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A geometric approach to stability of linear reset systems

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Abstract— **In this paper, a class of linear dynamical systems, called linear reset systems, is studied from a geometric point of view. Their state satisfies a system of linear differential equations (with constant coefficients) but they are provided with a mechanism which resets the state when a certain condition is met.**

In particular, when the dynamical system without reset is stable, sufficient conditions on the reset structure are given, which guarantee asymptotic stability of the corresponding reset system.

I. INTRODUCTION

In order to overcome some limitations and enhance the performance of linear controllers, the so-called Clegg integrator was introduced by Clegg in 1958 [1]: the output of this integrator resets to zero when its input is zero, thus justifying the name *reset controllers* for a large class of control systems which exhibit an analogous behavior.

An improvement of the Clegg integrator was developed by Horowitz and coworkers during the seventies [2], [3] when they introduced and investigated the first-order reset element (FORE).

However, only at the end of the 1990s, thanks to important contributions of Chait, Hollot and coworkers [4], many research groups started working in this field [5], [6], [7]. Finally, one of the first monographs on the subject was published in 2012 [8].

In the meantime, a great number of results were published in different areas as, for instance, impulsive [9], [10], hybrid [11], [12], and switching systems [13], [14], which are strictly related to reset systems. Indeed, reset systems can be considered as particular cases of these general classes of dynamical systems.

One of the main disadvantages of reset controllers, due to the interaction between continuous time and discrete event driven dynamics (which characterizes this class of dynamical systems), is that the reset action may destabilize the system. Therefore, the analysis of stability is at the same time non trivial and of fundamental importance.

The effect of the reset structure on stability is investigated in this paper following a geometric point of view. This leads to an original interpretation of the sufficient conditions for well-posedness and for stability which will be determined.

A thorough analysis of an application of reset control to asymptotic tracking is presented to justify some initial assumptions on the abstract model of linear reset systems and to motivate the study of stability.

II. LINEAR RESET SYSTEMS

The main goal of this paper is to study stability of linear systems with state reset in a general context.

Consider a linear system of differential equations represented by

$$\dot{x} = Ax, \tag{1}$$

whose state, or part of it, is reset by the operator Π whenever the state trajectory hits a given subspace $\mathcal{V}$. This situation is depicted in Figure 1 using a hybrid automaton diagram.

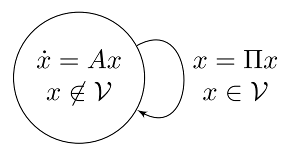

Fig. 1. Model of a linear reset system

So, the reset system may be defined as follows:

$$\begin{cases} \dot{x} = Ax & \text{if } x \notin \mathcal{V}, \\ x^+ = \Pi x & \text{if } x \in \mathcal{V}, \end{cases} \tag{2}$$

where x^+ is the state of the system immediately after reset.

The following assumptions are made on the reset system:

- the state-space has dimension n, i.e., $x \in \mathbb{R}^n$;
- the subspace $\mathcal{V} \subset \mathbb{R}^n$, called *reset* (or *switching*) *space*, has dimension $m < n$ and is the image of a full (column) rank matrix M:
$$\mathcal{V} = \operatorname{img} M = \{My \in \mathbb{R}^n : y \in \mathbb{R}^m\}; \tag{3}$$
- the matrix operator Π is a projection, i.e., it satisfies the equation
$$\Pi^2 = \Pi; \tag{4}$$
- the projection Π has rank k and its image is the subspace $\mathcal{W} \subset \mathbb{R}^n$, called *projection space*:
$$\mathcal{W} = \operatorname{img} \Pi = \{\Pi x \in \mathbb{R}^n : x \in \mathbb{R}^n\}. \tag{5}$$

Moreover, some additional hypotheses are introduced.

1) Since the main goal is to achieve asymptotic stability of the reset system, we assume that this property

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already holds for the system without reset (1), i.e., A is a Hurwitz matrix.

2) The projection Π satisfies condition
$$\Pi\mathcal{I} = \mathcal{I} \tag{6}$$
for some given subspace $\mathcal{I} \subseteq \mathbb{R}^n$.
Observe that when (6) holds, since Π is a projection, its restriction to $\mathcal{I}$ is the identity operator. Thus, equation (6) is equivalent to
$$(I - \Pi)\mathcal{I} = \{0\}. \tag{7}$$

3) If $\mathcal{V} \cap \mathcal{W} \neq \{0\}$, after each reset the system is forced to
   a) follow the continuous-time dynamics (1) and to
   b) wait at least δ time units, for some $\delta > 0$ (called *dwell-time*), before the state can be reset again.

Some commentaries may be useful to understand the meaning of these hypotheses.

Hypothesis 1) is in fact not necessary, since the reset mechanism may also be used to stabilize an unstable linear system, as shown in [4]. However, this papers is concerned with the effects on an already stable system of a reset control which aims at some kind of performance improvement — compare, for instance, Figures 5 and 6 of Example 4 of Section IV.

Hypothesis 2) is just an abstract formalization of a linear constraint, *represented* by the subspace $\mathcal{I} \subseteq \mathbb{R}^n$ in Equation 6, which could be imposed on the reset control. Such a constraint depends on each particular situation. When it is not present, the hypothesis holds true for any choice of the projection Π by defining $\mathcal{I} = \{0\}$.

In the specific case of Example 4, it will be shown that when hypothesis 2) is not fulfilled, the reset structure may give rise to limit cycles and even instability. These behaviors are displayed in Figures 7 and 8.

Finally, the purpose of hypotheses 3) consists in preventing the system from being ill-posed.

Actually, without hypothesis 3a), after reaching any state $x \in \mathcal{V} \cap \mathcal{W}$, the system would lock and, if $x \neq 0$, it would be stable but not asymptotically stable. Usually, to avoid this situation, the *reset surface* $\mathcal{M} = \mathcal{V} \setminus \mathcal{W}$ is considered instead of the whole space $\mathcal{V}$: indeed, by definition, if $x \in \mathcal{M}$, then $x^+ \notin \mathcal{M}$. The two approaches, i.e., reset as in definition (2) and hypothesis 3a) or reset on the *reset surface*, are equivalent. Nevertheless, only the first one is considered in this paper, since the emphasis will be placed on the vector space structure of $\mathcal{V}$ and $\mathcal{W}$.

On the other hand, a dwell-time, as in hypothesis 3b), is often required in the literature to impede the system to exhibit a "Zeno behaviour" (also called *beating*), i.e., a sequence of infinite resets in a finite time.

It is clear that when $\mathcal{V} \cap \mathcal{W} = \{0\}$ the system cannot be blocked by a reset, making hypothesis 3a) unnecessary. However, it is less obvious that in this case also no dwell-time is needed: this fact is a direct consequence of the following result.

Proposition 1: Let $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$ be subspaces of $\mathbb{R}^n$ satisfying $\mathcal{V} \cap \mathcal{W} = \{0\}$ and $\mathcal{W} \neq \{0\}$. Then, for every $A \in \mathbb{R}^{n\times n}$, there exists $\delta > 0$ such that $\mathrm{e}^{At}x \notin \mathcal{V}$ whenever $x \in \mathcal{W} \setminus \{0\}$ and $|t| < \delta$.

See, for the proof, the analogous statement in [15].

Remark 2: Even if condition $\mathcal{V}\cap\mathcal{W} = \{0\}$ would simplify the assumptions on the problem (and some results that will be stated next too), it is rather restrictive, as we will show in Section IV.

III. The linear reset controller

Before giving a detailed example of linear reset system, a simple controller implementing a reset mechanism will be introduced in this section.

A reset controller $\mathcal{C}$ is a linear time invariant system represented by a classical state-space model; however, some components of its state are instantaneously reset when a certain condition is met — this action is suggested by the particular shape of the corresponding block, as depicted in Figure 2.

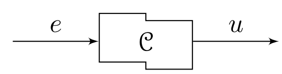

Fig. 2. Symbol of a reset controller $\mathcal{C}$ with input e and output u

In this paper, both the reset action and the reset condition are linear. In the specific case of the reset controller, the former acts only on the state and the latter depends only on the input. Mathematically, this means that two matrices $N_\mathcal{C}$ and $\Pi_\mathcal{C}$ are given, such that when $N_\mathcal{C}e = 0$ the state $\xi_\mathcal{C} \in \mathbb{R}^c$ is *immediately* reset to the new value $\xi_\mathcal{C}^+ = \Pi_\mathcal{C}\xi_\mathcal{C}$.

Hence, the reset controller $\mathcal{C}$ is described by
$$\begin{cases} \dot{\xi}_\mathcal{C} = A_\mathcal{C}\xi_\mathcal{C} + B_\mathcal{C}e & \text{if } N_\mathcal{C}e \neq 0, \\ \xi_\mathcal{C}^+ = \Pi_\mathcal{C}\xi_\mathcal{C} & \text{if } N_\mathcal{C}e = 0, \\ u = C_\mathcal{C}\xi_\mathcal{C} + D_\mathcal{C}e, \end{cases} \tag{8}$$
where $e \in \mathbb{R}^q$, $u \in \mathbb{R}^l$, and all the matrices have suitable dimensions.

By assumption, $\Pi_\mathcal{C}$ is a projection. Note that, by (4), this means that a repeated application of this operator would not change the state obtained by the the first reset, i.e., if $\xi_\mathcal{C}^+ = \Pi_\mathcal{C}\xi_\mathcal{C}$, then $\Pi_\mathcal{C}\xi_\mathcal{C}^+ = \xi_\mathcal{C}^+$.

In the most basic situation, as in the Clegg integrator, $\Pi_\mathcal{C} = 0$ and the whole state vector is reset to zero; also *partial zero resets* of z out of c components of the state have been considered, which correspond (after reordering) to projection matrices like this
$$\begin{bmatrix} I_{c-z} & 0 \\ 0 & 0_{z\times z} \end{bmatrix}. \tag{9}$$
However, being more convenient from a mathematical point of view, a general projection $\Pi_\mathcal{C}$ is here considered.

In any case, the final implementation of the controller can always be simplified by choosing a suitable realization which reduces any reset matrix to (9).

Actually, after changing the state to $\widetilde{\xi}_{\mathcal{C}} = S_{\mathcal{C}}\xi_{\mathcal{C}}$, where $S_{\mathcal{C}}$ is any invertible matrix, the following reset condition is obtained.

$$\widetilde{\xi}_{\mathcal{C}}^{+} = S_{\mathcal{C}}\xi_{\mathcal{C}}^{+} = S_{\mathcal{C}}\Pi_{\mathcal{C}}\xi_{\mathcal{C}} = S_{\mathcal{C}}\Pi_{\mathcal{C}}S_{\mathcal{C}}^{-1}\widetilde{\xi}_{\mathcal{C}} = \widetilde{\Pi}_{\mathcal{C}}\widetilde{\xi}_{\mathcal{C}},$$

where the new reset matrix $\widetilde{\Pi}_{\mathcal{C}}$ is still a projection that, by choosing a diagonalizing $S_{\mathcal{C}}$, has the form of (9).

IV. Asymptotic tracking with reset controller

In this section the solution of a tracking problem using a reset controller will be linked to the stability analysis of an autonomous reset system.

Consider the system illustrated in Figure 3, where the reference signal r generated by an exosystem $\mathcal{R}$ enters a feedback loop consisting of a reset controller $\mathcal{C}$ and a plant $\mathcal{P}$. The objective is to track the reference signal r by the output y of the plant or, equivalently, to let the steady-state error $e = r - y$ between the reference signal and the controlled plant output tend to zero.

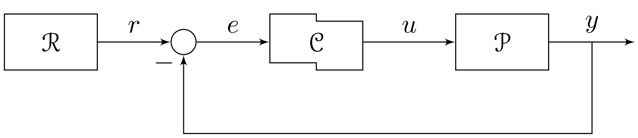

Fig. 3. Block diagram of the system

The following assumptions are made on the system.

- The state-space realization of the plant $\mathcal{P}$ is
$$\begin{cases} \dot{\xi}_{\mathcal{P}} = A_{\mathcal{P}}\xi_{\mathcal{P}} + B_{\mathcal{P}}u \\ y = C_{\mathcal{P}}\xi_{\mathcal{P}} + D_{\mathcal{P}}u, \end{cases} \tag{10}$$
where $\xi_{\mathcal{P}} \in \mathbb{R}^p$ is the state of plant.
- The exosystem $\mathcal{R}$ is an autonomous system with state $\xi_{\mathcal{R}} \in \mathbb{R}^\rho$ and state-space realization
$$\begin{cases} \dot{\xi}_{\mathcal{R}} = A_{\mathcal{R}}\xi_{\mathcal{R}} \\ r = C_{\mathcal{R}}\xi_{\mathcal{R}}. \end{cases} \tag{11}$$
Observe that, for the tracking problem to make sense, this system should not be asymptotically stable. Thus, as is usual in the literature, we shall suppose that the eigenvalues of $A_{\mathcal{R}}$ do not have negative real part.
- The reset controller $\mathcal{C}$ is described by (8), where $\Pi_{\mathcal{C}}$ is a projection.
Moreover, for the sake of simplicity, $D_{\mathcal{C}} = 0$. However, even if $D_{\mathcal{C}} \neq 0$, the nonsingularity of the matrix $I + D_{\mathcal{C}}D_{\mathcal{P}}$ would be a (quite reasonable) sufficient condition for the correctness of this example.

For the tracking problem to be feasible (without reset), it is known that, by the internal model principle, the transfer function of the series interconnection of $\mathcal{C}$ and $\mathcal{P}$ must have the poles of the exosystem $\mathcal{R}$.

Therefore, the only assumption that can be made about stability concerns the closed loop system.

In the equivalent system of Figure 4, the exosystem has been simplified and $\mathcal{R}'$ is just described by the autonomous equation $\dot{\xi}_{\mathcal{R}} = A_{\mathcal{R}}\xi_{\mathcal{R}}$. On the other hand, the closed loop system Σ is obtained through the corresponding transformation of the original system, in order to have input $\xi_{\mathcal{R}}$ and output e. It can be represented by the state space model with reset defined by

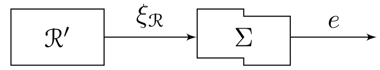

Fig. 4. Equivalent closed loop system with state reset

$$\begin{cases} \dot{\xi} = A\xi + B\xi_{\mathcal{R}} & \text{if } N_{\mathcal{C}}e \neq 0, \\ \xi^{+} = \Pi\xi & \text{if } N_{\mathcal{C}}e = 0, \\ e = C\xi + D\xi_{\mathcal{R}}, \end{cases} \tag{12}$$

where the state is $\xi = \begin{bmatrix} \xi_{\mathcal{P}} \\ \xi_{\mathcal{C}} \end{bmatrix}$ and the new matrices are

$$A = \begin{bmatrix} A_{\mathcal{P}} & B_{\mathcal{P}}C_{\mathcal{C}} \\ -B_{\mathcal{C}}C_{\mathcal{P}} & A_{\mathcal{C}} - B_{\mathcal{C}}D_{\mathcal{P}}C_{\mathcal{C}} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_{\mathcal{C}}C_{\mathcal{R}} \end{bmatrix},$$
$$C = \begin{bmatrix} -C_{\mathcal{P}} & -D_{\mathcal{P}}C_{\mathcal{C}} \end{bmatrix}, \quad D = \begin{bmatrix} C_{\mathcal{R}} \end{bmatrix},$$
$$\Pi = \begin{bmatrix} I_p & 0 \\ 0 & \Pi_{\mathcal{C}} \end{bmatrix}. \tag{13}$$

Notice that the controller $\mathcal{C}$ of Figure 3 solves the asymptotic tracking problem if and only if the (autonomous) system of Figure 4 is asymptotically stable, i.e., A is Hurwitz (see, for instance, [16, Ch. 2] for a solution of this problem). However, even when Σ is stable, the system is internally unstable due to the presence of the exosystem $\mathcal{R}'$. To obtain an internally stable system, an additional transformation must be carried out.

So, assume that A is Hurwitz and let T be the solution of the Sylvester equation

$$TA_{\mathcal{R}} = AT + B, \tag{14}$$

which exists and is unique, since $A_{\mathcal{R}}$ and A do not have common eigenvalues (see for example [17, Ch. 12.5]). It is easy to check that the new state $x = \xi - T\xi_{\mathcal{R}}$ satisfies the system of equations

$$\begin{cases} \dot{x} = Ax, \\ e = Cx + (CT + D)\xi_{\mathcal{R}}. \end{cases} \tag{15}$$

Observe that $T\xi_{\mathcal{R}}$ represents the (unstable) internal model of the exosystem. In particular, by partitioning x and T like ξ, it follows that $x_{\mathcal{P}} = \xi_{\mathcal{P}} - T_{\mathcal{P}}\xi_{\mathcal{R}}$ and $x_{\mathcal{C}} = \xi_{\mathcal{C}} - T_{\mathcal{C}}\xi_{\mathcal{R}}$. Roughly speaking, the internal model can be found partly in the plant ($T_{\mathcal{P}}\xi_{\mathcal{R}}$) and partly in the controller ($T_{\mathcal{C}}\xi_{\mathcal{R}}$) and x tends to zero when both the plant's and the controller's state tend to the corresponding internal model state.

So, after *removing* the internal model $T\xi_{\mathcal{R}}$, system (15) is internally stable and the state x converges to zero. Therefore, the initial tracking problem is solved if

$$CT + D = 0. \tag{16}$$

Now, consider the reset control of (12). An analogous reset equation can be found for system (15): indeed, since $\xi = x + T\xi_{\mathcal{R}}$ and $\xi_{\mathcal{R}}^+ = \xi_{\mathcal{R}}$, it follows that

$$\begin{aligned}\xi^+ = \Pi\xi \iff & x^+ + T\xi_{\mathcal{R}}^+ = \Pi x + \Pi T\xi_{\mathcal{R}} \\ \iff & x^+ = \Pi x + (\Pi - I)T\xi_{\mathcal{R}}. \end{aligned} \quad (17)$$

So, by the freeness of $\xi_{\mathcal{R}}$, the *new* reset equation $x^+ = \Pi x$ is equivalent to the original one if and only if $(\Pi - I)T = 0$. To understand the significance of this condition, it is sufficient to rewrite it in a different way: by the definition of Π in (13), it follows that

$$\begin{aligned}(\Pi - I)T = 0 \iff & (\Pi_{\mathcal{C}} - I)T_{\mathcal{C}} = 0 \\ \iff & \Pi_{\mathcal{C}}T_{\mathcal{C}} = T_{\mathcal{C}}. \end{aligned} \quad (18)$$

In particular, this means that the controller's internal model state, $T_{\mathcal{C}}\xi_{\mathcal{R}}$, is not altered by the reset action. Hence, the reset mechanism does not interfere with the tracking goal, permitting $\xi_{\mathcal{C}}$ to tend to $T_{\mathcal{C}}\xi_{\mathcal{R}}$.

Note that condition (18) is one of the motivations for not requiring a highly structured projection matrix as in (9). Actually, in that case, the number of resettable states z would be bounded from above by the number of null lines of $T_{\mathcal{C}}$, which could be zero even for a small rank matrix, thus leading to an underused or disabled reset mechanism. Conversely, a generic projection satisfying (18), can have rank bounded from below by the rank of $T_{\mathcal{C}}$, thus maximizing the number of states that can be reset.

Finally, if the system of Figure 3 admits a controller such that A is Hurwitz and condition (16) is satisfied, then it is not necessary to consider the error equation $e = Cx$ in (15) and also the reset condition can be directly expressed in terms of the state. Indeed, if $N = N_{\mathcal{C}}C$, then

$$N_{\mathcal{C}}e = 0 \iff Nx = 0. \quad (19)$$

Thus, instead of the asymptotic tracking problem, the asymptotic stability of the following autonomous system can be analysed:

$$\begin{cases} \dot{x} = Ax & \text{if } Nx \neq 0, \\ x^+ = \Pi x & \text{if } Nx = 0, \end{cases} \quad (20)$$

where A is stable, Π is a projection, i.e., $\Pi^2 = \Pi$, and $\Pi T = T$ (note the equivalence of (20) with (2) when $\mathcal{V} = \ker N$).

When these conditions are met, reset control allows to reduce overshoot in the step response, overcoming limitations which are present in all linear controllers [18].

Remark 3: Looking at the previous problem from a more geometric point of view, observe that, if $\mathcal{I} = \operatorname{img} T$, then

$$\Pi T = T \iff \Pi\mathcal{I} = \mathcal{I},$$

i.e., condition (6) is satisfied and hypothesis 2) of Section II holds true.

Observe also that the projection space is

$$\mathcal{W} = \mathbb{R}^p \oplus \operatorname{img} \Pi_{\mathcal{C}}$$

and, taking for instance $D_{\mathcal{P}} = 0$, the reset space is

$$\mathcal{V} = \ker N = \ker N_{\mathcal{C}}C_{\mathcal{P}} \oplus \mathbb{R}^c.$$

In this case, the condition $\mathcal{V} \cap \mathcal{W} = \{0\}$ is met only when the reset condition is $x_{\mathcal{P}} = 0$ and $\Pi_{\mathcal{C}} = 0$. Such a complete zero reset would be hardly compatible with (18).

Example 4: Consider the system of Figure 3, where the exosystem $\mathcal{R}$, the controller $\mathcal{C}$, and the plant $\mathcal{P}$, are defined by

$$\mathcal{R} : \begin{cases} \dot{\xi}_{\mathcal{R}} = 0, \\ r = \xi_{\mathcal{R}}, \end{cases}$$

$$\mathcal{C} : \begin{cases} \dot{\xi}_{\mathcal{C}} = \begin{bmatrix} -5 & -5 \\ 0 & 0 \end{bmatrix} \xi_{\mathcal{C}} + \begin{bmatrix} -9 \\ 1 \end{bmatrix} e & \text{if } e \neq 0, \\ \xi_{\mathcal{C}}^+ = \Pi_{\mathcal{C}}\xi_{\mathcal{C}} & \text{if } e = 0, \\ u = \begin{bmatrix} 2 & 1 \end{bmatrix} \xi_{\mathcal{C}}, \end{cases}$$

$$\mathcal{P} : \begin{cases} \dot{\xi}_{\mathcal{P}} = 2\xi_{\mathcal{P}} + u, \\ y = -\xi_{\mathcal{P}}. \end{cases}$$

Note that, according to the internal model principle, the controller has a pole at zero. The initial conditions are assumed to be zero, with the exception of the exosystem's state, which represents the constant reference $r = \xi_{\mathcal{R}} = -1$.

Next, the description of system Σ (Figure 4) is found by applying formulas (13):

$$A = \begin{bmatrix} 2 & 2 & 1 \\ -9 & -5 & -5 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -9 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = 1, \quad \Pi = \begin{bmatrix} 1 & 0 \\ 0 & \Pi_{\mathcal{C}} \end{bmatrix}.$$

The eigenvalues of A are -1 and $-1 \pm 2i$, thus the system is stable and the Sylvester equation (14) has a unique solution $T = \begin{bmatrix} -1 & 2 & -2 \end{bmatrix}^\top$, which also satisfies equation (16).

After defining the new state $x = \xi - T\xi_{\mathcal{R}} = \xi + T$ (with initial state $x_0 = T$), the tracking error $e = Cx = x_{\mathcal{P}}$ tends to zero by the asymptotic stability of the autonomous system $\dot{x} = Ax$, as shown in Figure 5.

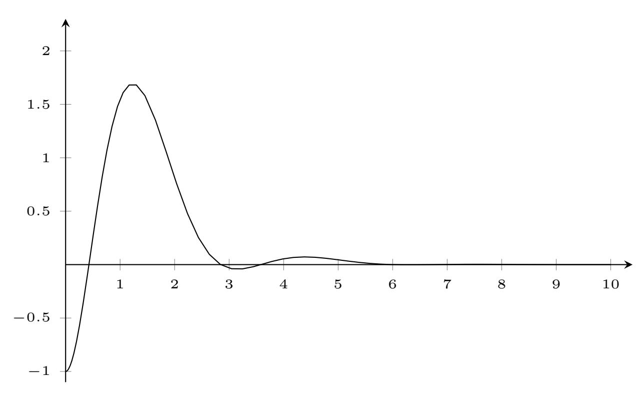

Fig. 5. Tracking error without reset mechanism.

This solves the tracking problem without reset control, i.e., with reset matrix $\Pi_\mathcal{C} = I$.

Now, to enable the reset mechanism, a matrix $\Pi_\mathcal{C} \neq I$ must be chosen, which satisfies condition (18). This means that $T_\mathcal{C} = \begin{bmatrix} 2 & -2 \end{bmatrix}^\top$ is an eigenvector of $\Pi_\mathcal{C}$ relative to the eigenvalue 1. Since the other eigenvalue must be 0 (a projection cannot have other eigenvalues), every admissible reset matrix is characterized as follows:

$$\Pi_\mathcal{C}(\alpha) = \begin{bmatrix} \alpha & \alpha - 1 \\ -\alpha & 1-\alpha \end{bmatrix},\ \alpha \in \mathbb{R}. \tag{21}$$

The error for $\alpha = -3$, for example, is presented in Figure 6: the overshoot is clearly reduced, while the convergence rate is maintained.

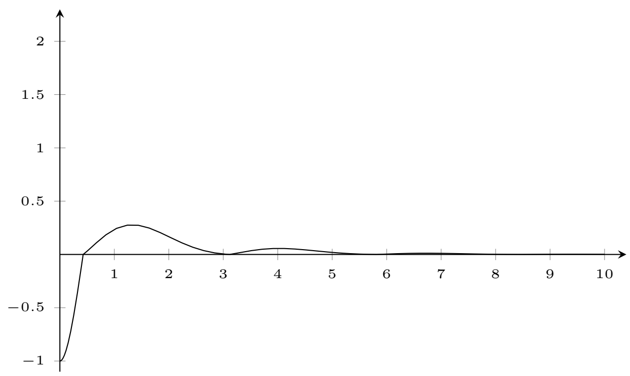

Fig. 6. Tracking error with projection matrix $\Pi_\mathcal{C} = \Pi_\mathcal{C}(-3)$.

However, not every value of α guarantees a better result. Actually, not even the solution of the tracking problem, or the asymptotic stability of system (20), is guaranteed: for $\alpha = 19.3$, simple stability is obtained (the system has a limit cycle) and for $\alpha = 19.4$ the system is unstable.

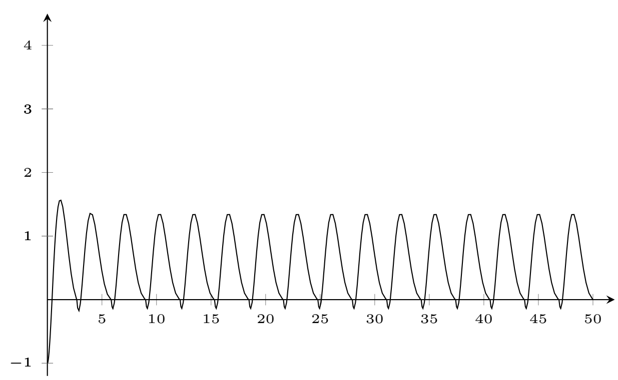

Fig. 7. Tracking error with $\Pi_\mathcal{C} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, similar to $\Pi_\mathcal{C}(19.3)$.

The system presents similar behaviors when condition (18), i.e., hypothesis 2) of Section II does not hold. Indeed, Figures 7 and 8 were obtained using the partial reset projection matrices $\Pi_\mathcal{C} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\Pi_\mathcal{C} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, respectively. In both cases, $\Pi_\mathcal{C} T_\mathcal{C} \neq T_\mathcal{C}$.

As for condition $\mathcal{V} \cap \mathcal{W} = \{0\}$, it does not hold in this example. Actually, $\mathcal{V} = \ker N = \ker \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $\mathcal{W} =$

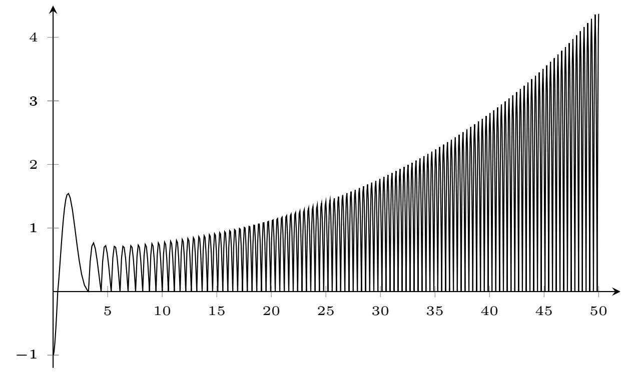

Fig. 8. Tracking error with $\Pi_\mathcal{C} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$,similar to $\Pi_\mathcal{C}(19.4)$.

$\operatorname{img} \Pi = \ker I - \Pi = \ker \begin{bmatrix} 0 & 1-\alpha & 1-\alpha \\ 0 & \alpha & \alpha \end{bmatrix}$. Therefore, $\mathcal{V} \cap \mathcal{W} = \operatorname{img} \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^\top \neq \{0\}$.

The role of the projection matrix in the stability of system (20), which is crucial, as the latter example showed, will be analysed in the following sections.

V. Stability criteria

In this section, sufficient conditions for the stability of reset system (2) are formulated. In what follows, the dynamics of the system without reset, i.e., matrix A of (1), the switching plane $\mathcal{V}$, and the projection matrix Π, hence also the projection plane $\mathcal{W}$, are given.

First of all, existence of global solutions and stability can be readily proven by invoking Theorem 1 in [5].

Theorem 5: Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously-differentiable, positive-definite, radially-unbounded function such that

$$\dot V(x) = \frac{\partial V(x)}{\partial x} Ax < 0, \qquad \text{if } x \neq 0, \tag{22}$$

$$V(\Pi x) - V(x) \leq 0, \qquad \text{if } x \in \mathcal{V}. \tag{23}$$

Then

- system (2) admits a left-continuous state function for all $t \geq 0$;
- the equilibrium point $x = 0$ is globally asymptotically stable.

Observe that the existence of a Lyapunov function V satisfying (22) is equivalent to asymptotic stability of the autonomous system (1), thus justifying the additional hypothesis 1) of Section II.

Moreover, in the linear case, this amounts to say that a *quadratic* Lyapunov function exists, i.e., there is a positive definite matrix $P = P^\top > 0$ such that $V(x) = x^\top P x$. Therefore, condition (22) will be expressed by the usual Lyapunov equation

$$A^\top P + PA < 0, \tag{24}$$

which is just the symmetric version of (22).

As for condition (23) of Theorem 5, an intuitive geometric interpretation can be given. To this aim, define the ellipsoidal set

$$\mathcal{E} = \{x \in \mathbb{R}^n : x^\top P x \leqslant 1\} \tag{25}$$

and, given a subspace $\mathcal{S}$ of $\mathbb{R}^n$, let

$$\mathcal{S}^\bullet = \mathcal{S} \cap \mathcal{E} = \{x \in \mathcal{S} : x^\top P x \leqslant 1\} \tag{26}$$

denote its intersection with $\mathcal{E}$.

In particular, we are interested in the sets $\mathcal{V}^\bullet$ and $\mathcal{W}^\bullet$. Intuitively, their relevance in relation to stability is as follows. Consider a (continuous) state trajectory starting in $\mathcal{W}^\bullet$. Since P defines a quadratic Lyapunov function for the unswitched system, the trajectory will stay within the ellipsoid $\mathcal{E}$. So, the state trajectory intersects the switching plane $\mathcal{V}$ as soon as it enters $\mathcal{V}^\bullet$ and, after the projection, the state will be in $\Pi\mathcal{V}^\bullet \subseteq \mathcal{W}$. Stability is guaranteed if the state still lies inside the ellipsoid, i.e., if $\Pi\mathcal{V}^\bullet$ is contained in $\mathcal{W}^\bullet$.

The following lemma relates this inclusion property to a linear matrix inequality.

Lemma 6: $\Pi\mathcal{V}^\bullet \subseteq \mathcal{W}^\bullet$ if and only if $M^\top\Pi^\top P\Pi M - M^\top P M \leq 0$.

Proof: ($\Rightarrow$) For any $0 \neq y \in \mathbb{R}^m$, let $k = y^\top M^\top P M y$ and define $\bar{y} = \frac{1}{\sqrt{k}} y$. Then, $\bar{y}^\top M^\top P M \bar{y} = 1$ and, therefore, $M\bar{y} \in \mathcal{V}^\bullet$. By hypothesis, since $\Pi\mathcal{V}^\bullet \subseteq \mathcal{W}^\bullet$, it follows that $\Pi M \bar{y} \in \Pi\mathcal{V}^\bullet \subseteq \mathcal{W}^\bullet$, i.e.,

$$\bar{y}^\top M^\top \Pi^\top P \Pi M \bar{y} \leqslant 1 = \bar{y}^\top M^\top P M \bar{y}.$$

After multiplying both sides by $k > 0$, we get that $y^\top(M^\top\Pi^\top P\Pi M - M^\top P M) y \leqslant 0$, thus proving the sufficiency.

($\Leftarrow$) For any $x \in \Pi\mathcal{V}^\bullet \subseteq \mathcal{W}$, there exists $y \in \mathbb{R}^m$ such that $x = \Pi M y$ and $y^\top M^\top P M y \leqslant 1$. Therefore, by hypothesis, we have that

$$x^\top P x = y^\top M^\top \Pi^\top P \Pi M y \leqslant y^\top M^\top P M y \leqslant 1.$$

Hence, $x \in \mathcal{W}^\bullet$. ∎

Remark 7: Condition $\Pi\mathcal{V}^\bullet \subseteq \mathcal{W}^\bullet$ of this lemma can be given in an equivalent (but, apparently, weaker) form. Indeed, let

$$\mathcal{V}^\circ = \mathcal{V} \cap \partial\mathcal{E} = \{x \in \mathcal{V} : x^\top P x = 1\}$$

denote the intersection of the boundary of $\mathcal{E}$ with $\mathcal{V}$. Then we have that

$$\Pi\mathcal{V}^\bullet \subseteq \mathcal{W}^\bullet \iff \Pi\mathcal{V}^\circ \subseteq \mathcal{W}^\bullet. \tag{27}$$

Roughly speaking, the intersection with a vector space and the projection maintain the convexity of the ellipsoidal set. Therefore, the inclusion of the 'boundary' is equivalent to the inclusion of the whole set.

The results presented so far can be now joined to formulate a geometric condition for stability.

Theorem 8: Consider the reset control system (2) and the notation defined by (3), (5) and (26). If there exists $P = P^\mathrm{T} > 0$ such that $A^\mathrm{T} P + PA < 0$ and

$$\Pi\mathcal{V}^\circ \subseteq \mathcal{W}^\bullet, \tag{28}$$

then

- the reset control system admits a left-continuous solution $x(t)$ for all $t \geq 0$;
- the equilibrium point $x = 0$ is asymptotically stable.

Proof: The quadratic Lyapunov function $V(x) = x^\top P x$ is a smooth, positive-definite, and radially-unbounded function which satisfies (22), since, by hypothesis, $\dot{V}(x) = x^\top(A^\top P + PA)x < 0$ for every $x \neq 0$.

Finally, note that for any $x \in \mathcal{V}$, $x = My$ for some y. Thus,

$$\begin{aligned} V(\Pi x) - V(x) &= x^\top \Pi^\top P \Pi x - x^\top P x \\ &= y^\top (M^\top \Pi^\top P \Pi M - M^\top P M) y \leqslant 0 \end{aligned}$$

holds true by Lemma 6 and Remark 7. The statement then follows by applying Theorem 5. ∎

VI. Geometric constraints on the projection

Given the reset control system (2), assume that there exists $P = P^\mathrm{T} > 0$ such that $A^\mathrm{T} P + PA < 0$ holds. The inclusion $\Pi\mathcal{V}^\circ \subseteq \mathcal{W}^\bullet$, which guarantees stability by Theorem 8, will be analysed in this section, giving rise to a condition on the projection kernel.

More precisely, $\ker \Pi$ must contain a subspace with a fixed dimension, which has to be chosen inside a certain set or is fixed.

Example 9: Consider the following situation: let $x \in \mathbb{R}^2$ be the state of system (2) and be $\mathcal{I}$ in (6) a one-dimensional space. In order to have a proper reset system, both the reset space $\mathcal{V}$ and the projection space $\mathcal{W}$ must have dimension one.

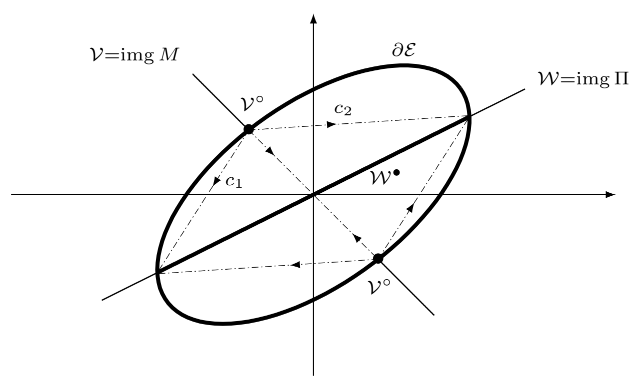

Fig. 9. State space ($\mathbb{R}^2$) of system (2) with $\dim \mathcal{V} = \dim \mathcal{W} = 1$.

If $\mathcal{V} \cap \mathcal{W} \neq \{0\}$, then $\mathcal{V} = \mathcal{W}$ and condition (28) holds true for any choice of the one-dimensional kernel of Π.

On the other hand, if $\mathcal{V} \cap \mathcal{W} = \{0\}$, as shown in Figure 9, then $\mathcal{W} = \mathcal{I}$, $\partial\mathcal{E}$ is an ellipse, $\mathcal{V}^\circ$ are two points on $\partial\mathcal{E}$ and $\mathcal{W}^\bullet$ is a line segment inside $\mathcal{E}$.

Therefore, for condition (28) to be satisfied, the kernel of Π must be chosen inside the cone defined by vectors c_1 and c_2. Notice that $\ker \Pi = \mathcal{V}$ is always a possible choice.

Example 10: Consider now system (2) with state $x \in \mathbb{R}^3$ and suppose that the reset space $\mathcal{V}$ is a plane and the projection space $\mathcal{W}$ is a line. When $\mathcal{V} \cap \mathcal{W} = \{0\}$, the two-dimensional $\ker \Pi$ is generated by two vectors that, as in

Example 9, lie in a bounded range. As before, $\ker \Pi = \mathcal{V}$, thus projecting $\mathcal{V}$ to 0, is a possible choice.

The situation is rather different, when $\mathcal{U} = \mathcal{V} \cap \mathcal{W} \neq \{0\}$, which means that, in this case, $\mathcal{U} = \mathcal{W}$. Moreover, $\ker \Pi$ is generated by a vector c in $\mathcal{V}$, but not in $\mathcal{W}$, and another one which does not belong to $\mathcal{V}$.

The latter can be freely chosen, but only one direction in $\mathcal{V}$ guarantees that condition (28) is satisfied. As shown in Figure 10, where just the plane $\mathcal{V}$ is represented, the restriction on the direction of vector c is given by the points of $\mathcal{V}^\circ$ in a neighborhood of the intersection of $\mathcal{V}^\circ$ with $\mathcal{W}^\bullet$, i.e., $\mathcal{U}^\circ$.

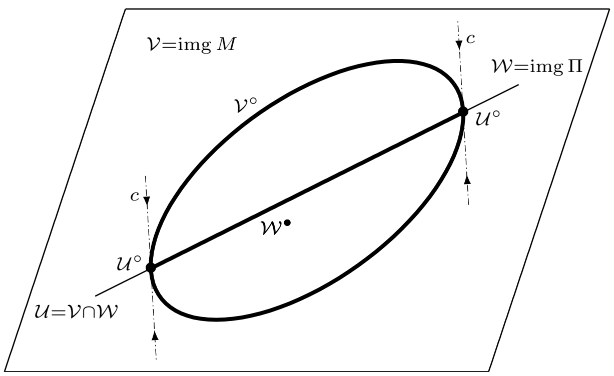

Fig. 10. Switching space of system (2) with $\mathcal{W} \subset \mathcal{V} \subset \mathbb{R}^3$ (state space).

Actually, for each point of $\mathcal{V}^\circ$, the direction of an admissible projection is given by a cone as in Figure 9. However, the direction c of the projection has to admissible for every point of $\mathcal{V}^\circ$ and, in a neighborhood of $\mathcal{U}^\circ$, the common direction tends to the tangent vector at $\mathcal{U}^\circ$.

Summing up, $\ker \Pi$ is generated by any vector not belonging to $\mathcal{V}$ and by the tangent vector to curve $\mathcal{V}^\circ$ at its intersection with $\mathcal{W}$.

Example 11: Finally, let system (2) have state $x \in \mathbb{R}^3$ and suppose that both the reset space $\mathcal{V}$ and the projection space $\mathcal{W}$ are planes, being $\ker \Pi$ a line generated by one vector c. Note that the intersection $\mathcal{U} = \mathcal{V} \cap \mathcal{W}$ has positive dimension.

Since condition (28) is trivially satisfied when $\mathcal{V} = \mathcal{W}$, let us consider the case where $\mathcal{V} \neq \mathcal{W}$ and, therefore, $\mathcal{U}$ is a line.

Since $\mathcal{W}^\bullet$ is a planar region delimited by an ellipse (shaded in Figure 11), the admissible directions for the projection of a point $p \in \mathcal{V}^\circ$ belong to a cone whose vertex is p. Therefore, apparently, the common direction c is not necessarily a vector of $\mathcal{V}$.

However, analysing the projections in a neighborhood of $\mathcal{U}^\circ = \mathcal{V}^\circ \cap \mathcal{W}$, it turns out that, as in Example 10, there is only one admissible direction c, which is exactly the tangent vector to $\mathcal{V}^\circ$ at $\mathcal{U}^\circ$. Therefore, $c \in \mathcal{V}$.

Observe that in this example, once $\mathcal{V}$, $\mathcal{W}$ and P are given, the stability condition (28) completely determines the projection matrix Π.

By the examples, it is clear that the kernel of a projection matrix Π such that (28) holds true, may have to satisfy some condition (besides, obviously, $\mathcal{W} \oplus \ker \Pi = \mathbb{R}^n$).

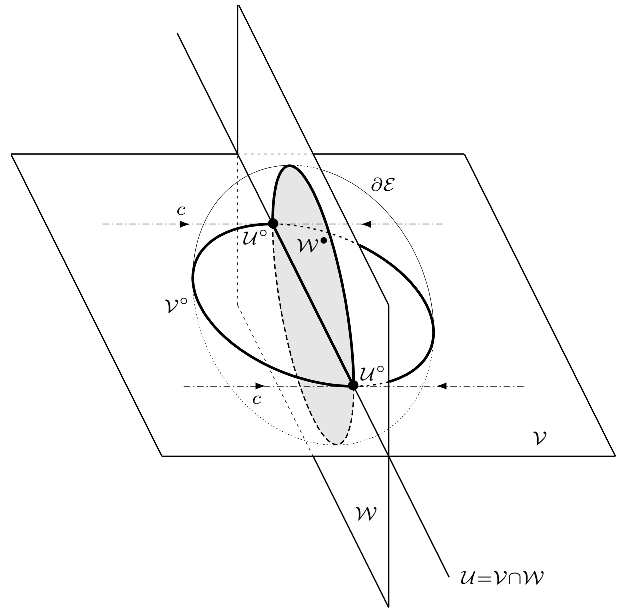

Fig. 11. State space ($\mathbb{R}^3$) of system (2) with $\dim \mathcal{V} = \dim \mathcal{W} = 2$.

The general picture is resumed in the following proposition.

Proposition 12: Define $l = \dim \mathcal{V} \cap \mathcal{W}$ and be $m = \dim \mathcal{V}$, $k = \dim \mathcal{W}$ and n the state space dimension, as stated in Section II. Then, condition (28) is satisfied only when one of these situations occurs.

1) If $l = 0$ (void intersection of the switching and the projection space), m vectors of a basis of $\ker \Pi$ are chosen within a suitable set, which depends on $\mathcal{V}^\circ$ and $\mathcal{W}^\bullet$, as the cone of Example 9; anyways, one possible choice is always a basis of $\mathcal{V}$. If $m + k < n$, there is no restriction on the other $n - m - k$ vectors of the basis.
2) If $l = m$ (the projection space contains the switching space), then (28) holds true for any projection.
3) If $0 < l < m$, define $\mathcal{U}^\circ = \mathcal{V}^\circ \cap \mathcal{W}$ and the vector space

$$\mathcal{V}^\circ_{\mathcal{W}} = \bigcap_{x_0 \in \mathcal{U}^\circ} \mathcal{V}^\circ_{x_0}, \text{ where} \tag{29}$$
$$\mathcal{V}^\circ_{x_0} = \{v \text{ tangent to } \mathcal{V}^\circ \text{ at } x_0\},\ x_0 \in \mathcal{U}^\circ.$$

Then, the subspace $\mathcal{V}^\circ_{\mathcal{W}}$, with dimension $m - l$, is contained in $\ker \Pi$, while there is no restriction on the other $n + l - k - m \geqslant 0$ vectors of its basis.

Observe that, in each of these three cases, the projection Π is *admissible* if $\ker \Pi$ contains some fixed subspace, which may depend on the reset space $\mathcal{V}$, on the projection space $\mathcal{W}$ and on the matrix P defining the quadratic Lyapunov function V.

In particular, as concerns the third situation, this subspace is $\mathcal{V}^\circ_{\mathcal{W}}$, defined in (29). The following lemma provides an easy way to compute it.

Lemma 13: Using the previous notation, let $N \in \mathbb{R}^{n-m\times n}$ and $L \in \mathbb{R}^{l\times n}$ be such that $\ker N = \mathcal{V}$ and $\operatorname{img} L = \mathcal{V} \cap \mathcal{W}$. Then

$$\mathcal{V}^\circ_{\mathcal{W}} = \ker \begin{bmatrix} L^\top P \\ N \end{bmatrix}.$$

Proof: Observe first that a vector v is tangent to $\mathcal{E}$ at x_0 if $x_0^\top P v = 0$, thus $\ker x_0^\top P$ is the vector space of tangent vectors at x_0. Since $\mathcal{V}^\circ$ is contained in $\mathcal{V}$, any vector tangent to $\mathcal{V}^\circ$ belongs to $\mathcal{V}$ too. Therefore, by its definition,

$$\mathcal{V}^\circ_{x_0} = \mathcal{V} \cap \ker x_0^\top P.$$

To conclude the proof, note that $x_0 \in \operatorname{img} L$, thus

$$\mathcal{V}^\circ_{\mathcal{W}} = \mathcal{V} \cap \ker L^\top P = \ker N \cap \ker L^\top P.$$

■

We may now conclude the discussion about the geometric meaning of condition (28), reformulating Theorem 8 with a simple condition on the projection operator.

Theorem 14: Let $N \in \mathbb{R}^{n-m\times n}$ and $L \in \mathbb{R}^{l\times n}$ be full (row) rank matrices such that $\ker N = \mathcal{V}$ and $\operatorname{img} L = \mathcal{V} \cap \mathcal{W}$ ($L = 0$ when $\mathcal{V} \cap \mathcal{W} = \{0\}$) and define

$$\mathcal{K} = \ker \begin{bmatrix} L^\top P \\ N \end{bmatrix}. \tag{30}$$

Then, the statement of Theorem 8 holds when condition (28) is replaced by

$$\mathcal{K} \subseteq \ker \Pi.$$

Proof: By Proposition 12, when $0 < l < m$, condition (28) is satisfied if $\mathcal{K} = \mathcal{V}^\circ_{\mathcal{W}}$, which is true by Lemma 13.

When $l = 0$, definition (30) reduces to $\mathcal{K} = \ker N = \mathcal{V}$ and, by Proposition 12.1), when $\mathcal{V} \subseteq \ker \Pi$, then condition (28) is satisfied.

Finally, when $l = m$, by Proposition 12.2) any projection should be admissible. Actually, it will be shown that, in this case, $\mathcal{K} = \{0\}$. Indeed, given any $z \in \mathcal{K}$, it follows that $Nz = 0$, thus $z \in \mathcal{V}$ and therefore, since $\operatorname{img} L = \mathcal{V} \cap \mathcal{W} = \mathcal{V}$, there exist some y such that $z = Ly$. Hence, $L^\top P L y = L^\top P z = 0$ and, being $L^\top P L$ invertible, $y = 0$ and also $z = Ly = 0$. ■

VII. CONCLUSIONS

Linear reset systems were investigated in this paper adopting a geometric approach. In particular, the stability problem of a general linear reset system was studied.

The main result is the following: if the stability of the system without reset is verified by a quadratic Lyapunov equation and the state is reset by a matrix projection Π, then the reset system is well-posed and stability is maintained if $\mathcal{I} \subseteq \operatorname{img} \Pi$ and $\mathcal{K} \subseteq \ker \Pi$, where $\mathcal{I}$ and $\mathcal{K}$ are two suitable subspaces, whose definition depends on the structure of the linear system (for $\mathcal{I}$) and on the reset structure and on the aforementioned quadratic Lyapunov function (for $\mathcal{K}$).

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