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Functional model realizations for Schur functions on $\mathbb{C}^+$

Mikael Kurula[1], Joseph A. Ball[2], Olof J. Staffans[1], and Hans Zwart[3]

***Abstract*— For an arbitrary given operator Schur function defined on the complex right-half plane, we give a controllable energy-preserving and an observable co-energy-preserving de Branges-Rovnyak functional model realization. Topics appearing only in the right-half-plane setting, such as the extrapolation space, are also discussed.**

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***Index Terms*— Schur function, right half-plane, functional model, de Branges-Rovnyak, reproducing kernel**

I. INTRODUCTION

Let $\mathcal{U}$ and $\mathcal{Y}$ be separable Hilbert spaces and let $\mathcal{B}(\mathcal{U},\mathcal{Y})$ denote the class of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$. It is by now very well known that any rational function ϕ holomorphic in a neighborhood of the origin with values in $\mathcal{B}(\mathcal{U},\mathcal{Y})$ can be realized as the transfer function of an input/state/output linear system. This means that there is a Hilbert space $\mathcal{X}$ (the state space) and a bounded operator *system matrix* (connecting matrix)

$$\mathbf{U} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

so that $\phi(z)$ has the representation

$$\phi(z) = D + zC(1 - zA)^{-1}B. \tag{1}$$

If we associate with $\mathbf{U}$ the discrete-time input/state/output system

$$\Sigma_{\mathbf{U}} : \quad \left\{ \begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array} \right. , \tag{2}$$

then the meaning of (1) is that ϕ is the *transfer function* of the i/s/o system $\Sigma_{\mathbf{U}}$.

We shall focus on one particular theory that describes concrete realizations of the form (1) for contraction-valued functions ϕ which is due to de Branges and Rovnyak [14], [15] and relies on reproducing kernel Hilbert spaces. Over the years numerous extensions of the de Branges-Rovnyak theory has been developed, and this still continues to be a very active field of research. To mention a few works in this direction, see e.g. [2] and its references or [8], [10], [13]. For other related work see e.g. [1], [3], [4], [5], [6], [7], [9], [12], [17].

We work in the complex right-half plane, but our exposition follows the disk case closely. We avoid the use of linear fractional transformations in order to expose the essential technical differences to the disk setting. Because of this choice, we need to use continuous-time systems theory rather than the discrete-time theory which corresponds to the disk case. We refer to [11] for proofs and more details.

II. CONTINUOUS-TIME INFINITE-DIMENSIONAL SYSTEMS THEORY

A. Rigged Hilbert spaces

The generator A of a C_0 semigroup, see e.g. [18, Chapter 3], is closed and its domain $\operatorname{dom}(A)$ is dense in $\mathcal{X}$. Moreover, the resolvent set $\rho(A)$ of a C_0 semigroup generator A contains some complex right-half plane. For such an operator, $\operatorname{dom}(A)$ is a Hilbert space with the inner product

$$\langle x, z\rangle_{\operatorname{dom}(A)} = \langle (\beta - A)x, (\beta - A)z\rangle_{\mathcal{X}} , \tag{3}$$

where β is some fixed but arbitrary complex number in $\rho(A)$.

Thus $\mathcal{X}_1 := \operatorname{dom}(A)$ with the norm $\|x\|_1 := \|(\beta - A)x\|_{\mathcal{X}}$ is a dense subspace of $\mathcal{X}$. It follows from (3) that A maps $\operatorname{dom}(A) = \mathcal{X}_1$ equipped with this norm continuously into $\mathcal{X}$. Denote by $\mathcal{X}_{-1}$ the completion of $\mathcal{X}$ with respect to the norm $\|x\|_{-1} = \|(\beta - A)^{-1}x\|_{\mathcal{X}}$. The operator A can then also be considered as a continuous operator which maps the dense subspace $\mathcal{X}_1$ of $\mathcal{X}$ into $\mathcal{X}_{-1}$, and we denote the unique extension of A to an operator in $\mathcal{B}(\mathcal{X},\mathcal{X}_{-1})$ by $A|_{\mathcal{X}}$.

The triple $\mathcal{X}_1 \subset \mathcal{X} \subset \mathcal{X}_{-1}$ constructed above is called a *Gelfand triple*, and the three spaces are also said to be *rigged*.

The (usually unbounded) adjoint A^* of a semigroup generator A also generates a C_0 semigroup on the same space as A. We denote the Gelfand triple associated to A^* by $\mathcal{X}_1^d \subset \mathcal{X} \subset \mathcal{X}_{-1}^d$. Moreover, we identify $\mathcal{X}_{\pm 1}^d$ and the dual of $\mathcal{X}_{\mp 1}$ with pivot space $\mathcal{X}$, so that, e.g.,

$$\left(x, x^d\right)_{\mathcal{X}_1, \mathcal{X}_{-1}^d} = \left\langle x, x^d\right\rangle_{\mathcal{X}}, \quad x \in \mathcal{X}_1,\, x^d \in \mathcal{X};$$

see Proposition 2.3 below.

B. Definition of a system node and its transfer function

A system node is the appropriate continuous-time analogue of the bounded connecting operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of a linear discrete-time system (2).

Definition 2.1: A linear operator

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \supset \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right) \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

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[1] Mikael Kurula and Olof J. Staffans are with the Department of mathematics, Åbo Akademi University, Fänriksgatan 3B, FIN-20500 Åbo, Finland. `mkurula@abo.fi, staffans@abo.fi`

[2] Joseph A. Ball is with the Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA. `joball@math.vt.edu`

[3] Hans Zwart is with the Department of applied mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands. `h.j.zwart@utwente.nl`

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is called a *system node* on the triple $(\mathcal{U},\mathcal{X},\mathcal{Y})$ of Hilbert spaces if it has all of the following properties:

1) The operator $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ is closed.
2) The operator
$$Ax := \begin{bmatrix}A\&B\end{bmatrix}\begin{bmatrix}x\\0\end{bmatrix} \quad \text{defined on}$$
$$\operatorname{dom}(A) := \left\{x\in\mathcal{X} \mid \begin{bmatrix}x\\0\end{bmatrix}\in\operatorname{dom}\left(\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]\right)\right\},$$
is the generator of a C_0-semigroup on $\mathcal{X}$.
3) The operator $\begin{bmatrix}A\&B\end{bmatrix}$ can be extended to an operator $\begin{bmatrix}A|_{\mathcal{X}} & B\end{bmatrix}$ that maps $\left[\begin{smallmatrix}\mathcal{X}\\\mathcal{U}\end{smallmatrix}\right]$ continuously into $\mathcal{X}_{-1}$.
4) The domain of $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ satisfies the condition
$$\operatorname{dom}\left(\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]\right)=\left\{\begin{bmatrix}x\\u\end{bmatrix}\in\begin{bmatrix}\mathcal{X}\\\mathcal{U}\end{bmatrix} \mid A|_{\mathcal{X}}x+Bu\in\mathcal{X}\right\}.$$

When these conditions are satisfied, $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$ are called the *input space*, *state space*, and *output space*, respectively, of the system node.

The conditions imposed on a system node guarantee that the continuous-time linear system
$$\begin{bmatrix}\frac{\mathrm{d}x}{\mathrm{d}t}(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}A\&B\\C\&D\end{bmatrix}\begin{bmatrix}x(t)\\u(t)\end{bmatrix},\quad t\geq 0,$$
has enough solutions to give rise to a meaningful theory; see [18, §4.6].

The continuous-time analogue of the transfer function (1) is presented in the following definition. This relies on the fact that $\left[\begin{smallmatrix}(\mu-A|_{\mathcal{X}})^{-1}B\\1\end{smallmatrix}\right]$ maps $\mathcal{U}$ into the domain of $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ for every $\mu\in\rho(A)$; see [18, Lemma 4.7.3].

Definition 2.2: The operators A and B in Definition 2.1 are the *main operator* and *control operator* of the system node $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$, respectively. The *observation operator* $C:\operatorname{dom}(A)\to\mathcal{Y}$ of $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ is the operator
$$Cx := \begin{bmatrix}C\&D\end{bmatrix}\begin{bmatrix}x\\0\end{bmatrix},\quad x\in\operatorname{dom}(A),\tag{4}$$
and the *transfer function* $\widehat{\mathfrak{D}}:\rho(A)\to\mathcal{B}(\mathcal{U},\mathcal{Y})$ of $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ is the operator-valued holomorphic function
$$\widehat{\mathfrak{D}}(\mu) := \begin{bmatrix}C\&D\end{bmatrix}\begin{bmatrix}(\mu-A|_{\mathcal{X}})^{-1}B\\1\end{bmatrix},\quad \mu\in\rho(A).\tag{5}$$

As is customary in continuous-time systems theory (see [19]), we identify any two $\mathcal{B}(\mathcal{U},\mathcal{Y})$-valued analytic functions that coincide on some complex right half-plane (for some $\omega\in\mathbb{R}$)
$$\mathbb{C}^+_\omega := \{\mu\in\mathbb{C} \mid \operatorname{Re}\mu>\omega\}.$$

This identification defines an equivalence relation on the set of transfer functions of system nodes. By a *realization* of a given analytic function φ, we mean a system node $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ with transfer function $\widehat{\mathfrak{D}}$ identified with φ in this way.

C. Controllability, observability, and passivity

Let $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ be a system node and denote the component of $\rho(A)$ that contains some right-half plane by $\rho_\infty(A)$. We say that $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ is *controllable* if
$$\operatorname{span}\left\{(\lambda-A|_{\mathcal{X}})^{-1}Bu \mid \lambda\in\rho_\infty(A),\, u\in\mathcal{U}\right\}$$
is dense in the state space $\mathcal{X}$. The system node $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ is *observable* if
$$\bigcap_{\lambda\in\rho_\infty(A)}\ker\left(C(\lambda-A)^{-1}\right)=\{0\}.$$

Proposition 2.3: Every system node $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ on the triple $(\mathcal{U},\mathcal{X},\mathcal{Y})$ of Hilbert-spaces has the following properties:

1) The adjoint $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]^*$ is a system node on $(\mathcal{Y},\mathcal{X},\mathcal{U})$. The main operator of $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]^*$ is $A^d=A^*$, the control operator is $B^d=C^*\in\mathcal{B}(\mathcal{Y},\mathcal{X}^d_{-1})$, the observation operator is $C^d=B^*\in\mathcal{B}(\mathcal{X}^d_1,\mathcal{U})$, and the transfer function satisfies $\widehat{\mathfrak{D}}^d(\mu)=\widehat{\mathfrak{D}}(\overline{\mu})^*$ for all $\mu\in\rho(A^*)$, where $\widehat{\mathfrak{D}}$ is the transfer function of $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$.
2) The system node $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ is controllable if and only if $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]^*$ is observable and vice versa.

A system node $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ is called *(scattering) passive* if it satisfies for all $\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]\in\operatorname{dom}\left(\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]\right)$ and $\left[\begin{smallmatrix}z\\y\end{smallmatrix}\right]=\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]$:
$$\langle z,x\rangle_{\mathcal{X}}+\langle x,z\rangle_{\mathcal{X}}\leq\langle u,u\rangle_{\mathcal{U}}-\langle y,y\rangle_{\mathcal{Y}}.$$

If this holds holds with equality rather than with inequality then $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ is called *(scattering) energy preserving*. We say that $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ is *(scattering) co-energy preserving* if the dual system node $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]^*$ is energy preserving. By a *(scattering) conservative* system node we mean one that is both energy preserving and co-energy preserving.

Partly due to the following result, see [18, Theorem 11.1.5(viii)], we may in our context replace $\rho_\infty(A)$ by $\mathbb{C}^+$ in the definitions of controllability and observability:

Lemma 2.4: For a passive system node with state space $\mathcal{X}$ and main operator A, we have $\mathbb{C}^+\subset\rho(A)$.

We are now ready to present the first functional model.

III. THE CONTROLLABLE ENERGY PRESERVING MODEL

We denote the class of functions $f:\mathbb{C}^+\to\mathcal{B}(\mathcal{U},\mathcal{Y})$, such that $\|f(\mu)\|\leq 1$ for all $\mu\in\mathbb{C}^+$, i.e., the Schur class over $\mathbb{C}^+$, by $\mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$. In analogy to the disk case we obtain the following fundamental result:

Theorem 3.1: If $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$ then the $\mathcal{B}(\mathcal{U})$-valued function

$$K_c(\mu,\lambda) = \frac{1-\varphi(\overline{\mu})^*\varphi(\overline{\lambda})}{\mu+\overline{\lambda}},$$

is a positive kernel on $\mathbb{C}^+$.

Let now $\mathcal{H}_c$ denote the Hilbert space whose reproducing kernel is K_c and let $e_c(\cdot)$ be the point-evaluation mapping on $\mathcal{H}_c$, so that $e_c(\lambda)^*u = K_c(\cdot,\lambda)u$ for all $\lambda \in \mathbb{C}^+$ and $u \in \mathcal{U}$. Introduce the mapping (for $u \in \mathcal{U}$, $\lambda \in \mathbb{C}^+$):

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c : \begin{bmatrix} e_c(\overline{\lambda})^*u \\ u \end{bmatrix} \mapsto \begin{bmatrix} \lambda e_c(\overline{\lambda})^*u \\ \varphi(\lambda)u \end{bmatrix}. \qquad (6)$$

Theorem 3.2: The formula (6) extends via linearity and limit-closure to an energy-preserving system node with input space $\mathcal{U}$, state space $\mathcal{H}_c$, and output space $\mathcal{Y}$. In the sequel we let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ denote this extension.

Denoting the main and control operators of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ by A_c and B_c, respectively, we obtain that

$$(\lambda - A_c|_{\mathcal{H}_c})^{-1}B_c = e_c(\overline{\lambda})^*, \quad \lambda \in \mathbb{C}^+.$$

In addition, $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is controllable:

$$\overline{\operatorname{span}}\left\{(\lambda - A_c|_{\mathcal{H}_c})^{-1}B_c u \mid u \in \mathcal{U},\, \lambda \in \mathbb{C}^+\right\} = \mathcal{H}_c,$$

and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ realizes φ:

$$\begin{bmatrix} C_c\&D_c \end{bmatrix} \begin{bmatrix} (\mu - A_c|_{\mathcal{H}_c})^{-1}B_c \\ 1 \end{bmatrix} = \varphi(\mu), \quad \mu \in \mathbb{C}^+.$$

Note that $\mathcal{H}_c$ is a Hilbert space of functions. Moreover, the next result says that every controllable energy-preserving realization of φ is unitarily similar to $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$. This justifies the terminology *canonical functional-model* system node for $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$.

Theorem 3.3: Let $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$ and let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a controllable and energy preserving realization of φ with state space $\mathcal{X}$. Then the mapping $\Delta : \mathcal{H}_c \to \mathcal{X}$ defined by

$$\Delta e_c(\overline{\lambda})^*u := (\lambda - A|_{\mathcal{X}})^{-1}Bu, \quad \lambda \in \mathbb{C}^+,\, u \in \mathcal{U},$$

extends by linearity and limit-closure to a unitary operator $\mathcal{H}_c \to \mathcal{X}$. Moreover, Δ *intertwines* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$:

$$\operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) = \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c\right) \quad \text{and}$$

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c,$$

so that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ are unitarily similar.

The following theorem gives the action of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ on generic elements in its domain (as opposed to only linear combinations of elements of the form $\left[\begin{smallmatrix} e_c(\overline{\lambda})^*u \\ u \end{smallmatrix}\right]$), cf. (6):

Theorem 3.4: A pair $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{U} \end{smallmatrix}\right]$ lies in $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ if and only if for some, or equivalently for all, $\lambda \in \mathbb{C}^+$, the function $x - e_c(\lambda)^*u$ lies in $\operatorname{dom}(A_c)$. For an arbitrary $\lambda \in \mathbb{C}^+$, the operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ maps an arbitrary $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ in its domain into

$$\begin{bmatrix} \mu \mapsto -\mu x(\mu) - \varphi(\overline{\mu})^*\gamma_\lambda + \big(1 - \varphi(\overline{\mu})^*\varphi(\overline{\lambda})\big)u \\ \gamma_\lambda + \varphi(\overline{\lambda})u \end{bmatrix}, \qquad (7)$$

where $\gamma_\lambda = C_c\big(x - e_c(\lambda)^*u\big)$.

IV. THE OBSERVABLE CO-ENERGY PRESERVING MODEL

A. Introduction and uniqueness of the observable model

In this section we present an observable co-energy-preserving functional model realization of an arbitrary given $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$. This realization uses the Hilbert space $\mathcal{H}_o$ with reproducing kernel

$$K_o(\mu,\lambda) = \frac{1-\varphi(\mu)\varphi(\lambda)^*}{\mu+\overline{\lambda}}$$

as state space.

Theorem 4.1: Suppose that we are given a function $\varphi \in \mathcal{S}(\mathbb{C}_+;\mathcal{U},\mathcal{Y})$ and define $\mathcal{H}_o = \mathcal{H}(K_o)$ as above.

1) The following operator is an observable, co-energy-preserving system node with transfer function equal to φ on $\mathbb{C}^+$:

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o : \begin{bmatrix} x \\ u \end{bmatrix} \mapsto \begin{bmatrix} z \\ y \end{bmatrix}, \quad \text{where}$$

$$z(\mu) := \mu x(\mu) + \varphi(\mu)u - y, \quad \mu \in \mathbb{C}^+, \qquad (8)$$

$$y := \lim_{\operatorname{Re}\eta \to \infty} \eta x(\eta) + \varphi(\eta)u. \qquad (9)$$

The domain of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ is

$$\left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_o \\ \mathcal{U} \end{bmatrix} \mid \exists y \in \mathcal{Y} :\ z \text{ in (8) lies in } \mathcal{H}_o \right\}.$$

For every $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o\right)$, the $y \in \mathcal{Y}$ such that z given in (8) is in $\mathcal{H}_o$ is unique and it is given by (9).

2) The kernel functions $K_o(\cdot,\lambda) = e_o(\lambda)^*$, $\lambda \in \mathbb{C}^+$, for the Hilbert space $\mathcal{H}_o$ are given by

$$e_o(\lambda)^* = (\overline{\lambda} - A_o^*|_{\mathcal{H}_o})^{-1}C_o^*, \quad \lambda \in \mathbb{C}^+,$$

Note that the action of the operator $\begin{bmatrix} C_o\&D_o \end{bmatrix}$ is to take a limit, and that the limit $\lim_{\operatorname{Re}\eta\to\infty} \varphi(\eta)u$ does not exist in general. The limit in (9) exists, however, for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o\right)$, since z in (8) is an element of $\mathcal{H}_o \subset H^2(\mathbb{C}^+;\mathcal{Y})$ and therefore z has the limit zero at infinity. We have the following uniqueness result:

Theorem 4.2: Let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be an observable and co-energy-preserving realization of φ with state space $\mathcal{X}$. The mapping

$$\Delta : e_o(\lambda)^*y \mapsto \left(\overline{\lambda} - A^*|_{\mathcal{X}}\right)^{-1} C^*y, \quad \lambda \in \mathbb{C}^+,\, y \in \mathcal{Y}.$$

extends into a unitary operator from $\mathcal{H}_o$ onto $\mathcal{X}$. The operator $\left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$ maps $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o\right)$ one-to-one onto $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$, and

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o.$$

B. The extrapolation space

Similar to the disk case one can obtain the following formula for the resolvent of A_o:

$$\left((\alpha - A_o)^{-1}x\right)(\mu) = \frac{x(\mu) - x(\alpha)}{\alpha - \mu}, \quad \alpha, \mu \in \mathbb{C}^+,\, x \in \mathcal{H}_o.$$

This suggests a way to concretely identify the (-1)-scaled rigged space (also called "extrapolation space") $\mathcal{H}_{o,-1}$ defined abstractly as the completion of the space $\mathcal{H}_o$ in the norm

$$\|x\| = \|(\beta - A_o)^{-1}\|_{\mathcal{H}_o}.$$

More precisely, following [16] we define

$$\mathcal{H}_{o,-1} = \left\{ x : \mathbb{C}^+ \to \mathcal{Y} \mid \mu \mapsto \frac{x(\mu) - x(\beta)}{\beta - \mu} \in \mathcal{H}_o \right\} \quad (10)$$

with norm given by

$$\|x\|_{\mathcal{H}_{o,-1}} = \left\| \mu \mapsto \frac{x(\mu) - x(\beta)}{\beta - \mu} \right\|_{\mathcal{H}_o}. \quad (11)$$

We emphasize again that the $\mathcal{H}_{o,-1}$ norm (and inner product) depends on the choice of $\beta \in \mathbb{C}_+$; different choices of β give different norms although all such norms are equivalent. The elements of $\mathcal{H}_{o,-1}$ are equivalence classes of functions modulo constant terms.

Theorem 4.3: Let the space $\mathcal{H}_{o,-1}$ be given by (10) and (11).

1) The map $\iota : x \mapsto [x]$ embeds $\mathcal{H}_o$ into $\mathcal{H}_{o,-1}$ as a dense subspace. A given element $[z] \in \mathcal{H}_{o,-1}$ is of the form $\iota(x)$ for some $x \in \mathcal{H}_o$ if and only if the function
$$\mu \mapsto \frac{z(\mu) - z(\beta)}{\beta - \mu}, \quad \mu \in \mathbb{C}^+,$$
is not only in $\mathcal{H}_o$ but in fact is in $\mathrm{dom}\,(A_o) = \mathcal{H}_{o,1} \subset \mathcal{H}_o$. When this is the case, the equivalence class representative x for $[z]$, for which $x \in \mathcal{H}_o$, is uniquely determined by the decay condition at infinity:
$$\lim_{\mathrm{Re}\,\eta \to \infty} x(\eta) = 0.$$
2) Define an operator $A_o|_{\mathcal{H}_o} : \mathcal{H}_o \to \mathcal{H}_{o,-1}$ by
$$A_o|_{\mathcal{H}_o} x := [\mu \mapsto \mu x(\mu)], \quad x \in \mathcal{H}_o,\, \mu \in \mathbb{C}^+.$$
When $\mathcal{H}_o$ is identified as a linear submanifold of $\mathcal{H}_{o,-1}$ as above, then $A_o|_{\mathcal{H}_o}$ is the unique extension of $A_o : \mathrm{dom}\,(A_o) \to \mathcal{H}_o$ to an operator in $\mathcal{B}(\mathcal{H}_o; \mathcal{H}_{o,-1})$.
3) With $\mathcal{H}_{o,-1}$ identified concretely as in (10), the action of $B_o : \mathcal{U} \to \mathcal{H}_{o,-1}$ is given by
$$B_o u := [\mu \mapsto \varphi(\mu) u], \quad u \in \mathcal{U},\, \mu \in \mathbb{C}^+.$$

Similar results can be obtained for the controllable energy-preserving model. Finally, we mention that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ can be connected to their discrete-time counterparts in using an internal Cayley system transformation; see [11] for details.

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