

Observer design for a general class of triangular systems

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Abstract—The paper deals with the observer design problem for a wide class of triangular nonlinear systems. Our main results generalize those obtained in the recent author’s works [2] and [3].

Index Terms—nonlinear triangular systems, observer design, switching dynamics.

I. INTRODUCTION

We derive sufficient conditions for the solvability of the observer design problem for time-varying single output systems of the form

$$\begin{aligned} \dot{x}_i &= f_i(t, x_1, \dots, x_{i+1}), i = 1, \dots, n-1 \\ \dot{x}_n &= f_n(t, x_1, \dots, x_n), \\ y &= x_1, (x_1, \dots, x_n) \in \mathbb{R}^n \end{aligned} \quad (1a)$$

$$(1b)$$

where the functions $f_i(\cdot)$, $i = 1, \dots, n$, are continuous and (locally) Lipschitz. It is known from Theorem 2.1 of Chapter 3 in [5] that for every $x_0 \in \mathbb{R}^n$ and for each fixed input, every single output control system which has a uniform canonical flag ([5, Chapter 2, Definition 2.1]) can be locally transformed in the above canonical form (1) in such a way that

$$\frac{\partial}{\partial x_{i+1}} f_i(t, x_1, \dots, x_{i+1}) \neq 0, \forall x = (x_1, \dots, x_n) \in \mathcal{U}, t \geq 0, \quad i = 1, \dots, n-1 \quad (2)$$

for certain neighborhood \mathcal{U} of x_0 . We also mention the works [4] and [8] where algebraic type necessary and sufficient conditions are established for feedback equivalence between a single input system and a triangular system whose dynamics have p -normal form. In our recent work [3], the observer design problem is studied for a subclass of systems (1) whose dynamics have p -normal form. The result of present work constitutes a generalization of previous results in the literature dealing with the observer design problem for triangular systems (see for instance [1], [5], [6] and relative references therein) and particularly generalizes the main result of the recent author’s work in [3].

We make the following assumption for the right hand side of system (1) which constitutes a (global) generalization of assumption (2).

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H1. For each $(t; x_1, \dots, x_i) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^i$, $i = 1, \dots, n-1$, the function $\mathbb{R} \ni z \rightarrow f_i(t, x_1, \dots, x_i, z) \in \mathbb{R}$ is strictly monotone.

The paper is organized as follows. We first provide notations and various concepts, including the concept of the *switching observer* that has been originally introduced in [2], for general time-varying systems:

$$\dot{x} = f(t, x), (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \quad (3a)$$

$$y = h(t, x), y \in \mathbb{R}^{\bar{n}} \quad (3b)$$

where $y(\cdot)$ plays the role of output. We then provide the precise statement of our main result (Theorem 1.1) concerning solvability of the observer design problem for (1). Section II contains some preliminary results concerning solvability of the observer design problem for the case (3) with linear output (Propositions 2.1 and 2.2). In Section III we use the results of Section II, in order to prove our main result.

Notations and definitions: We adopt the following notations. For a given vector $x \in \mathbb{R}^n$, x' denotes its transpose and $|x|$ its Euclidean norm. We use the notation $|A| := \max\{|Ax| : x \in \mathbb{R}^n; |x| = 1\}$ for the induced norm of a matrix $A \in \mathbb{R}^{m \times n}$. By N we denote the class of all increasing C^0 functions $\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. For given $R > 0$, denote by B_R the closed ball of radius $R > 0$, centered at $0 \in \mathbb{R}^n$. Consider a pair of metric spaces X_1, X_2 and a set-valued map $X_1 \ni x \rightarrow Q(x) \subset X_2$. We say that $Q(\cdot)$ satisfies the *Compactness Property (CP)*, if for every sequence $(x_\nu)_{\nu \in \mathbb{N}} \subset X_1$ and $(q_\nu)_{\nu \in \mathbb{N}} \subset X_2$ with $x_\nu \rightarrow x \in X_1$ and $q_\nu \in Q(x_\nu)$, there exist a subsequence $(x_{\nu_k})_{k \in \mathbb{N}}$ and $q \in Q(x)$ such that $q_{\nu_k} \rightarrow q$. We also invoke the well known fact, see [7], that the time-varying system (3a) is forward complete, if and only if there exists a function $\beta \in NN$ such that the solution $x(\cdot) := x(\cdot, t_0, x_0)$ of (3a) initiated from x_0 at time $t = t_0$ satisfies:

$$|x(t)| \leq \beta(t, |x_0|), \forall t \geq t_0 \geq 0, x_0 \in \mathbb{R}^n \quad (4)$$

provided that the dynamics of (3a) are C^0 and Lipschitz on $x \in \mathbb{R}^n$. It turns out that, under these regularity assumptions plus forward completeness for (3a), for each $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$ the corresponding output $y(t) = h(t, x(t, t_0, x_0))$ of (3) is defined for all $t \geq t_0$. For each $t_0 \geq 0$ and nonempty subset M of \mathbb{R}^n , we may consider the set $O(t_0, M)$ of all outputs of (3), corresponding to initial state $x_0 \in M$ and initial time $t_0 \geq 0$:

$$\begin{aligned} O(t_0, M) &:= \{y : [t_0, \infty) \rightarrow \mathbb{R}^{\bar{n}} \\ &: y(t) = h(t, x(t, t_0, x_0)); t \geq t_0, x_0 \in M\} \end{aligned} \quad (5)$$

$$A(t, q, x, e, y) := (A_{C_1}(t, q, x, e_1, y), A_{C_2}(t, q, x, e_1, e_2, y), \dots, A_{C_m}(t, q, x, e_1, \dots, e_m, y)) \quad (10)$$

For given $\emptyset \neq M \subset \mathbb{R}^n$, we say that the **Observer Design Problem (ODP) is solvable for (3) with respect to M** , if for every $t_0 \geq 0$ there exist a continuous map $G := G_{t_0}(t, z, w) : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^n$ and a nonempty set $\bar{M} \subset \mathbb{R}^n$ such that for every $z_0 \in \bar{M}$ and output $y \in O(t_0, M)$ the corresponding trajectory $z(\cdot) := z(\cdot, t_0, z_0; y)$; $z(t_0) = z_0$ of the observer $\dot{z} = G(t, z, y)$ exists for all $t \geq t_0$ and the error $e(\cdot) := x(\cdot) - z(\cdot)$ between the trajectory $x(\cdot) := x(\cdot, t_0, x_0)$, $x_0 \in M$ of (3a) and the trajectory $z(\cdot) := z(\cdot, t_0, z_0; y)$ of the observer satisfies:

$$\lim_{t \rightarrow \infty} e(t) = 0 \quad (6)$$

We say that the **Switching Observer Design Problem (SODP) is solvable for (3) with respect to M** , if for every $t_0 \geq 0$ there exist a strictly increasing sequence of times $(t_k)_{k \in \mathbb{N}}$ with $t_1 = t_0$ and $\lim_{k \rightarrow \infty} t_k = \infty$, a sequence of continuous mappings $G_k := G_{k, t_{k-1}}(t, z, w) : [t_{k-1}, t_{k+1}] \times \mathbb{R}^n \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^n$, $k \in \mathbb{N}$ and a nonempty set $\bar{M} \subset \mathbb{R}^n$ such that the solution $z_k(\cdot)$ of the system

$$\dot{z}_k = G_k(t, z_k, y), t \in [t_{k-1}, t_{k+1}] \quad (7)$$

with initial $z(t_{k-1}) \in \bar{M}$ and output $y \in O(t_0, M)$, is defined for every $t \in [t_{k-1}, t_{k+1}]$ and in such a way that, if we consider the piecewise continuous map $Z : [t_0, \infty) \rightarrow \mathbb{R}^n$ defined as $Z(t) := z_k(t)$, $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$, where for each $k \in \mathbb{N}$ the map $z_k(\cdot)$ denotes the solution of (7), then the error $e(\cdot) := x(\cdot) - Z(\cdot)$ between the trajectory $x(\cdot) := x(\cdot, t_0, x_0)$, of (3a) and $Z(\cdot)$ satisfies (6).

Our main result is the following theorem.

Theorem 1.1: For the system (1), assume that Hypothesis H1 is satisfied and (1a) is forward complete, i.e. there exists a function $\beta \in NN$, such that the solution $x(\cdot) := x(\cdot, t_0, x_0)$ of (1a) satisfies the estimation (4). Then:

- (i) the SODP is solvable for (1) with respect to \mathbb{R}^n .
- (ii) if in addition we assume that it is a priori known, that the initial states of (1) belong to a given ball B_R of radius $R > 0$ centered at zero $0 \in \mathbb{R}^n$, then the ODP is solvable for (1) with respect to B_R . \triangleleft

Theorem 1.1 constitutes a generalization of Theorem 1.1 in [3] for systems (1), whose dynamics are C^1 and have the particular form $f_i(t, x_1, \dots, x_{i+1}) := \tilde{f}_i(t, x_1, \dots, x_i) + a_i(t, x_1)x_{i+1}^{m_i}$, for certain $\tilde{f}_i \in C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}^i; \mathbb{R})$ and $a_i \in C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}; \mathbb{R})$, $i = 1, \dots, n-1$, where the constants m_i , $i = 1, \dots, n-1$ are odd integers and the functions $a_i(\cdot, \cdot)$, $i = 1, \dots, n-1$ satisfy the condition $|a_i(t, y)| > 0$, $\forall t \in \mathbb{R}_{\geq 0}$, $y \in \mathbb{R}$.

II. PRELIMINARY RESULTS

The proof of our main result concerning the case (1), is based on some preliminary results concerning the case of

systems (3) with linear output:

$$\dot{x} = f(t, x) := F(t, x, H(t)x), (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \quad (8a)$$

$$y = h(t, x) := H(t)x, y \in \mathbb{R}^{\bar{n}} \quad (8b)$$

where $H : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{\bar{n} \times n}$ is C^0 and $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^n$ is C^0 and Lipschitz on $(x, y) \in \mathbb{R}^n \times \mathbb{R}^{\bar{n}}$. We assume that system (8a) is forward complete, namely, the solution $x(\cdot) := x(\cdot, t_0, x_0)$ of (8a) satisfies (4) for certain $\beta \in NN$, hence, for every $R > 0$ and $t \geq 0$ we can define:

$$Y_R(t) := \{y \in \mathbb{R}^{\bar{n}} : y = H(t)x(t, t_0, x_0), \text{ for certain } t_0 \in [0, t] \text{ and } x_0 \in B_R\} \quad (9)$$

where $H(\cdot)$ is given in (8b). Obviously, $Y_R(t) \neq \emptyset$ for all $t \geq 0$ and, if (4) holds, the set-valued map $[0, \infty) \ni t \rightarrow Y_R(t) \subset \mathbb{R}^{\bar{n}}$ satisfies the CP and further $y(t) \in Y_R(t)$, for every $t \geq t_0 \geq 0$ and $y \in O(t_0, B_R)$; (see notations). Also, given integers $\ell, m, n, \bar{n} \in \mathbb{N}$ and a map $A : \mathbb{R}_{\geq 0} \times \mathbb{R}^\ell \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^{m \times m}$, we say that $A(\cdot, \cdot, \cdot, \cdot, \cdot)$ satisfies Property P1, if the following holds:

P1. $A(t, q, x, e, y)$ has the form (10) on top of the page, where each mapping $A_{C_i} : \mathbb{R}_{\geq 0} \times \mathbb{R}^\ell \times \mathbb{R}^n \times \mathbb{R}^i \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^{m \times m}$, $i = 1, \dots, m$ is continuous on $\mathbb{R}_{\geq 0} \times \mathbb{R}^\ell \times \mathbb{R}^n \times \{(e_1, \dots, e_i) \in \mathbb{R}^i : e_i \neq 0\} \times \mathbb{R}^{\bar{n}}$ and bounded on every compact subset of $\mathbb{R}_{\geq 0} \times \mathbb{R}^\ell \times \mathbb{R}^n \times \mathbb{R}^i \times \mathbb{R}^{\bar{n}}$.

We make the following hypothesis:

Hypothesis 2.1. There exist a function $g \in C^1(\mathbb{R}_{\geq 0}; \mathbb{R})$ satisfying:

$$0 < g(t) < 1, \forall t \geq 0; \quad (11a)$$

$$\dot{g}(t) \geq -g(t), \forall t \geq 0; \quad (11b)$$

$$\lim_{t \rightarrow \infty} g(t) = 0 \quad (11c)$$

an integer $\ell \in \mathbb{N}$, a map $A : \mathbb{R}_{\geq 0} \times \mathbb{R}^\ell \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^{n \times n}$ satisfying P1 and constants $L > 1$, $c_1, c_2 > 0$, $R > 0$ with

$$c_1 \geq \frac{1}{2} \quad (12)$$

such that the following properties hold:

A1. For every $\xi \geq 1$ there exists a set-valued map

$$[0, \infty) \ni t \rightarrow Q_R(t) := Q_{R, \xi}(t) \subset \mathbb{R}^\ell \quad (13)$$

with $Q_R(t) \neq \emptyset$ for any $t \geq 0$, satisfying the CP and such that

$$\begin{aligned} &\forall t \geq 0, y \in Y_R(t), x, z \in \mathbb{R}^n \text{ with } |x| \leq \beta(t, R) \\ &\text{and } |x - z| \leq \xi \Rightarrow \Delta F(t, x, z, y) := F(t, x, y) - F(t, z, y) \\ &= A(t, q, x, x - z, y)(x - z) \text{ for certain } q \in Q_R(t) \end{aligned} \quad (14)$$

with $Y_R(\cdot)$ as given by (9).

A2. For every $\xi \geq 1$ there exists a set-valued map $Q_R := Q_{R, \xi}$ as in Hypothesis A1, in such a way that for every $t_0 \geq 0$, a time-varying symmetric matrix $P_R :=$

$P_{R,\xi,t_0} \in C^1([t_0, \infty); \mathbb{R}^{n \times n})$ and a function $d_R := d_{R,\xi,t_0} \in C^0([t_0, \infty); \mathbb{R})$ can be found, satisfying:

$$P_R(t) \geq I_{n \times n}, \forall t \geq t_0; |P_R(t_0)| \leq L; \quad (15a)$$

$$d_R(t) > c_1, \forall t \geq t_0 + 1; \int_{t_1}^{t_2} d_R(s) ds > -c_2, \forall t_2 \geq t_1 \geq t_0; \quad (15b)$$

$$\begin{aligned} e' P_R(t) A(t, q, x, e, y) e + \frac{1}{2} e' \dot{P}_R(t) e &\leq -d_R(t) e' P_R(t) e, \\ \forall t \geq t_0, q \in Q_R(t), x \in \mathbb{R}^n, e \in \ker H(t), y \in Y_R(t) : \\ |x| \leq \beta(t, R), |e| \leq \xi, e' P_R(t) e &\geq g(t) \end{aligned} \quad (15c)$$

with $Y_R(\cdot)$ as given by (9).

The following proposition provides a Luenberger type observer, for the case where it is a priori known, that the initial state of the system belongs to a given nonempty bounded subset of the state space. This result constitutes a slight generalization of Proposition 2.1 in [3].

Proposition 2.1: Consider the system (8) and assume that it is forward complete, namely, (4) holds for certain $\beta \in NN$ and satisfies Hypothesis 2.1. Then, the following hold: For each $\bar{t}_0 \geq t_0 \geq 0$ and constant ξ satisfying

$$\xi \geq \sqrt{L} \exp\{2c_2\} \beta(\bar{t}_0, \bar{R}), \bar{R} := R + 1 \quad (16)$$

there exists a function $\phi_R := \phi_{R,\xi,\bar{t}_0} \in C^1([\bar{t}_0, \infty); \mathbb{R}_{>0})$, such that the solution $z(\cdot)$ of system

$$\begin{aligned} \dot{z} &= G_{\bar{t}_0}(t, z, y) \\ &:= F(t, z, y) + \phi_R(t) P_R^{-1}(t) H'(t) (y - H(t)z) \end{aligned} \quad (17a)$$

$$\text{with initial } z(\bar{t}_0) = 0 \quad (17b)$$

where $P_R(\cdot) := P_{R,\xi,\bar{t}_0}(\cdot)$ is given in A2, is defined for all $t \geq \bar{t}_0$ and the error $e(\cdot) := x(\cdot) - z(\cdot)$ between the trajectory $x(\cdot) := x(\cdot, t_0, x_0)$ of (8a), initiated from $x_0 \in B_R$ at time $t_0 \geq 0$ and the trajectory $z(\cdot) := z(\cdot, \bar{t}_0, 0; y)$ of (17) satisfies:

$$|e(t)| < \xi, \forall t \geq \bar{t}_0; \quad (18a)$$

$$\begin{aligned} |e(t)| &\leq \max\{\xi \exp\{-c_1(t - (\bar{t}_0 + 1))\}, \sqrt{g(t)}\}, \\ \forall t &\geq \bar{t}_0 + 1 \end{aligned} \quad (18b)$$

It follows from (11c) and (18b), that for $\bar{t}_0 := t_0$ the ODP is solvable for (8) with respect to B_R ; particularly the error $e(\cdot)$ between the trajectory $x(\cdot) := x(\cdot, t_0, x_0)$, $x_0 \in B_R$ of (8a) and the trajectory $z(\cdot) := z(\cdot, t_0, z_0; y)$, $z_0 = 0$ of the observer $\dot{z} = G_{t_0}(t, z, y)$ satisfies (6). \triangleleft

The following proposition constitutes a slight generalization of Proposition 2.2 in [3]. It establishes sufficient conditions for the existence of a switching observer exhibiting the state determination of (8), without any a priori information concerning the initial condition. We make the following hypothesis:

Hypothesis 2.2: There exist constants $L > 1$, $c_1, c_2 > 0$ such that (12) holds, an integer $\ell \in \mathbb{N}$, a function $g \in C^1(\mathbb{R}_{\geq 0}; \mathbb{R})$ satisfying (11) and a map $A : \mathbb{R}_{\geq 0} \times \mathbb{R}^\ell \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ satisfying P1, in such a way that for every

$R > 0$ Hypothesis 2.1 is fulfilled, namely, both A1 and A2 hold.

Proposition 2.2: In addition to the hypothesis of forward completeness for (8a), assume that system (8) satisfies Hypothesis 2.2. Then the SODP is solvable for (8) with respect to \mathbb{R}^n . \triangleleft

The proofs of Propositions 2.1 and 2.2 are based on the same approaches with those used for proofs of Propositions 2.1 and 2.2 in [3], respectively, plus some elementary modifications. Moreover a preliminary technical result (Lemma 2.1 below) is required, which also constitutes a slight modification of Lemma 2.1 in [3]. In order to present the precise statement of this preliminary result we consider a pair (H, A) of mappings:

$$H := H(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{\bar{n} \times m}; \quad (19a)$$

$$A := A(t, q, x, e, y) : \mathbb{R}_{\geq 0} \times \mathbb{R}^\ell \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^{m \times m} \quad (19b)$$

for certain $\ell, m, n, \bar{n} \in \mathbb{N}$, where $H(\cdot)$ is continuous and $A(\cdot, \cdot, \cdot, \cdot, \cdot)$ satisfies Property P1. We make the following hypothesis:

Hypothesis 2.3: Let $g(\cdot) \in C^0(\mathbb{R}_{\geq 0}; \mathbb{R})$ satisfying (11a) and assume that for certain constant $R > 0$ and for every $\xi \geq 1$, there exist a function $\beta_R := \beta_{R,\xi} \in N$ and set-valued mappings $[0, \infty) \ni t \rightarrow Y_R(t) := Y_{R,\xi}(t) \subset \mathbb{R}^{\bar{n}}$ and $[0, \infty) \ni t \rightarrow Q_R(t) := Q_{R,\xi}(t) \subset \mathbb{R}^\ell$ with $Y_R(t) \neq \emptyset$ and $Q_R(t) \neq \emptyset$ for all $t \geq 0$, satisfying the CP, in such a way that for every $t_0 \geq 0$, a time-varying symmetric matrix $P_R := P_{R,\xi,t_0} \in C^1([t_0, \infty); \mathbb{R}^{m \times m})$ can be found, satisfying $P_R(t) \geq I_{m \times m}, \forall t \geq t_0$ and a function $d_R := d_{R,\xi,t_0} \in C^0([t_0, \infty); \mathbb{R})$, in such a way that

$$\begin{aligned} e' P_R(t) A(t, q, x, e, y) e + \frac{1}{2} e' \dot{P}_R(t) e &\leq -d_R(t) e' P_R(t) e, \\ \forall t \geq t_0, q \in Q_R(t), x \in \mathbb{R}^n, e \in \ker H(t), y \in Y_R(t) : \\ |x| \leq \beta_R(t), |e| \leq \xi, e' P_R(t) e &\geq g(t) \end{aligned} \quad (20)$$

Lemma 2.1: Consider the pair (H, A) of the time-varying mappings in (19) and assume that Hypothesis 2.3 is fulfilled for certain $R > 0$. Then for every $\xi \geq 1$, $t_0 \geq 0$ and $\bar{d}_R := \bar{d}_{R,\xi,t_0} \in C^0([t_0, \infty); \mathbb{R})$ with $\bar{d}_R(t) < d_R(t), \forall t \geq t_0$, there exists a function $\phi_R := \phi_{R,\xi,t_0} \in C^1([t_0, \infty); \mathbb{R}_{>0})$ such that

$$\begin{aligned} e' P_R(t) A(t, q, x, e, y) e + \frac{1}{2} e' \dot{P}_R(t) e &\leq \phi_R(t) |H(t)e|^2 \\ &- \bar{d}_R(t) e' P_R(t) e, \forall t \geq t_0, q \in Q_R(t), x \in \mathbb{R}^n, e \in \mathbb{R}^m, \\ y \in Y_R(t) : |x| \leq \beta_R(t), |e| \leq \xi, e' P_R(t) e &\geq g(t) \end{aligned} \quad (21)$$

III. PROOF OF THEOREM 1.1

In this section we apply the results of Section II to prove our main result concerning the solvability of the SODP(ODP) for triangular systems (1).

Proof of Theorem 1.1: The proof of both statements is based on the results of Propositions 2.1 and 2.2 and is based on a generalization of the methodology employed in the proof of the main result in [3]. Without loss of generality, we may assume that, instead of Assumption H1 it holds:

$$\delta_{i,2}(t, y, x_2, \dots, x_{i+1}, e_2) \begin{cases} := \frac{f_i(t, y, x_2, x_3, \dots, x_{i+1}) - f_i(t, y, x_2 - e_2, x_3, \dots, x_{i+1})}{e_2}, \\ \text{for } e_2 \neq 0 \text{ and } 2 \leq i \leq n; n \geq 3 \\ := \frac{f_i(t, y, x_2) - f_i(t, y, x_2 - e_2)}{e_2}, \text{ for } e_2 \neq 0 \text{ and } i = 1, 2; n = 2 \text{ or } i = 1; n \geq 3 \\ := 0, \text{ for } e_2 = 0 \text{ and } n, i, j \text{ in each case above} \end{cases}$$

$$(t; y; x_2, \dots, x_{i+1}; e_2) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^i \times \mathbb{R} \quad (23a)$$

$$\delta_{i,j}(t, y, x_2, \dots, x_{i+1}, e_2, \dots, e_j) \begin{cases} := \frac{f_i(t, y, x_2 - e_2, \dots, x_{j-1} - e_{j-1}, x_j, x_{j+1}, \dots, x_{i+1}) - f_i(t, y, x_2 - e_2, \dots, x_{j-1} - e_{j-1}, x_j - e_j, x_{j+1}, \dots, x_{i+1})}{e_j}, \\ \text{for } e_j \neq 0 \text{ and } 3 \leq i \leq n-1; 3 \leq j \leq i; n \geq 4 \text{ or } i = n; 3 \leq j \leq n-1; n \geq 4 \\ := \frac{f_n(t, y, x_2 - e_2, \dots, x_{n-1} - e_{n-1}, x_n) - f_n(t, y, x_2 - e_2, \dots, x_{n-1} - e_{n-1}, x_n - e_n)}{e_n}, \\ \text{for } e_n \neq 0 \text{ and } i = j = n; n \geq 3 \\ := 0, \text{ for } e_j = 0, \text{ and } n, i, j \text{ in each case above} \end{cases}$$

$$(t; y; x_2, \dots, x_{i+1}; e_2, \dots, e_j) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^i \times \mathbb{R}^{j-1} \quad (23b)$$

$$\delta_{i,i+1}(t, y, x_2, \dots, x_{i+1}, e_2, \dots, e_{i+1}) \begin{cases} := \frac{f_i(t, y, x_2 - e_2, \dots, x_i - e_i, x_{i+1}) - f_i(t, y, x_2 - e_2, \dots, x_i - e_i, x_{i+1} - e_{i+1})}{e_{i+1}}, \text{ for } e_{i+1} \neq 0 \\ := 0, \text{ for } e_{i+1} = 0 \end{cases}$$

$$(t; y; x_2, \dots, x_{i+1}; e_2, \dots, e_{i+1}) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^i \times \mathbb{R}^i, \quad (23c)$$

H1.' For each $(t; x_1, \dots, x_i) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^i$, $i = 1, \dots, n-1$, the function $\mathbb{R} \ni z \rightarrow f_i(t, x_1, \dots, x_i, z) \in \mathbb{R}$ is strictly increasing.

The proof of the first statement, is based on the establishment of Hypothesis 2.2 for system (1). Hence, we show that there exist an integer $\ell \in \mathbb{N}$, a map $A : \mathbb{R}_{\geq 0} \times \mathbb{R}^\ell \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ satisfying P1, constants $L > 1$, $c_1, c_2 > 0$ such that (12) holds and a function $g(\cdot)$ satisfying (11), in such a way that for each $R > 0$, both A1 and A2 hold for (1). Let $R > 0$, $\xi \geq 1$ and define:

$$F(t, x, y) := (f_1(t, y, x_2), f_2(t, y, x_2, x_3), \dots, f_{n-1}(t, y, x_2, \dots, x_n), f_n(t, y, x_2, \dots, x_n))',$$

$$(t, x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R} \quad (22)$$

Also, for every pair of indices (i, j) with $2 \leq j \leq n$, $j-1 \leq i \leq n$ we define the functions $\delta_{i,j}(\cdot)$ as in (23a)-(23c) on top of the page (where we have used the notation $x_{i+1}|_{i=n} := x_n$ and $\mathbb{R}^i|_{i=n} := \mathbb{R}^{n-1}$, in (23a), (23b)). By exploiting the Lipschitz continuity assumption for the functions $f_i(\cdot)$, $i = 1, \dots, n$ it follows that for every $2 \leq j \leq n$, $j-1 \leq i \leq n$, the following properties hold:

S1. $\delta_{i,j}(\cdot)$ is continuous on the set

$$\mathcal{O}_{i,j} := \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^i \times \{(e_2, \dots, e_j) \in \mathbb{R}^{j-1} : e_j \neq 0\} \quad (24)$$

S2. $\delta_{i,j}(\cdot)$ is bounded on every compact subset of $\mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^i \times \mathbb{R}^{j-1}$. (where we have used the notation $\mathbb{R}^i|_{i=n} := \mathbb{R}^{n-1}$ in both S1 and S2)

Furthermore, from H1', (24), (23a) and (23c) it follows:

S3.

$$\delta_{i,i+1}(t, y, x_2, \dots, x_{i+1}, e_2, \dots, e_{i+1}) > 0, \quad (25)$$

$$\forall (t; y; x_2, \dots, x_{i+1}; e_2, \dots, e_{i+1}) \in \mathcal{O}_{i,i+1}, i = 1, \dots, n-1$$

From (23a)-(23c) we deduce that for $i = 1, \dots, n$ it holds:

$$f_i(t, y, x_2, \dots, x_{i+1}) - f_i(t, y, z_2, \dots, z_{i+1})$$

$$= \sum_{j=2}^{i+1} \delta_{i,j}(t, y, x_2, \dots, x_{i+1}, x_2 - z_2, \dots, x_j - z_j)(x_j - z_j)$$

$$\forall t \in \mathbb{R}_{\geq 0}, y \in \mathbb{R}, (x_2, \dots, x_{i+1}), (z_2, \dots, z_{i+1}) \in \mathbb{R}^i \quad (26)$$

(where we have used the notation $\sum_{j=2}^{i+1}|_{i=n} := \sum_{j=2}^n$, $x_{i+1}|_{i=n} := x_n$ and $z_{i+1}|_{i=n} := z_n$). Also, by invoking Property S2, the following property holds.

S4. There exists a function $\sigma_R := \sigma_{R,\xi} \in N \cap C^1([0, \infty); \mathbb{R})$ satisfying:

$$\sigma_R(t) \geq \sum_{i=2}^n \sum_{j=2}^i \sup\{|\delta_{i,j}(t, y, x_2, \dots, x_{i+1}, e_2, \dots, e_j)| : |y| \leq \beta(t, R), |(x_2, \dots, x_{i+1})| \leq \beta(t, R), |(e_2, \dots, e_j)| \leq \xi\}, \forall t \geq 0 \quad (27)$$

(where we have used the notation $x_{i+1}|_{i=n} := x_n$). Next, consider the set-valued map $[0, \infty) \ni t \rightarrow Q_R(t) := Q_{R,\xi}(t) \subset \mathbb{R}^\ell$, $\ell := \frac{n(n+1)}{2}$ defined as

$$Q_R(t) := \{q = (q_{1,1}; q_{2,1}, q_{2,2}; \dots; q_{n,1}, q_{n,2}, \dots, q_{n,n}) \in \mathbb{R}^\ell : |q| \leq \sigma_R(t)\} \quad (28)$$

that obviously satisfies the CP and consider the set valued mappings $\mathbb{R}_{\geq 0} \times (0, \xi] \ni (t, r) \rightarrow Q_{R,i}(t, r) := Q_{R,\xi,i}(t, r) \subset \mathbb{R} \times \mathbb{R}^i \times \mathbb{R}^i$, $i = 1, \dots, n-1$ given as

$$Q_{R,i}(t, r) := \{(y; x_2, \dots, x_{i+1}; e_2, \dots, e_{i+1}) \in \mathbb{R} \times \mathbb{R}^i \times \mathbb{R}^i : |y| \leq \beta(t, R), |(x_2, \dots, x_{i+1})| \leq \beta(t, R), |(e_2, \dots, e_{i+1})| \leq \xi, |e_{i+1}| \geq r\} \quad (29)$$

Finally, for $i = 1, \dots, n-1$ define:

$$\begin{aligned} D_{R,\xi,i}(t,r) &:= D_{R,i}(t,r) \\ &= \min\{\delta_{i,i+1}(t,y,x_2,\dots,x_{i+1},e_2,\dots,e_{i+1}) : \\ &\quad (y;x_2,\dots,x_{i+1};e_2,\dots,e_{i+1}) \in Q_{R,i}(t,r)\} \\ &\quad (t,r) \in \mathbb{R}_{\geq 0} \times (0,\xi] \end{aligned} \quad (30)$$

By exploiting (29), (30) and Properties S1, S3, it follows that the functions $D_{R,i}(\cdot, \cdot)$, $i = 1, \dots, n-1$ are continuous and the following hold:

$$\begin{aligned} 0 < D_{R,i}(t,r) &\leq \delta_{i,i+1}(t,y,x_2,\dots,x_{i+1},e_2,\dots,e_{i+1}), \\ &\quad \forall (y;x_2,\dots,x_{i+1};e_2,\dots,e_{i+1}) \in Q_{R,i}(t,r), \\ &\quad (t,r) \in \mathbb{R}_{\geq 0} \times (0,\xi] \end{aligned} \quad (31)$$

$$\begin{aligned} D_{R,i}(t,r_1) &\leq D_{R,i}(t,r_2), \forall t \in \mathbb{R}_{\geq 0}, \\ r_1, r_2 &\in (0,\xi] \text{ with } r_1 < r_2 \end{aligned} \quad (32)$$

Now, let $Y_R(\cdot)$ as given by (9) with

$$H := (1, 0, \dots, 0) \in \mathbb{R}^{1 \times n} \quad (33)$$

and notice that, due to (4), (9) and (33), it holds:

$$|y| \leq \beta(t, R), \text{ for every } y \in Y_R(t), t \geq 0 \quad (34)$$

From (22), (26), (28), (34) and Property S4, it follows that for every $t \geq 0$, $y \in Y_R(t)$ and $x, z \in \mathbb{R}^n$ with $|x| \leq \beta(t, R)$ and $|x - z| \leq \xi$ we have:

$$\begin{aligned} F(t, x, y) - F(t, z, y) &= A(t, q, x, x - z, y)(x - z), \\ &\text{for some } q \in Q_R(t) \text{ with } q_{i,1} = 0, i = 1, \dots, n; \end{aligned} \quad (35a)$$

with $A(\cdot, \cdot, \cdot, \cdot, \cdot)$ as given in (35b) on top of the next page. Notice that this map has the form (10), and, due to S1 and S2, satisfies Property P1. Hence, A1 is satisfied.

In order to establish A2, we prove that there exist constants $L > 1$, $c_1, c_2 > 0$ such that (12) holds and a function $g(\cdot)$ satisfying (11), in such a way that for every $R > 0$, $\xi \geq 1$ and $t_0 \geq 0$, a time-varying symmetric matrix $P_R := P_{R,\xi,t_0} \in C^1([t_0, \infty); \mathbb{R}^{n \times n})$ and a function $d_R := d_{R,\xi,t_0} \in C^0([t_0, \infty); \mathbb{R})$ can be found satisfying all conditions (15a), (15b), (15c) with H , $A(\cdot, \cdot, \cdot, \cdot, \cdot)$, $Y_R(\cdot)$ and $Q_R(\cdot)$, as given by (33), (35b), (9) and (28), respectively and with $\beta(\cdot, \cdot)$ as given in (4) for the case of system (1). We proceed by induction as follows. Pick $L > 1$, $c_1 := 1$, $c_2 := n$ and let $g(\cdot)$ be a C^1 function satisfying (11). Also, let $R > 0$ and for $k = 2, \dots, n$ define:

$$\begin{aligned} H_k &:= (1, 0, \dots, 0) \in \mathbb{R}^{1 \times k}, e := (e_{n-k+1}; \hat{e}') \in \mathbb{R} \times \mathbb{R}^{k-1} \\ \hat{e} &:= (e_{n-k+2}, \dots, e_n)' \in \mathbb{R}^{k-1} \end{aligned} \quad (36a)$$

and consider the map $A_k : \mathbb{R}_{\geq 0} \times \mathbb{R}^\ell \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \rightarrow \mathbb{R}^{k \times k}$ with components:

$$\begin{aligned} (A_k(t, q, x, e, y))_{i,j} & \quad (36b) \\ \left\{ \begin{array}{l} := q_{n-k+i, n-k+j}, \text{ for } j \leq i \\ := \delta_{n-k+i, n-k+j}(t, y, x_2, \dots, x_{n-k+j}, 0, \dots, 0, \\ \quad e_{n-k+1}, \dots, e_{n-k+j}), \text{ for } j = i+1; k < n-1 \\ := \delta_{n-k+i, n-k+j}(t, y, x_2, \dots, x_{n-k+j}, e_2, \dots, e_{n-k+j}), \\ \quad \text{for } j = i+1; k = n-1, n \\ := 0, \text{ for } j > i+1 \end{array} \right. \end{aligned}$$

Claim 1 (Induction Hypothesis): Let $k \in \mathbb{N}$ with $2 \leq k \leq n$. Then for L, R and $g(\cdot)$ as above and for every $\xi \geq 1$ and $t_0 \geq 0$, there exist a time-varying symmetric matrix $P_{R,k} := P_{R,\xi,t_0,k} \in C^1([t_0, \infty); \mathbb{R}^{k \times k})$ and a mapping $d_{R,k} := d_{R,\xi,t_0,k} \in C^0([t_0, \infty); \mathbb{R})$, in such a way that the following hold:

$$P_{R,k}(t) > I_{k \times k}, \forall t \geq t_0; |P_{R,k}(t_0)| \leq L; \quad (37a)$$

$$\begin{aligned} d_{R,k}(t) &> n - k + 1, \forall t \geq t_0 + 1; \int_{t_1}^{t_2} d_{R,k}(s) ds > -k, \\ &\quad \forall t_2 \geq t_1, t_1, t_2 \in [t_0, t_0 + 1] \end{aligned} \quad (37b)$$

$$\begin{aligned} &e' P_{R,k}(t) A_k(t, q, x, e, y) e + \frac{1}{2} e' \dot{P}_{R,k}(t) e \\ &\leq -d_{R,k}(t) e' P_{R,k}(t) e, \forall t \geq t_0, q \in Q_R(t), x \in \mathbb{R}^n, \\ &\quad e \in \ker H_k, y \in Y_R(t) : |x| \leq \beta(t, R), |e| \leq \xi, \\ &e' P_{R,k}(t) e \geq g(t) \end{aligned} \quad (37c)$$

with H_k , $A_k(\cdot, \cdot, \cdot, \cdot, \cdot)$, $Y_R(\cdot)$ and $Q_R(\cdot)$ as given in (36a), (36b), (9) and (28), respectively.

By taking into account (36), it follows that the mappings H_n and $A_n(\cdot, \cdot, \cdot, \cdot, \cdot)$ coincide with H and $A(\cdot, \cdot, \cdot, \cdot, \cdot)$ as given by (33) and (35b), respectively, hence, A2 is a consequence of Claim 1 with $H := H_n$ and $A(\cdot, \cdot, \cdot, \cdot, \cdot) := A_n(\cdot, \cdot, \cdot, \cdot, \cdot)$ and with $d_R := d_{R,n}$ and $P_R := P_{R,n}$ as given in (37b), (37a). Indeed, relations (15a), (15c) follow directly from (37a), (37c) and both inequalities of (15b) are a consequence of (37b) with $k = n$, if we take into account that $c_1 = 1$ and $c_2 = n$.

Proof of Claim 1 for $k := 2$: For reasons of notational simplicity, we may assume that $n > 3$. In that case we may define:

$$H_2 := (1, 0), e := (e_{n-1}, e_n)' \in \mathbb{R}^2 \quad (38a)$$

$$\begin{aligned} A_2(t, q, x, e, y) &:= \\ &\begin{pmatrix} q_{n-1, n-1} & \delta_{n-1, n}(t, y, x_2, \dots, x_n, 0, \dots, 0, e_{n-1}, e_n) \\ q_{n, n-1} & q_{n, n} \end{pmatrix} \end{aligned} \quad (38b)$$

Also, consider the constants L, R and the function $g(\cdot)$ as above and let $\xi \geq 1$ and $t_0 \geq 0$. We establish existence of a time-varying symmetric matrix $P_{R,2} := P_{R,\xi,t_0,2} \in C^1([t_0, \infty); \mathbb{R}^{2 \times 2})$ and a mapping $d_{R,2} := d_{R,\xi,t_0,2} \in C^0([t_0, \infty); \mathbb{R})$ in such a way that

$$P_{R,2}(t) > I_{2 \times 2}, \forall t \geq t_0; |P_{R,2}(t_0)| \leq L; \quad (39a)$$

$$\begin{aligned} d_{R,2}(t) &> n - 1, \forall t \geq t_0 + 1; \int_{t_1}^{t_2} d_{R,2}(s) ds > -2, \\ &\quad \forall t_2 \geq t_1, t_1, t_2 \in [t_0, t_0 + 1]; \end{aligned} \quad (39b)$$

$$\begin{aligned} &e' P_{R,2}(t) A_2(t, q, x, e, y) e + \frac{1}{2} e' \dot{P}_{R,2}(t) e \\ &\leq -d_{R,2}(t) e' P_{R,2}(t) e, \forall t \geq t_0, q \in Q_R(t), x \in \mathbb{R}^n, \\ &\quad e = (e_{n-1}, e_n)' \in \mathbb{R}^2, y \in Y_R(t) : |x| \leq \beta(t, R), \\ &\quad e \in \ker H_2, |e| \leq \xi, e' P_{R,2}(t) e \geq g(t) \end{aligned} \quad (39c)$$

$$A(t, q, x, e, y) := \begin{pmatrix} q_{1,1} & \delta_{1,2}(t, y, x_2, e_2) & 0 & \cdots & 0 \\ q_{2,1} & q_{2,2} & \delta_{2,3}(t, y, x_2, x_3, e_2, e_3) & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ q_{n-1,1} & q_{n-1,2} & q_{n-1,3} & \cdots & \delta_{n-1,n}(t, y, x_2, \dots, x_n, e_2, \dots, e_n) \\ q_{n,1} & q_{n,2} & q_{n,3} & \cdots & q_{n,n} \end{pmatrix} \quad (35b)$$

with H_2 , $A_2(\cdot, \cdot, \cdot, \cdot, \cdot)$, $Y_R(\cdot)$ and $Q_R(\cdot)$ as given in (38a), (38b), (9) and (28), respectively. Define:

$$P_{R,2}(t) := \begin{pmatrix} p_{R,1}(t) & p_R(t) \\ p_R(t) & L \end{pmatrix}, t \geq t_0 \quad (40)$$

for certain $p_{R,1}, p_R \in C^1([t_0, \infty); \mathbb{R})$, to be determined in the sequel and notice, that due to (38a) and (40), we have:

$$\begin{aligned} \{e \in \ker H_2 : |e| \leq \xi \text{ and } e' P_{R,2}(t) e \geq g(t)\} \\ = \{e = (0, e_n)' : \sqrt{g(t)/L} \leq |e_n| \leq \xi\} \end{aligned} \quad (41)$$

Then, by taking into account (38), (40) and (41), the desired (39c) is written:

$$\begin{aligned} p_R(t) \delta_{n-1,n}(t, y, x_2, \dots, x_n, 0, \dots, 0, e_n) + L q_{n,n} \\ \leq -L d_{R,2}(t), \forall t \geq t_0, q \in Q_R(t), x \in \mathbb{R}^n, e = (0, e_n)' \in \mathbb{R}^2, \\ y \in Y_R(t) : |x| \leq \beta(t, R), \sqrt{g(t)/L} \leq |e_n| \leq \xi \end{aligned} \quad (42)$$

By invoking (28), (29), (34) and the equivalence between (39c) and (42), it follows that, in order to prove (39c), it suffices to determine $p_{R,1}, p_R \in C^1([t_0, \infty); \mathbb{R})$ and $d_{R,2} \in C^0([t_0, \infty); \mathbb{R})$ in such a way that (39a) and (39b) are fulfilled, and further:

$$\begin{aligned} p_R(t) \delta_{n-1,n}(t, y, x_2, \dots, x_n, 0, \dots, 0, e_n) + L \sigma_R(t) \\ \leq -L d_{R,2}(t), \forall t \geq t_0, (y; x_2, \dots, x_n; 0, \dots, 0, e_n) \in \\ Q_{R,n-1}(t, \sqrt{g(t)/L}) \end{aligned} \quad (43)$$

We also require, that the candidate function $p_R(\cdot)$ satisfies:

$$p_R(t) \leq 0, \forall t \geq t_0; p_R(t_0) = 0 \quad (44)$$

Then, by taking into account (31), (44) and (43), it suffices to prove:

$$p_R(t) D_{R,n-1}(t, \sqrt{g(t)/L}) + L \sigma_R(t) \leq -L d_{R,2}(t), \forall t \geq t_0 \quad (45)$$

for certain $p_{R,1}, p_R \in C^1([t_0, \infty); \mathbb{R})$ and $d_{R,2} \in C^0([t_0, \infty); \mathbb{R})$ satisfying (39a), (39b) and (44).

Construction of p_R and $d_{R,2}$: First, notice that the mapping $t \rightarrow D_{R,n-1}(t, \sqrt{g(t)/L})$, $t \geq t_0$ is continuous, and due to (31) we can find a function $\mu_2 \in C^1([t_0, \infty); \mathbb{R})$, satisfying:

$$0 < \mu_2(t) \leq D_{R,n-1}(t, \sqrt{g(t)/L}), \text{ for every } t \geq t_0 \quad (46)$$

Let

$$M_2 := \max\{\sigma_R(t) : t \in [t_0, t_0 + \frac{1}{2}]\} \quad (47a)$$

$$\tau_2 := \min\left\{\frac{1}{M_2}, 1\right\} \quad (47b)$$

and define $\theta := \theta_{R,\xi,t_0} \in C^1([t_0, \infty); \mathbb{R})$, $p_R \in C^1([t_0, \infty); \mathbb{R})$ and $d_{R,2} \in C^0([t_0, \infty); \mathbb{R})$ as follows:

$$\theta(t) \begin{cases} := 0, & t = t_0 \\ \in [0, 1], & t \in [t_0, t_0 + \frac{\tau_2}{2}] \\ := 1, & t \in [t_0 + \frac{\tau_2}{2}, \infty) \end{cases} \quad (48)$$

$$p_R(t) := -\theta(t) \frac{L(n + \sigma_R(t))}{\mu_2(t)}, t \geq t_0 \quad (49)$$

$$d_{R,2}(t) \begin{cases} := -M_2, & t \in [t_0, t_0 + \frac{\tau_2}{2}] \\ \in [-M_2, n], & t \in [t_0 + \frac{\tau_2}{2}, t_0 + \tau_2] \\ := n, & t \in [t_0 + \tau_2, \infty) \end{cases} \quad (50)$$

We show that (39b), (44) and (45) are fulfilled, with $p_R(\cdot)$ and $d_{R,2}(\cdot)$ as given by (49) and (50), respectively. Indeed, (44) follows directly by recalling (46), (48) and (49). Both inequalities of (39b) are a direct consequence of (47b) and (50). We next show that (45) holds as well, with $p_R(\cdot)$ and $d_{R,2}(\cdot)$ as above and consider two cases:

Case 1: $t \in [t_0, t_0 + \frac{\tau_2}{2}]$. In that case, (45) follows directly from (44), (46), (47) and (50).

Case 2: $t \in [t_0 + \frac{\tau_2}{2}, \infty)$. Then from (46), (48), (49) and (50) it follows that:

$$\begin{aligned} -\frac{L(n + \sigma_R(t))}{\mu_2(t)} D_{R,n-1}(t, \sqrt{g(t)/L}) + L \sigma_R(t) \leq \\ -L n \leq -L d_{R,2}(t) \end{aligned}$$

namely, (45) again holds for all $t \in [t_0 + \frac{\tau_2}{2}, \infty)$.

We therefore conclude that (45) is fulfilled for all $t \geq t_0$.

Finally, the construction of $p_{R,1}(\cdot)$ included in (40), is the same with that given in proof of Theorem 1.1 in [3] and is omitted. This completes the proof of Claim 1 for $k = 2$.

Proof of Claim 1 (general step of induction procedure):

Assume now that Claim 1 is fulfilled for certain integer k with $2 \leq k < n$. We prove that Claim 1 also holds for $k := k + 1$. Consider the pair (H, A) as given in (19) with $H(t) := H_k$, $A(t, q, x, e, y) := A_k(t, q, x, e, y)$, $\ell = \frac{n(n+1)}{2}$, $m := k$, $n := n$ and $\bar{n} := 1$, where H_k and A_k are defined by (36a) and (36b), respectively. Notice, that the map A_k as given by (36b) has the form (10) and due to S1 and S2, satisfies Property P1. Hence, by the first inequality of (37a) and (37c), we conclude that Hypothesis 2.3 of the previous section holds, with R and $g(\cdot)$ as above, $Y_R(\cdot)$, $Q_R(\cdot)$ and $\beta_R(\cdot) := \beta(\cdot, R)$ as given in (9), (28) and (4), respectively, and with $d_R(\cdot) := d_{R,k}(\cdot)$ and $P_R(\cdot) := P_{R,k}(\cdot)$ as given in (37a), (37b). Finally, for every $\xi \geq 1$ and $t_0 \geq 0$, consider the function $\bar{d}_{R,k} := \bar{d}_{R,\xi,t_0,k}$ defined as:

$$\bar{d}_{R,k}(t) := d_{R,k}(t) - \frac{1}{2}, t \geq t_0 \quad (51)$$

$$A_{k+1}(t, q, x, e, y) = \begin{pmatrix} q_{n-k, n-k} & \delta_{n-k, n-k+1}(t, y, x_2, \dots, x_{n-k+1}, 0, \dots, 0, e_{n-k+1}) & 0 & \cdots & 0 \\ q_{n-k+1, n-k} & & & & \\ \vdots & & & & \\ q_{n, n-k} & & \boxed{A_k(t, q, x, \hat{e}, y)} & & \end{pmatrix} \quad (56)$$

which satisfies $\bar{d}_{R,k}(t) < d_{R,k}(t)$ for all $t \geq t_0$. It follows that all requirements of Lemma 2.1 are fulfilled and therefore, there exists a function $\phi_{R,k} := \phi_{R,\xi,t_0,k} \in C^1([t_0, \infty); \mathbb{R}_{>0})$ such that

$$\begin{aligned} & e' P_{R,k}(t) A_k(t, q, x, e, y) e + \frac{1}{2} e' \dot{P}_{R,k}(t) e \\ & \leq \phi_{R,k}(t) |H_k e|^2 - \bar{d}_{R,k}(t) e' P_{R,k}(t) e, \\ & \forall t \geq t_0, q \in Q_R(t), x \in \mathbb{R}^n, e \in \mathbb{R}^k, y \in Y_R(t) : \\ & |x| \leq \beta(t, R), |e| \leq \xi, e' P_{R,k}(t) e \geq g(t) \end{aligned} \quad (52)$$

Furthermore, due to (37b) and (51), the map $\bar{d}_{R,k}(\cdot)$ satisfies:

$$\begin{aligned} \bar{d}_{R,k}(t) & > n - k + \frac{1}{2}, \forall t \geq t_0 + 1; \int_{t_1}^{t_2} \bar{d}_{R,k}(s) ds > -(k + \frac{1}{2}), \\ & \forall t_2 \geq t_1, t_1, t_2 \in [t_0, t_0 + 1] \end{aligned} \quad (53)$$

In the sequel, we exploit (52) and (53), in order to establish that Claim 1 is fulfilled for $k = k + 1$. Specifically, for the same L, R and $g(\cdot)$ as above and for any $\xi \geq 1$ and $t_0 \geq 0$, we show that there exist a time-varying symmetric matrix $P_{R,k+1} \in C^1([t_0, \infty); \mathbb{R}^{(k+1) \times (k+1)})$ and a map $d_{R,k+1} \in C^0([t_0, \infty); \mathbb{R})$, such that both (37a) and (37b) are fulfilled with $k = k + 1$ and further:

$$\begin{aligned} & e' P_{R,k+1}(t) A_{k+1}(t, q, x, e, y) e + \frac{1}{2} e' \dot{P}_{R,k+1}(t) e \\ & \leq -d_{R,k+1}(t) e' P_{R,k+1}(t) e, \forall t \geq t_0, q \in Q_R(t), x \in \mathbb{R}^n \\ & e \in \ker H_{k+1}, y \in Y_R(t) : |x| \leq \beta(t, R), |e| \leq \xi, \\ & e' P_{R,k+1}(t) e \geq g(t) \end{aligned} \quad (54)$$

where

$$\begin{aligned} H_{k+1} & := (1, 0, \dots, 0) \in \mathbb{R}^{1 \times (k+1)}, e := (e_{n-k}; \hat{e})' \in \mathbb{R} \times \mathbb{R}^k, \\ \hat{e} & := (e_{n-k+1}, \dots, e_n)' \in \mathbb{R}^k \end{aligned} \quad (55a)$$

the components of the map $A_{k+1} : \mathbb{R}_{\geq 0} \times \mathbb{R}^\ell \times \mathbb{R}^n \times \mathbb{R}^{k+1} \times \mathbb{R} \rightarrow \mathbb{R}^{(k+1) \times (k+1)}$ are defined as:

$$(A_{k+1}(t, q, x, e, y))_{i,j} \quad (55b) \quad \begin{cases} := q_{n-k-1+i, n-k-1+j}, & \text{for } j \leq i \\ := \delta_{n-k-1+i, n-k-1+j}(t, y, x_2, \dots, x_{n-k+i}, 0, \dots, 0, \\ \quad e_{n-k}, \dots, e_{n-k+i}), & \text{for } j = i + 1; k < n - 2 \\ := \delta_{n-k-1+i, n-k-1+j}(t, y, x_2, \dots, x_{n-k+i}, e_2, \dots, e_{n-k+i}), \\ \quad \text{for } j = i + 1; k = n - 2, n - 1 \\ := 0, & \text{for } j > i + 1 \end{cases}$$

$$P_{R,k+1}(t) := \begin{pmatrix} p_{R,1}(t) & p_R(t) & 0 & \cdots & 0 \\ p_R(t) & & & & \\ 0 & & \boxed{P_{R,k}(t)} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix} \quad (55c)$$

and where $Y_R(\cdot), Q_R(\cdot)$ are given in (9) and (28), respectively. Again, for reasons of notational simplicity, we may assume that $k < n - 1$. It then follows from (36b) and (55b) that for every $e = (0, \hat{e})' = (0, e_{n-k+1}, \dots, e_n)' \in \ker H_{k+1} := (1, 0, \dots, 0) \in \mathbb{R}^{1 \times (k+1)}$, the map $A_{k+1}(\cdot, \cdot, \cdot, \cdot, \cdot)$ takes the form (56) on top of the page.

Let $\xi \geq 1$ and $t_0 \geq 0$. We determine functions $p_{R,1}, p_R \in C^1([t_0, \infty); \mathbb{R})$ and $d_{R,k+1} \in C^0([t_0, \infty); \mathbb{R})$ such that (37a) and (37b) are fulfilled with $k = k + 1$, and further (54) holds, with $H_{k+1}, A_{k+1}(\cdot, \cdot, \cdot, \cdot, \cdot)$ and $P_{R,k+1}(\cdot)$ as given by (55). By taking into account (56), it follows that (54) is equivalent to:

$$\begin{aligned} & e_{n-k+1}^2 p_R(t) \delta_{n-k, n-k+1}(t, y, x_2, \dots, x_{n-k+1}, 0, \dots, 0, \\ & e_{n-k+1}) + e' P_{R,k}(t) A_k(t, q, x, \hat{e}, y) \hat{e} + \frac{1}{2} e' \dot{P}_{R,k}(t) \hat{e} \\ & \leq -d_{R,k+1} e' P_{R,k}(t) \hat{e}, \forall t \geq t_0, q \in Q_R(t), x \in \mathbb{R}^n, \\ & e := (e_{n-k}; \hat{e})' \in \mathbb{R} \times \mathbb{R}^k, y \in Y_R(t) : |x| \leq \beta(t, R), \\ & e \in \ker H_{k+1}, |e| \leq \xi, e' P_{R,k+1}(t) e \geq g(t) \end{aligned} \quad (57)$$

Notice that, according to (55a) and (55c), we have $e' P_{R,k+1}(t) e = e' P_{R,k}(t) \hat{e}$ for every $e = (0, \hat{e})' = (0, e_{n-k+1}, \dots, e_n)' \in \ker H_{k+1}$, thus, by taking into account (36a) and (52), it suffices, instead of (57), to show that

$$\begin{aligned} & e_{n-k+1}^2 (p_R(t) \delta_{n-k, n-k+1}(t, y, x_2, \dots, x_{n-k+1}, 0, \dots, 0, \\ & e_{n-k+1}) + \phi_{R,k}(t)) \leq (\bar{d}_{R,k}(t) - d_{R,k+1}(t)) e' P_{R,k}(t) \hat{e}, \\ & \forall t \geq t_0, x \in \mathbb{R}^n, \hat{e} \in \mathbb{R}^k, y \in Y_R(t) : |x| \leq \beta(t, R), \\ & |\hat{e}| \leq \xi, e' P_{R,k}(t) \hat{e} \geq g(t) \end{aligned} \quad (58)$$

Establishment of (58) plus (37a) and (37b) for $k = k + 1$: We impose the following additional requirements for the candidate functions $p_R(\cdot)$ and $d_{R,k+1}(\cdot)$:

$$p_R(t) \leq 0, \forall t \geq t_0; p_R(t_0) = 0; \quad (59a)$$

$$d_{R,k+1}(t) \leq \bar{d}_{R,k}(t), \forall t \geq t_0 \quad (59b)$$

Then, by taking into account (59b) and the fact that the desired inequality in (58) should be valid for those $\hat{e} \in \mathbb{R}^k$ for which $|\hat{e}| \leq \xi$ and $e' P_{R,k}(t) \hat{e} \geq g(t)$, it follows that, in order to show (58) and that (37a), (37b) are valid with $k = k + 1$, it suffices to show that

$$\begin{aligned} & e_{n-k+1}^2 (p_R(t) \delta_{n-k, n-k+1}(t, y, x_2, \dots, x_{n-k+1}, 0, \dots, 0, \\ & e_{n-k+1}) + \phi_{R,k}(t)) \leq (\bar{d}_{R,k}(t) - d_{R,k+1}(t)) g(t), \\ & \forall t \geq t_0, x \in \mathbb{R}^n, e_{n-k+1} \in \mathbb{R}, y \in Y_R(t) : \\ & |x| \leq \beta(t, R), |e_{n-k+1}| \leq \xi \end{aligned} \quad (60)$$

for suitable functions $p_{R,1}, p_R \in C^1([t_0, \infty); \mathbb{R})$ and $d_{R,k+1} \in C^0([t_0, \infty); \mathbb{R})$, in such a way that (37a), (37b) hold with $k = k + 1$, and in addition $p_R(\cdot)$ and $d_{R,k+1}(\cdot)$

satisfy (59). We proceed to the explicit construction of these functions. Due to (29), (31), (34), (59a) and the fact that, due to requirement (59b), equation (60) holds trivially for $e_{n-k+1} = 0$, it suffices, instead of (60), to show that

$$\begin{aligned} & r^2(p_R(t)D_{R,n-k}(t, r) + \phi_{R,k}(t)) \\ & \leq (\bar{d}_{R,k}(t) - d_{R,k+1}(t))g(t), \forall t \geq t_0, r \in (0, \xi] \end{aligned} \quad (61)$$

Construction of the mappings p_R and $d_{R,k+1}$: Let

$$M_{k+1} := \max \left\{ |\bar{d}_{R,k}(t)| + \frac{1}{4} + \frac{\xi^2 \phi_{R,k}(t)}{g(t)} : t \in [t_0, t_0 + \frac{1}{2}] \right\} \quad (62a)$$

$$\tau_{k+1} := \min \left\{ \frac{1}{4M_{k+1}}, \frac{1}{2} \right\} \quad (62b)$$

and consider a function $\theta := \theta_{R,\xi,t_0} \in C^1([t_0, \infty); \mathbb{R})$ defined as:

$$\theta(t) \begin{cases} := 0, & t = t_0 \\ \in [0, 1], & t \in [t_0, t_0 + \frac{\tau_{k+1}}{2}] \\ := 1, & t \in [t_0 + \frac{\tau_{k+1}}{2}, \infty) \end{cases} \quad (63)$$

By taking into account (62), it follows that:

$$\bar{d}_{R,k}(t) - \frac{1}{4} \geq -M_{k+1}, \forall t \in [t_0, t_0 + \tau_{k+1}] \quad (64)$$

hence, by exploiting (64), we can construct a function $d_{R,k+1} \in C^0([t_0, \infty); \mathbb{R})$, satisfying:

$$\begin{cases} d_{R,k+1}(t) \\ := -M_{k+1}, & t \in [t_0, t_0 + \frac{\tau_{k+1}}{2}] \\ \in [-M_{k+1}, \bar{d}_{R,k}(t) - \frac{1}{4}], & t \in [t_0 + \frac{\tau_{k+1}}{2}, t_0 + \tau_{k+1}] \\ := \bar{d}_{R,k}(t) - \frac{1}{4}, & t \in [t_0 + \tau_{k+1}, \infty) \end{cases} \quad (65)$$

Notice that (59b), follows from (64) and (65). Also, define:

$$\zeta(t) := \frac{1}{2} \sqrt{\frac{g(t)}{\phi_{R,k}(t) + 1}}, t \geq t_0 \quad (66)$$

Due to (30) and (31), the map $t \rightarrow D_{R,n-k}(t, \zeta(t))$, $t \geq t_0$ is continuous and there exists a function $\mu_{k+1} \in C^1([t_0, \infty); \mathbb{R})$ satisfying:

$$0 < \mu_{k+1}(t) \leq D_{R,n-k}(t, \zeta(t)), \text{ for every } t \geq t_0 \quad (67)$$

Finally, define $p_R \in C^1([t_0, \infty); \mathbb{R})$ as:

$$p_R(t) := -\frac{\theta(t)\phi_{R,k}(t)}{\mu_{k+1}(t)}, t \geq t_0 \quad (68)$$

which due to (63) and (67), satisfies (59a).

Proof of (61): We consider two cases:

Case 1: $t \in [t_0, t_0 + \frac{\tau_{k+1}}{2}]$. By taking into account (62a), it follows that $M_{k+1} \geq -\bar{d}_{R,k}(t) + \frac{\phi_{R,k}(t)\xi^2}{g(t)}$ for every $t \in [t_0, t_0 + \frac{1}{2}]$, which in conjunction with (62b) and (65) imply:

$$\bar{d}_{R,k}(t) - d_{R,k+1}(t) \geq \frac{\phi_{R,k}(t)\xi^2}{g(t)}, \forall t \in [t_0, t_0 + \frac{\tau_{k+1}}{2}] \quad (69)$$

Hence, from (31), (59a) and (69) we deduce that

$$\begin{aligned} & r^2(p_R(t)D_{R,n-k}(t, r) + \phi_{R,k}(t)) \leq r^2\phi_{R,k}(t) \\ & \leq \frac{\xi^2\phi_{R,k}(t)}{g(t)}g(t) \leq (\bar{d}_{R,k}(t) - d_{R,k+1}(t))g(t), \forall r \in (0, \xi] \end{aligned}$$

which implies (61) for $t \in [t_0, t_0 + \frac{\tau_{k+1}}{2}]$.

Case 2: $t \in [t_0 + \frac{\tau_{k+1}}{2}, \infty)$. We consider two further subcases.

Subcase 1: $0 < r \leq \zeta(t)$. Due to (65), it holds $\bar{d}_{R,k}(t) - d_{R,k+1}(t) \geq \frac{1}{4}$ for every $t \in [t_0 + \frac{\tau_{k+1}}{2}, \infty)$, hence, by exploiting (31), (59a) and (66) we have

$$\begin{aligned} & r^2(p_R(t)D_{R,n-k}(t, r) + \phi_{R,k}(t)) \leq \zeta^2(t)\phi_{R,k}(t) \\ & \leq (\bar{d}_{R,k}(t) - d_{R,k+1}(t))g(t) \end{aligned} \quad (70)$$

Subcase 2: $\zeta(t) \leq r \leq \xi$. By taking into account (59b), (63), (32), (66), (67) and (68), we deduce that

$$\begin{aligned} & r^2(p_R(t)D_{R,n-k}(t, r) + \phi_{R,k}(t)) \leq r^2\phi_{R,k}(t) \\ & \times \left(-\frac{D_{R,n-k}(t, \zeta(t))}{\mu_{k+1}(t)} + 1 \right) \leq 0 \leq (\bar{d}_{R,k}(t) - d_{R,k+1}(t))g(t) \end{aligned}$$

The latter, in conjunction with (70) asserts that (61) is fulfilled for every $t \in [t_0 + \frac{\tau_{k+1}}{2}, \infty)$. Both cases above guarantee that (61) holds for all $t \in [t_0, \infty)$ as required.

Proof of (37a) and (37b) for $k = k+1$: The proof of (37b) is the same with that given in proof of Theorem 1.1 in [3] and is omitted. Finally, the proof of (37a) is based on the construction of the map $p_{R,1}(\cdot)$ as involved in (55c), and is also the same with that given in proof of Theorem 1.1 in [3].

We have shown that all requirements of Claim 1 hold, which, as was pointed out, establishes that for every $R > 0$ Hypothesis A2 is fulfilled. We therefore conclude, that for system (1) Hypothesis 2.2 is satisfied hence, by invoking the result of Proposition 2.2, it follows that the SODP is solvable for (1) with respect to \mathbb{R}^n . The establishment of the second statement of Theorem 1.1, follows directly from Proposition 2.1. \square

Example: Consider the two dimensional system $\dot{x}_1 = x_2^3 a_1(t, x_1) + x_2 a_2(t, x_1) + a_3(t, x_1)$; $\dot{x}_2 = x_2 b(t, x_1)$; $y = x_1$, where $a_i(t, x_1) > 0, i = 1, 2, 3, \forall (t, x_1) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ and assume that there exist $C \in N$ and a constant $L > 0$ such that $|a_i(t, x_1)| \leq C(t)(|x_1| + L), i = 1, 2, 3; |b(t, x_1)| \leq L, \forall (t, x_1) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. It is easy to check that the system satisfies all conditions of Theorem 1.1, therefore the SODP is solvable with respect to \mathbb{R}^2 .

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