{\mathcal{Z}_1} \widehat{\mathfrak{B}}_2(\lambda), \\ \widehat{\mathfrak{C}}_1(\lambda) &= \widehat{\mathfrak{C}}_2(\lambda)|_{\mathcal{X}_1}, & \widehat{\mathfrak{D}}_1(\lambda) &= \widehat{\mathfrak{D}}_2(\lambda). \end{aligned} \tag{11}$$

If Ω is connected, then the same statement remains true if "for all $\lambda \in \Omega$*" is replaced by "for* some $\lambda \in \Omega$*".*

Analogous results are also true for Ω-restrictions and Ω-projections.

Definition 13. Let $\Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$ be two regular i/s/o pseudo-systems with i/o resolvent functions $\widehat{\mathfrak{D}}_i$, and let Ω be an open subset of $\rho(\Sigma_1) \cap \rho(\Sigma_2)$. We say that Σ_1 and Σ_2 are *externally Ω-equivalent* if $\widehat{\mathfrak{D}}_1(\lambda) = \widehat{\mathfrak{D}}_2(\lambda)$ for all $\lambda \in \Omega$.

This condition can be reformulated in terms of Ω-trajectories of Σ_1 and Σ_2 as follows: Σ_1 and Σ_2 are externally Ω-equivalent if and only if they satisfy the following condition: If $\begin{bmatrix} \hat{x}_i \\ \hat{u}_i \\ \hat{y}_i \end{bmatrix}$ is an Ω-trajectory of Σ_i, $i = 1, 2$, with initial state zero, and if $\hat{u}_1(\lambda) = \hat{u}_2(\lambda)$ for all $\lambda \in \Omega$, then $\hat{y}_1(\lambda) = \hat{y}_2(\lambda)$ for all $\lambda \in \Omega$.

Lemma 14. *If the i/s/o pseudo-system Σ_1 is a Ω-compression of the i/s/o pseudo-system Σ_2, then Σ_1 and Σ_2 are externally Ω-equivalent.*

Definition 15. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular i/s/o pseudo-system with i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}$, and let Ω be an open subset of $\rho(\Sigma)$.

1) The subspace $\mathfrak{R}_\Sigma^\Omega := \bigvee_{\lambda \in \Omega} \operatorname{im}(\widehat{\mathfrak{B}}(\lambda))$ is called the Ω-*reachable subspace* of Σ.
2) The subspace $\mathfrak{U}_\Sigma^\Omega := \bigcap_{\lambda \in \Omega} \ker \widehat{\mathfrak{C}}(\lambda)$ is called the Ω-*unobservable subspace* of Σ.
3) Σ is Ω-*controllable* if $\mathfrak{R}_\Sigma^\Omega = \mathcal{X}$, and Σ is Ω-*observable* if $\mathfrak{U}_\Sigma^\Omega = \{0\}$.

Definition 16. A regular i/s/o pseudo-system Σ is Ω-minimal, where Ω is some open subset of $\rho(\Sigma)$, if Σ does not have any non-trivial Ω-compression (i.e., Σ is not a non-trivial dilation of any other i/s/o pseudo-system).

Theorem 17. *Every regular i/s/o pseudo-system Σ has a minimal Ω-compression, where Ω is an arbitrary open subset of $\rho(\Sigma)$, i.e., there exists a minimal regular i/s/o pseudo-system Σ_1 which is an Ω-compression of Σ. (This compression is not unique in general.)*

Note that we do not claim that every i/s/o system Σ has a minimal Ω-compression that is also an i/s/o system. But it follows from the above result that Σ does have a minimal Ω-compression in the form of an i/s/o pseudo-system, i.e., an Ω-compression were the generating operator S of the Ω-compression is allowed to be multi-valued and to have a non-dense domain. (It is also possible to give additional conditions under which the minimal Ω-compression is, indeed, an i/s/o system.)

Theorem 18. *An i/s/o pseudo-system Σ is Ω-minimal if and only if Σ is both Ω-controllable and Ω-observable, where Ω is an arbitrary open subset of $\rho(\Sigma)$.*

VI. Frequency Domain Intertwinements of I/S/O Pseudo-Systems

All the notions that we discussed in the preceding section, i.e., restrictions, projections, compressions, and dilations, are special cases of the more general notion of *intertwinements* of two i/s/o pseudo-system. This notion is defined as follows.

Definition 19. Let $\Sigma_i = (S_i, \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$, $i = 1, 2$, be two regular i/s/o pseudo-systems (with the same input and output spaces), and let Ω be an open subset of $\rho(\Sigma_1) \cap \rho(\Sigma_2)$. We say that Σ_1 and Σ_2 are Ω-intertwined by the multi-valued operator $R\colon \mathcal{X}_1 \to \mathcal{X}_2$ if the following condition holds: If

$\begin{bmatrix} \hat{x}_i \\ \hat{u}_i \\ \hat{y}_i \end{bmatrix}$ are Ω trajectories of Σ_i with initial states $x_0^i \in \mathcal{X}_i$, $i = 1, 2$, and if $x_0^2 \in R x_0^1$ and $\hat{u}_1(\lambda) = \hat{u}_2(\lambda)$ for all $\lambda \in \Omega$, then $\hat{x}_2(\lambda) \in R x_1(\lambda)$ and $\hat{y}_1(\lambda) = \hat{y}_2(\lambda)$ for all $\lambda \in \Omega$.

(Above we have used the convention that the conditions $x_0^2 \in R x_0^1$ and $\hat{x}_2(\lambda) \in R x_1(\lambda)$ imply that $x_0^1 \in \operatorname{dom}(R)$ and $x_1(\lambda) \in \operatorname{dom}(R)$.)

Lemma 20. *Let $\Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$ be two regular i/s/o pseudo-systems with i/s/o resolvent matrices $\widehat{\mathfrak{S}}_i = \begin{bmatrix} \widehat{\mathfrak{A}}_i & \widehat{\mathfrak{B}}_i \\ \widehat{\mathfrak{C}}_i & \widehat{\mathfrak{D}}_i \end{bmatrix}$, $i = 1, 2$, and let Ω be an open subset of $\rho(\Sigma_1) \cap \rho(\Sigma_2)$. Then Σ_1 and Σ_2 are Ω-intertwined by the multi-valued operator $R\colon \mathcal{X}_1 \to \mathcal{X}_2$ if and only if the following four conditions hold for all $\lambda \in \Omega$:*

1) $\widehat{\mathfrak{A}}_2(\lambda) x_2 \in R \widehat{\mathfrak{A}}_1(\lambda) x_1$ *for all* $x_2 \in R x_1$.
2) $\widehat{\mathfrak{B}}_2(\lambda) u_0 \in R \widehat{\mathfrak{B}}_1(\lambda) u_0$ *for all* $u_0 \in \mathcal{U}$.
3) $\widehat{\mathfrak{C}}_2(\lambda) x_2 = \widehat{\mathfrak{C}}_1(\lambda) x_1$ *for all* $x_2 \in R x_1$.
4) $\widehat{\mathfrak{D}}_2(\lambda) = \widehat{\mathfrak{D}}_1(\lambda)$.

If Ω is connected, then the same statement remains true if "for all $\lambda \in \Omega$*" is replaced by "for* some $\lambda \in \Omega$*".*

Lemma 21. *Let $\Sigma_1 = (S_1, \mathcal{X}_1; \mathcal{U}, \mathcal{Y})$ and $\Sigma_2 = (S_2, \mathcal{X}_2, \mathcal{U}, \mathcal{Y})$ be two regular i/s/o pseudo-systems (with the same input and output spaces), and suppose that $\mathcal{X}_1$ is a closed subspace of $\mathcal{X}_2$ with a complement $\mathcal{Z}_1$ in $\mathcal{X}_2$, so that $\mathcal{X}_2 = \mathcal{X}_1 \dotplus \mathcal{Z}_1$. Let Ω be an open subset of $\rho(\Sigma_1) \cap \rho(\Sigma_2)$.*

1) *Σ_1 is the Ω-restriction of Σ_2 to $\mathcal{X}_1$ if and only if Σ_1 and Σ_2 are intertwined by the embedding operator $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$.*
2) *Σ_1 is the Ω-projection of Σ_2 onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ if and only if Σ_2 and Σ_1 are intertwined by the projection operator $P_{\mathcal{X}_1}^{\mathcal{Z}_1}$.*

The corresponding result for Ω-compressions and Ω-dilations is more complicated to explain (see [1] for details). A particular consequence of that result is the following:

Lemma 22. *All Ω-restrictions, Ω-projections, Ω-compressions, and Ω-dilations can be interpreted as Ω-intertwinements, where the intertwining operator R is closed and single-valued and has closed domain and closed range.*

Theorem 23. *Let $\Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$, $i = 1, 2$, be two regular i/s/o pseudo-systems (with the same input and output spaces) with i/s/o resolvent matrices $\widehat{\mathfrak{S}}_i = \begin{bmatrix} \widehat{\mathfrak{A}}_i & \widehat{\mathfrak{B}}_i \\ \widehat{\mathfrak{C}}_i & \widehat{\mathfrak{D}}_i \end{bmatrix}$, $i = 1, 2$, and let Ω be an open subset of $\rho(\Sigma_1) \cap \rho(\Sigma_2)$.*

1) *Σ_1 and Σ_2 are Ω-intertwined by some multi-valued operator $R\colon \mathcal{X}_1 \to \mathcal{X}_2$ if and only if Σ_1 and Σ_2 are externally Ω-equivalent.*
2) *If Σ_1 and Σ_2 are Ω-intertwined by some multi-valued operator $R\colon \mathcal{X}_1 \to \mathcal{X}_2$, then Σ_1 and Σ_2 are also Ω-intertwined by the closure of R.*

Definition 24. Two regular i/s/o pseudo-systems $\Sigma_i = (S_i, \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$, $i = 1, 2$, are Ω-*pseudo-similar* if they are Ω-intertwined by some (single-valued) injective linear operator with dense domain and dense range.

Lemma 25. *Let $\Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$, $i = 1, 2$, be two minimal regular i/s/o pseudo-systems (with the same input and output spaces) with i/s/o resolvent matrices $\widehat{\mathfrak{S}}_i = \begin{bmatrix} \widehat{\mathfrak{A}}_i & \widehat{\mathfrak{B}}_i \\ \widehat{\mathfrak{C}}_i & \widehat{\mathfrak{D}}_i \end{bmatrix}$, $i = 1, 2$, and let Ω be an open subset of $\rho(\Sigma_1) \cap \rho(\Sigma_2)$. Then Σ_1 and Σ_2 are Ω-pseudo-similar if and only if Σ_1 and Σ_2 are externally Ω-equivalent.*

VII. THE WELL-POSED CASE

Analogous results to those presented above are valid for restrictions, projections, compressions, dilations, intertwinements, and pseudo-similarities for linear stationary well-posed i/s/o systems. In this setting the components of the i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}$ is replaced by the evolution semigroup $\mathfrak{A}$, the input map $\mathfrak{B}$, the output map $\mathfrak{C}$, and the input/output map $\mathfrak{D}$ of the well-posed i/s/o system, and frequency domain Ω-trajectories are replaced by time domain *generalised trajectories* of Σ on $\mathbb{R}^+$. By a generalised trajectory of Σ on $\mathbb{R}^+$ we mean a triple of functions $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$, where x is continuous and u and y belong locally to L^2, which is the limit of a sequence of classical trajectories $\begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix}$ of Σ on $\mathbb{R}^+$ in the sense that $x_n \to x$ in $C(\mathbb{R}^+; \mathcal{X})$, $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{U})$, and $y_n \to y$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{Y})$ as $n \to \infty$. In the well-posed case there is a one-to-one correspondence between the time domain versions of the notions mentioned above and the corresponding frequency domain versions if one chooses the set Ω to be the right half-plane $\{\lambda \in \mathbb{C} \mid \Re\lambda > \omega(\Sigma)\}$, where $\omega(\Sigma)$ is the growth bound of Σ. See [1] for details.

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