

# Partial synchronization manifolds for linearly time-delay coupled systems

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**Abstract**— Sometimes a network of dynamical systems shows a form of incomplete synchronization characterized by synchronization of some but not all of its systems. This type of incomplete synchronization is called *partial synchronization*. Partial synchronization is associated with the existence of *partial synchronization manifolds*, which are linear invariant subspaces of  $\mathcal{C}$ , the state space of the network of systems. We focus on partial synchronization manifolds in networks of systems that interact via linear coupling functions subject to time-delays. For such networks a number of necessary and sufficient conditions for the existence of partial synchronization manifolds are presented. A simple, illustrative example demonstrates the application of our results.

## I. INTRODUCTION

Network synchronization, understood as the “stable time-correlated behavior” of the systems in the network, finds applications in fields varying from biology to engineering. Examples include the simultaneous flashing of thousands of fireflies [1], the synchronous firing of pacemaker neurons that regulate our heartbeat [2], entrainment of our circadian rhythms to the 24-hour day-night cycle [3], platooning of vehicles [4], cooperation in robotic systems [5], [6] and secure communication [7], [8], [9]. Many more examples are found in [10], [11], [12] and the references therein.

In order to achieve synchronization, coupled systems share information about their state and update their states according to specific coupling law(s). Because the exchange and processing of information takes a certain amount of time, time-delays appear naturally in these coupling law(s). For instance, the finite propagation speed of the membrane potential through the axon of a neuron [13] makes instantaneous interaction between neurons impossible; When drivers (try to) synchronize the velocities of their vehicles, they experience time-delays related to their response latency, which is in this situation known to vary between 0.6 s and 2 s [14].

Much of the research on network synchronization of (time-delay) coupled systems is devoted to what we call *full synchronization*, that is, the asymptotically match of the state variables of *all* systems in the network. However, full synchronization cannot always be achieved, for instance when the coupling between the systems is not strong enough or time-delays are too large. The impossibility to fully

synchronize is typical in large networks. Moreover, full synchronization in networks is not always desirable; An abnormal amount of synchronization in brain networks is a signature of brain disorders like epilepsy and Parkinson’s disease [15]. A fundamental question is what happens in case full synchronization does not occur. One possibility is that there is no synchronization at all, but a network of coupled systems may also show a form of incomplete synchronization, which we call *partial synchronization*. More precisely, we say that a network partially synchronizes if the state variables of some, but not all, of its systems asymptotically match.

For analysis of partial synchronization in networks the first step to take is to find linear invariant subspaces of the coupled systems’ state-space  $\mathcal{C}$  that correspond to particular modes of partial synchronization. Such linear invariant subspaces of  $\mathcal{C}$  we call *partial synchronization manifolds*. We focus on partial synchronization manifolds that are induced by the structure of the network and the specific coupling law. These partial synchronization manifolds are robust in the sense that their existence does not depend on the specific dynamics of the systems. In the search for partial synchronization manifolds we restrict ourselves to linear coupling functions defined by weighted differences of the (time-delayed) outputs of the systems. These linear coupling functions appear in networks of coupled neurons [16], [17], [18], networks of biological systems [19], [20], coupled mechanical systems [6], [21], [22], [23] and electrical systems [8], [24]. The coupling for system  $i$  depends either on the difference  $y_j(t - \tau) - y_i(t)$  or  $y_j(t - \tau) - y_i(t - \tau)$ , where  $y_i(t)$ ,  $y_i(t - \tau)$  are the output, respectively time-delayed output of system  $i$ , and  $y_j(t - \tau)$  is the time-delayed output of system  $j$  that connects to system  $i$ . We remark that there is a fundamental difference between coupling defined by differences  $y_j(t - \tau) - y_i(t)$  and the type of coupling that depends on differences  $y_j(t - \tau) - y_i(t - \tau)$ ; The former type of coupling is *invasive*, i.e. the coupling does not vanish when systems  $i$  and  $j$  are synchronized, while the latter type of coupling is *non-invasive*. Note that delay-free coupling, i.e. coupling with  $\tau = 0$ , is non-invasive.

We present equivalent conditions for the existence of partial synchronization manifolds in networks of systems with linear time-delay coupling. Each condition has its advantages and drawbacks. First we establish a condition involving the row-sums of certain blocks in the weighted adjacency matrix of the network. This condition is very useful for the design of networks that should exhibit partial synchronization. Moreover, checking this condition is com-

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putationally efficient ( $\mathcal{O}(k^2)$  for a network of  $k$  systems) and, in case of rational interaction weights (to be defined in the next section), it can be done in exact arithmetic. However, the required block-structure is often not present in a given weighted adjacency matrix, and finding a rearrangement of the nodes that brings this matrix in the desired block-form may be difficult. We therefore present additional conditions for the existence of partial synchronization manifolds. A second condition is expressed in terms of invariant subspaces of the weighted adjacency matrix and the weighted degree matrix. This condition allows identification of partial synchronization manifolds from repeating patterns in the eigenvectors of these matrices. We remark that verification of this condition is computationally less efficient than verification of the first condition because determining the eigenvectors of a  $k \times k$  matrix has a computational cost of  $\mathcal{O}(k^3)$ . Moreover, eigenvectors can only be numerically computed with finite precision. A third condition shows that the existence of a partial synchronization manifold is equivalent to the existence of a solution of a matrix equation. It is shown in [25] that this matrix solution allows one to assess the stability of a partial synchronization manifold. (The question about the stability of the partial synchronization manifold, which is the next step in the analysis of partial synchronization in networks, is however not considered in this paper.) For coupling defined by differences of the form  $y_j(t-\tau) - y_i(t)$  we present a fourth condition, which involves so-called *balanced equivalence relations* [26]. The benefit of the latter approach is that it can be verified graphically. Moreover, there is an algorithm to detect the minimal set of balanced equivalence relations in networks with uniform interaction weights [27].

## II. PROBLEM SETTING

It is convenient to represent the network by a *directed graph*  $\mathcal{G}$ , being an ordered pair  $(\mathcal{V}, \mathcal{E})$ , with  $\mathcal{V} = \{1, 2, \dots, k\}$  the set of *nodes* and  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$  the set of *edges*. An edge  $e_\ell = (i, j) \in \mathcal{E}$  defines a connection from node  $j \in \mathcal{V}$  to node  $i \in \mathcal{V}$ . The edge  $e_\ell = (i, j)$  may be represented graphically as an arrow with its tail at  $j$  and its head at  $i$ . We assume that the graph  $\mathcal{G}$  is *strongly connected*, that is, for every two nodes  $i, j \in \mathcal{V}$  there exists a path in  $\mathcal{G}$  that joins  $i$  and  $j$ . This assumption ensures that the network can not be decomposed into two or more disjoint networks. In addition we assume that  $\mathcal{G}$  is *simple*, which means that there are no edges from any  $i \in \mathcal{V}$  to itself, i.e.  $(i, i) \notin \mathcal{E}$  for all  $i \in \mathcal{V}$ . We denote by  $\mathcal{N}_i$  the set of *neighbors* of node  $i$ ,

$$\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}.$$

For the graph  $\mathcal{G}$  shown in Figure 1 we have  $\mathcal{N}_1 = \{4\}$ ,  $\mathcal{N}_2 = \{1\}$ ,  $\mathcal{N}_3 = \{4\}$  and  $\mathcal{N}_4 = \{2, 3\}$ .

Let  $u_i(t)$  and  $y_i(t)$  be the input, respectively, output of system  $i$ . For a constant  $\tau > 0$ ,  $y_i(t-\tau)$  is the *time-delayed* output of system  $i$ . We assume that each system can measure its own (time-delayed) output and we consider the following two linear time-delay coupling laws,

$$u_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} [y_j(t-\tau) - y_i(t)], \quad (1)$$

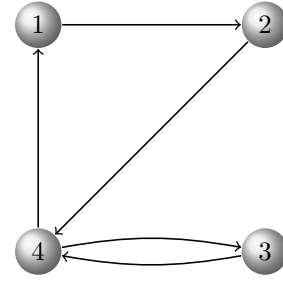


Fig. 1. A simple strongly connected graph  $\mathcal{G}$  with  $\mathcal{V} = \{1, 2, 3, 4\}$  and  $\mathcal{E} = \{(1, 4), (2, 1), (4, 2), (3, 4), (4, 3)\}$ .

and

$$u_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} [y_j(t-\tau) - y_i(t-\tau)], \quad (2)$$

where the positive constants  $a_{ij}$  are the *interaction weights*. We let the matrix  $A \in \mathbb{R}^{k \times k}$  have its  $ij^{\text{th}}$  entry equal to  $a_{ij}$  if  $(i, j) \in \mathcal{E}$  and 0 otherwise. The matrix  $A$  is known as the *weighted adjacency matrix*. We let  $D \in \mathbb{R}^{k \times k}$ ,

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_k \end{pmatrix},$$

with  $d_i = \sum_{j \in \mathcal{N}_i} a_{ij}$ , be the *weighted degree matrix*. The relation  $L = D - A$  defines the *weighted Laplacian matrix*.

The systems' dynamics are governed by the equations

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + Bu_i(t), \\ y_i(t) = Cx_i(t), \quad i = 1, 2, \dots, k. \end{cases} \quad (3)$$

Here  $x_i(t) \in \mathbb{R}^n$  is the state of system  $i$ ,  $u_i(t) \in \mathbb{R}^m$ ,  $1 \leq m \leq n$ , is its input and  $y_i(t) \in \mathbb{R}^m$  is its output. Matrices  $B$  and  $C$  are of appropriate dimension and  $\text{rank}(BC) = m$ . Function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is assumed to satisfy the usual conditions that ensure existence and uniqueness of solutions of the coupled systems (3), (1) or (3), (2), cf. [28]. The state-space of the coupled systems is  $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^{kn})$ , the space of continuous functions mapping the interval  $[-\tau, 0]$  into  $\mathbb{R}^{kn}$ . We let, for  $x_t \in \mathcal{C}$ ,  $x_t(\theta) = x(t+\theta)$ ,  $\theta \in [-\tau, 0]$ . A solution of the coupled systems through  $(t_0, \phi) \in \mathbb{R} \times \mathcal{C}$  is denoted by  $x_t(t_0, \phi)$  (or  $x(t; t_0, \phi) = x_t(t_0, \phi)(0)$ ),  $t \geq t_0$ . A set  $\mathcal{M} \subset \mathcal{C}$  is *flow invariant* if  $\phi \in \mathcal{M}$  implies  $x_t(t_0, \phi) \in \mathcal{M}$  for all  $t \geq t_0$ .

We are interested in flow invariant subspaces of  $\mathcal{C}$  of the form

$$\mathcal{P}(\Pi) := \{\phi \in \mathcal{C} | \phi = \text{col}(\phi_1, \dots, \phi_k) \text{ and } (I_{kn} - \Pi \otimes I_n)\phi(\theta) = 0, -\tau \leq \theta \leq 0\},$$

where  $\Pi \in \mathbb{R}^{k \times k}$  is a *permutation matrix* other than identity.

*Definition 1:*  $\mathcal{P}(\Pi)$  is *partial synchronization manifold* if  $\mathcal{P}(\Pi)$  is *flow invariant* with respect to *all* systems of the form (3) that are coupled via either (1) or (2).  $\blacktriangle$

It readily follows that the partial synchronization manifold induced by the permutation matrix  $\Pi$  corresponds to a unique

partitioning of  $\mathcal{V}$ . For instance, for  $\mathcal{V} = \{1, 2, 3\}$ ,  $\mathcal{P}(\Pi)$  with

$$\Pi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

corresponds to the partition  $\{\{1, 2\}, \{3\}\}$ . So  $\Pi$  defines a set of equivalence relations on  $\mathcal{V}$ , that are directly related to the partitioning of  $\mathcal{V}$  induced by  $\Pi$ . We denote such an equivalence relation by  $\sim_{\Pi}$ . Thus in the example above we have  $1 \sim_{\Pi} 2$ .

### III. BLOCK-STRUCTURED ADJACENCY MATRIX

We denote by

$$K = \dim \ker(I - \Pi)$$

the number of clusters in the network. Note that in this case a cluster is a proper subset of  $\mathcal{V}$  of systems that are synchronized (for identical initial data), *not* a cluster in the network topological sense as in, for instance, [29]. Note in addition that a cluster may consist of a single system. A cluster is characterized by synchronized states and synchronized outputs of the systems within the cluster. Let  $\bar{y}_{\ell}(t)$  be the synchronized output of a system in cluster  $\ell$ ,  $\ell = 1, 2, \dots, K$ . Since the dynamics of the systems are *identical* and  $\text{rank}(BC) = m$ , whence  $\text{rank}(CB) = m$ , i.e. each system is output-controllable, it is immediate that  $\mathcal{P}(\Pi)$  is flow invariant independent of the vectorfield  $f$  if and only if, amongst all clusters, each system in a cluster receives the same input  $\bar{u}_{\ell}(t)$  as the other systems in that cluster.

Suppose that we relabel the nodes such that systems  $1, \dots, k_1$  belong to cluster 1, systems  $k_1 + 1, \dots, k_2$  belong to cluster 2, and so on. Then the weighted adjacency matrix can be partitioned as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1K} \\ A_{21} & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{(K-1)K} \\ A_{K1} & \cdots & A_{K(K-1)} & A_{KK} \end{pmatrix}. \quad (4)$$

The diagonal blocks in the structured adjacency matrix (4) define the interactions of systems *within* a cluster, the off-diagonal blocks define the interactions of systems *amongst* clusters. Let  $u_{C_{\ell}}(t) = \text{col}(\bar{u}_{\ell}(t), \dots, \bar{u}_{\ell}(t))$ ,  $y_{C_{\ell}}(t) = \text{col}(\bar{y}_{\ell}(t), \dots, \bar{y}_{\ell}(t))$ . Then for coupling (1) the inputs of systems (3) on  $\mathcal{P}(\Pi)$  satisfy

$$\begin{pmatrix} u_{C_1}(t) \\ \vdots \\ u_{C_K}(t) \end{pmatrix} = - \left( \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_K \end{pmatrix} \otimes I_m \right) \begin{pmatrix} y_{C_1}(t) \\ \vdots \\ y_{C_K}(t) \end{pmatrix} + \left( \begin{pmatrix} A_{11} & \cdots & A_{1K} \\ \vdots & \ddots & \vdots \\ A_{K1} & \cdots & A_{KK} \end{pmatrix} \otimes I_m \right) \begin{pmatrix} y_{C_1}(t - \tau) \\ \vdots \\ y_{C_K}(t - \tau) \end{pmatrix}$$

with each block  $D_i$  a diagonal matrix of dimension  $\dim(A_{ii})$ ,

$$\begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_K \end{pmatrix} = \text{diag}(A\mathbf{1}).$$

Here  $\otimes$  denotes the Kronecker product of two matrices, cf. [30], and  $\mathbf{1}$  is a vector of appropriate dimension with ones as entries. It is immediately clear that the existence of clusters implies, and is implied by, the following condition on the block-structured weighted adjacency matrix (4).

*Lemma 1:* Given a network with block-structured weighted adjacency matrix (4) and let  $\Pi \in \mathbb{R}^{k \times k}$  be a permutation matrix other than identity of the form

$$\Pi = \begin{pmatrix} \Pi_1 & & \\ & \ddots & \\ & & \Pi_K \end{pmatrix}$$

with each  $\Pi_{\ell}$  a permutation matrix of dimension  $\dim(A_{\ell\ell})$ . Then  $\mathcal{P}(\Pi)$  is a partial synchronization manifold for the coupled systems (3), (1) if and only if *each* block  $A_{p\ell}$ ,  $p, \ell = 1, \dots, K$ , has constant row sums.

Note that the condition that every block has constant row sums implies that each block  $D_{\ell}$  is of the form  $\bar{d}_{\ell}I$  with positive constant  $\bar{d}_{\ell}$  being the weighted degree of a node in cluster  $\ell$ .

For coupling (2) we can write

$$\begin{pmatrix} u_{C_1}(t) \\ \vdots \\ u_{C_K}(t) \end{pmatrix} = \left( \begin{pmatrix} D_1 - A_{11} & \cdots & -A_{1K} \\ \vdots & \ddots & \vdots \\ -A_{K1} & \cdots & D_K - A_{KK} \end{pmatrix} \otimes I_m \right) \begin{pmatrix} y_{C_1}(t - \tau) \\ \vdots \\ y_{C_K}(t - \tau) \end{pmatrix}.$$

Recall that  $D\mathbf{1} = A\mathbf{1}$ . Then it is easy to see that the necessary and sufficient condition for the existence of a partial synchronization manifold is that each off-diagonal block has constant row sums. Formally:

*Lemma 2:* Given a network with block-structured weighted adjacency matrix (4) and let  $\Pi \in \mathbb{R}^{k \times k}$  be a permutation matrix other than identity of the form

$$\Pi = \begin{pmatrix} \Pi_1 & & \\ & \ddots & \\ & & \Pi_K \end{pmatrix}$$

with each  $\Pi_{\ell}$  a permutation matrix of dimension  $\dim(A_{\ell\ell})$ . Then  $\mathcal{P}(\Pi)$  is a partial synchronization manifold for the coupled systems (3), (2) if and only if *each off-diagonal* block  $A_{p\ell}$ ,  $p, \ell = 1, \dots, K$ ,  $p \neq \ell$ , has constant row sums.

### IV. ALGEBRAIC CONDITIONS

As remarked before, a given weighted adjacency matrix  $A$  may not have the desired block-structure (4), hence additional conditions for the existence of partial synchronization manifolds are desired. In this section we present for both

types of coupling two algebraic conditions that are necessary and sufficient for the existence of a partial synchronization manifold. One condition (the second condition in Lemmas 3 and 4) allows one to find a permutation matrix that induces a partial synchronization manifold by identifying repeating patterns in the eigenvectors of  $A$  (or  $L$ ). The other condition (the third condition in Lemmas 3 and 4) is particularly useful for assessing the stability of the partial synchronization manifold [25].

The following has been proved in [31].

*Lemma 3:* Consider the coupled systems (3), (2) and let  $\Pi$  be a permutation matrix other than identity. The following statements are equivalent:

- 1)  $\mathcal{P}(\Pi)$  is a partial synchronization manifold for the coupled systems (3), (2);
- 2)  $\ker(I - \Pi)$  is an invariant subspace of  $L = D - A$ ;
- 3) there exists a matrix  $X \in \mathbb{R}^{k \times k}$  such that

$$(I - \Pi)L = X(I - \Pi).$$

For coupling (1) we derive a similar result, whose proof is more involving because of the invasive nature of this type of coupling.

*Lemma 4:* Consider the coupled systems (3), (1) and let  $\Pi$  be a permutation matrix other than identity. The following statements are equivalent:

- 1)  $\mathcal{P}(\Pi)$  is a partial synchronization manifold for the coupled systems (3), (1);
- 2)  $\Pi D = D\Pi$  and  $\ker(I - \Pi)$  is an invariant subspace of  $A$ ;
- 3)  $\Pi D = D\Pi$  and there exists a matrix  $X \in \mathbb{R}^{k \times k}$  such that

$$(I - \Pi)A = X(I - \Pi).$$

Note that in both lemmas it is *not* assumed that  $A$  is of the form (4). The condition  $\Pi D = D\Pi$  in Lemma 4 states that  $\mathcal{P}(\Pi)$  can only be a partial synchronization manifold for (3), (1) if the systems in a cluster have the same degree  $d_i$ . We only prove Lemma 4; The proof of Lemma 3 can be obtained with minor modifications for the following proof.

*Proof:* The scheme of the proof is  $3 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3$ .

( $3 \Rightarrow 1$ ). Let  $x(t) = \text{col}(x_1(t), \dots, x_k(t))$ ,  $F(x(t)) = \text{col}(f(x_1(t)), \dots, f(x_k(t)))$  and write dynamics of the coupled systems (3), (1) as

$$\dot{x}(t) = F(x(t)) - (D \otimes BC)x(t) + (A \otimes BC)x(t - \tau).$$

Premultiplication by  $(I_{kn} - \Pi \otimes I_n)$  gives

$$\begin{aligned} (I_{kn} - \Pi \otimes I_n)\dot{x}(t) &= F(x(t)) - F((\Pi \otimes I_n)x(t)) \\ &\quad - (I_{kn} - \Pi \otimes I_n)(D \otimes BC)x(t) \\ &\quad + (I_{kn} - \Pi \otimes I_n)(A \otimes BC)x(t - \tau). \end{aligned} \quad (5)$$

Here we used  $(I_{kn} - \Pi \otimes I_n)F(x(t)) = F(x(t)) - F((\Pi \otimes I_n)x(t))$ . Invoking condition 3 and using the properties of the Kronecker product we find

$$\begin{aligned} (I_{kn} - \Pi \otimes I_n)\dot{x}(t) &= F(x(t)) - F((\Pi \otimes I_n)x(t)) \\ &\quad - (D \otimes BC)(I_{kn} - \Pi \otimes I_n)x(t) \\ &\quad + (X \otimes BC)(I_{kn} - \Pi \otimes I_n)x(t - \tau). \end{aligned}$$

For  $x(t + \theta) = (\Pi \otimes I_n)x(t + \theta)$ ,  $\theta \in [-\tau, 0]$ , the right hand side of the latter equation vanishes, hence  $\dot{x}(t) = (\Pi \otimes I_n)\dot{x}(t)$ , which proves condition 1.

( $1 \Rightarrow 2$ ). Since  $\phi(\theta) = (\Pi \otimes I_n)\phi(\theta)$ ,  $\theta \in [-\tau, 0]$ , for any  $\phi \in \mathcal{P}(\Pi)$ , equation (5) reduces to

$$\begin{aligned} 0 &= -(I_{kn} - \Pi \otimes I_n)(D \otimes BC)\phi(0) \\ &\quad + (I_{kn} - \Pi \otimes I_n)(A \otimes BC)\phi(-\tau). \end{aligned} \quad (6)$$

For any  $\phi \in \mathcal{P}(\Pi)$  we can write  $\phi(\theta) = u(\theta) \otimes v(\theta)$  with  $u \in \mathcal{C}([-\tau, 0], \mathbb{R}^k)$  and  $v \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ . We may assume that  $\phi$  is non-zero, hence  $v(\theta) \neq 0$  on  $[-\tau, 0]$ . Note that  $0 = (I_{kn} - \Pi \otimes I_n)\phi(\theta) = (I_{kn} - \Pi \otimes I_n)(u(\theta) \otimes v(\theta)) = (I_k - \Pi)u(\theta) \otimes v(\theta)$ , hence  $u(\theta) \in \ker(I_k - \Pi)$ . Substitute  $\phi(\theta) = u(\theta) \otimes v(\theta)$  in equation (6) to obtain

$$\begin{aligned} 0 &= -(I_k - \Pi)Du(0) \otimes BCv(0) \\ &\quad + (I_k - \Pi)Au(-\tau) \otimes BCv(-\tau). \end{aligned}$$

This equation can only hold for any  $u(\theta)$  if  $Du(0) \in \ker(I_k - \Pi)$  and  $Au(-\tau) \in \ker(I_k - \Pi)$ . Note that  $Du(0) \in \ker(I_k - \Pi)$  for  $u(0) \in \ker(I - \Pi)$  if and only if  $\Pi$  and  $D$  commute.

( $2 \Rightarrow 3$ ). Let  $K_1 = k - K$  be the rank of  $(I_k - \Pi)$  and consider its *singular value decomposition*

$$(I_k - \Pi) = V\Sigma W^\top,$$

with orthogonal matrices  $V$  and  $W$ , and  $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_{K_1}, 0, \dots, 0)$ ,  $\sigma_i$  are the singular values. Then the matrix equation in condition 3 can be written as

$$(I_k - \Pi)AW = XW\Sigma. \quad (7)$$

Let  $v_i$  and  $w_i$  be the columns of the matrices  $V$  and  $W$ , respectively. Note that

$$\ker(I - \Pi) = \text{span}\{w_{K_1+1}, \dots, w_k\}.$$

Invoking condition 2, we find

$$Aw_i \in \ker(I_k - \Pi) \Rightarrow (I_k - \Pi)Aw_i = 0 \quad \forall i = K_1+1, \dots, k.$$

Let  $XV = Z = \text{col}(z_1, \dots, z_k)$ , then (7) takes the form

$$\begin{aligned} \text{col}((I_k - \Pi)Aw_1, \dots, (I_k - \Pi)Aw_{K_1}, 0, \dots, 0) \\ = \text{col}(\sigma_1 z_1, \dots, \sigma_{K_1} z_{K_1}, 0, \dots, 0), \end{aligned}$$

which can be solved. Indeed, letting  $z_i = \frac{1}{\sigma_i}(I_k - \Pi)Aw_i$ ,  $i = 1, \dots, K_1$ , and  $z_{K_1+1}, \dots, z_k$  arbitrary, we find  $X = ZV^\top$ . ■

As an example we consider a network with weighted adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix}.$$

Clearly this weighted adjacency matrix has a block-structure that reveals that  $\mathcal{P}(\Pi_1)$  with

$$\Pi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is a partial synchronization manifold. The matrix  $A$  has eigenvectors

$$v_1 = \begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix}, v_2 = \begin{pmatrix} a \\ -a \\ -a \\ a \end{pmatrix}, v_3 = \begin{pmatrix} a \\ a \\ -a \\ -a \end{pmatrix}, v_4 = \begin{pmatrix} a \\ -a \\ a \\ -a \end{pmatrix},$$

$a \in \mathbb{R}$ , corresponding to eigenvalues  $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = -1, \lambda_4 = -3$ . Then the repeating patterns in the eigenvectors reveal that

$$\text{span}\{v_1, v_2\} = \ker(I - \Pi_2), \quad \Pi_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{span}\{v_1, v_4\} = \ker(I - \Pi_3), \quad \Pi_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

are invariant subspaces of  $A$ . Note that any eigenvector of the form  $a\mathbf{1} \in \ker(I - \Pi)$  for any permutation matrix  $\Pi$ . Since  $D = 3I$  commutes with every permutation matrix we find that  $\mathcal{P}(\Pi_i), i = 1, 2, 3$ , are partial synchronization manifolds. In addition we observe that  $A$  and  $\Pi_i, i = 1, 2, 3$ , commute. Thus  $X = A$  solves the equations  $(I - \Pi_i)A = X(I - \Pi_i)$ . It follows from the third condition of the lemmas that if  $A$  is symmetric and both  $A$  and  $D$  commute with  $\Pi$ , then  $\mathcal{P}(\Pi)$  is a partial synchronization manifold. However, we remark that commuting of  $\Pi$  with  $D$  and  $A$  is sufficient but not necessary for  $\mathcal{P}(\Pi)$  to be a partial synchronization manifold.

## V. BALANCED EQUIVALENCE RELATIONS

In this section we present an additional condition for  $\mathcal{P}(\Pi)$  to be a partial synchronization manifold for the coupled systems (3), (1). In particular, we show that the theory of *balanced equivalence relations* for graphs with multiple arrows [26] defines partial synchronization manifolds in networks of systems (3) with linear time-delayed coupling (1). A nice aspect of this method is that it does not rely of the computation of eigenvalues and eigenvectors. Instead this method provides a graphical way to determine the permutation matrix  $\Pi$  that defines the partial synchronization manifold  $\mathcal{P}(\Pi)$ . In section VII we discuss the reason for not including such a condition for systems (3) with coupling (2).

To apply the theory of balanced equivalence relations to networks with non-uniform interaction weights we first derive a graph  $\tilde{\mathcal{G}}$ , more precisely, a multigraph  $\tilde{\mathcal{G}}$ , from the original graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ . We do this as follows:

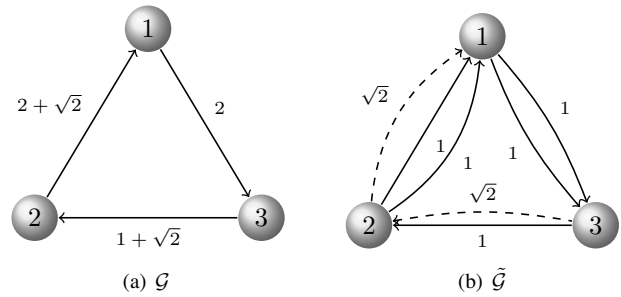


Fig. 2. Construction of  $\tilde{\mathcal{G}}$ .  $\bar{a}_1 = 1, \bar{a}_2 = \sqrt{2}$ .

- 1) determine  $r$  rationally independent base weights  $\bar{a}_\ell, \ell = 1, \dots, r$ , such that each  $a_{ij} = \sum_\ell \bar{a}_\ell k_{\ell ij}$  with non-negative integers  $k_{\ell ij}$ ;
- 2) construct a new set of edges  $\tilde{\mathcal{E}}$  by replacing each edge  $(i, j) \in \mathcal{E}$  with weight  $a_{ij} = \sum_\ell \bar{a}_\ell k_{\ell ij}$  by  $k_{\ell ij}$  edges  $(i, j)$  with weights  $\bar{a}_\ell, \ell = 1, \dots, r$ ;
- 3) define the multigraph  $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}})$ .

See Figure 2 for a graphical example. We note that in typical applications, where the interaction weights are estimated with finite precision, the weights are rational numbers. In that case  $r = 1$  and  $\bar{a}_1 = \bar{a}$  is the greatest common divisor of all weights. The next step is to define equivalence classes on the nodes and edges of  $\tilde{\mathcal{G}}$ .

*Definition 2:* Given the multigraph  $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}})$ , we let

- $\sim_E$  be an equivalence relation on the edges such any two edges  $e_i, e_j \in \tilde{\mathcal{E}}$  are equivalent,  $e_i \sim_E e_j$ , if and only if they have identical base-weight  $\bar{a}_\ell$ ;
- $\sim_V$  be an equivalence relation on the nodes such that any nodes  $i, j \in \mathcal{V}$  are equivalent,  $i \sim_V j$ , if and only if both nodes have the same number of equivalent edges pointing to them.  $\blacktriangle$

We remark that the equivalence relations on the nodes can also be formulated as follows: any two nodes  $i, j \in \mathcal{V}$  are equivalent,  $i \sim_V j$ , if and only if  $d_i = d_j$ .

We now define balanced equivalence relations using the notion of a *balanced coloring* [26] of the multigraph  $\tilde{\mathcal{G}}$ . (We prefer, for the sake of conciseness, to define balanced equivalence relations in terms of a balanced coloring rather than to use the concept of *input isomorphisms* as in the original definition of balanced equivalence presented in [26])

*Definition 3 (Balanced coloring and equivalence):* A coloring of the nodes of the multigraph  $\tilde{\mathcal{G}}$  with equivalence relations  $\sim_V$  and  $\sim_E$  with  $p$  colors is balanced if and only if, for all  $i, j = 1, \dots, p$ ,

- 1) all  $c_i$ -colored nodes belong to the same equivalence class on the nodes, and
- 2) every  $c_i$ -colored node receives edges of the same equivalence class from an equal number of nodes with color  $c_j$ .

An equivalence relation  $\bowtie$  on the nodes of  $\tilde{\mathcal{G}}$  is balanced if and only if the coloring, obtained by giving nodes of the same equivalence class defined by  $\bowtie$  the same color, is a balanced coloring.  $\blacktriangle$

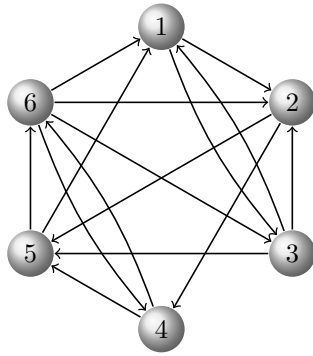


Fig. 3. The network of the example. Each edge has equal weight.

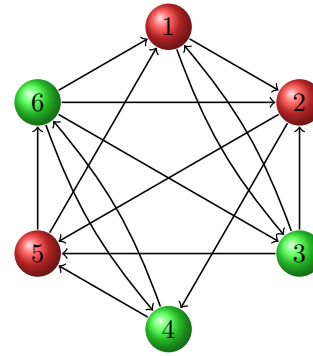


Fig. 4. Balanced coloring of the network

For the multigraph shown in Figure 2(b) the only balanced coloring is the one where each node is assigned a different color.

*Lemma 5:*  $\mathcal{P}(\Pi)$  is a partial synchronization manifold for the coupled system (3), (1) if and only if the equivalence relations  $\sim_{\Pi}$  induced by  $\Pi$  on  $(\tilde{\mathcal{G}}, \sim_V, \sim_E)$  are balanced. A balanced equivalence relation on  $(\tilde{\mathcal{G}}, \sim_V, \sim_E)$  induced by  $\Pi$  is denoted by  $\bowtie_{\Pi}$ .

*Proof:* The proof that the balanced equivalence relations imply that  $\mathcal{P}(\Pi)$  is a partial synchronization manifold for the coupled systems is straightforward; It follows from Definition 3 that balanced equivalence implies that the systems in the same equivalence class defined by  $\bowtie_{\Pi}$  receive identical inputs on  $\mathcal{P}(\Pi)$ . For proving that  $\mathcal{P}(\Pi)$  being a partial synchronization manifold implies that the network admits balanced equivalence relations we assume that the weighted adjacency matrix  $A$  is of the form (4). Assign nodes  $1, \dots, k_1$  color  $c_1$ , nodes  $k_1 + 1, \dots, k_2$  color  $c_2$ , and so on. Write each entry  $a_{ij}$  of the block-structured weighted adjacency matrix as a linear combination of the base weights  $\bar{a}_{\ell}$ ,  $\ell = 1, \dots, r$ . Lemma 1 states that  $\mathcal{P}(\Pi)$  being a partial synchronization manifold implies that each block  $A_{ij}$  has constant row sums. Then because all blocks have constant row sums the systems with the same color are equivalent (since they have the same degree  $d_i$ ). In addition it implies that each system with color  $c_i$  receives the same number of equivalent edges from systems with color  $c_j$ . By Definition 3 the equivalence relations are balanced. ■

## VI. EXAMPLE

We consider the coupled systems (3), (1) with the network structure shown in Figure 3. The systems in the network are equivalent if and only if they have the same degree. Thus  $1 \sim_V 2 \sim_V 5$  and  $3 \sim_V 4 \sim_V 6$ . Since all edges have the same weight all edges are equivalent. To determine whether the equivalence relations on the nodes are balanced we give systems 1, 2 and 5 color  $c_1$  (red), and systems 3, 4 and 6 are assigned color  $c_2$  (green). See Figure 4.

We see that each system with color  $c_1$  receives input from two systems with color  $c_2$  and one system with color  $c_1$ . Each system with color  $c_2$  receives input from one system with color  $c_1$  and one system with color  $c_2$ . Thus by Definition

3 we have the balanced equivalence relations  $1 \bowtie_{\Pi} 2 \bowtie_{\Pi} 5$ ,  $3 \bowtie_{\Pi} 4 \bowtie_{\Pi} 6$  with

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

By Lemma 5  $\mathcal{P}(\Pi)$  is a partial synchronization manifold.

The adjacency matrix and degree matrix of the network are

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 3 & \\ & & & & & 2 \end{pmatrix}.$$

It is easy to verify that  $D\Pi = \Pi D$ . Let  $v = (v_1 \ v_1 \ v_2 \ v_2 \ v_1 \ v_2)^{\top}$ ,  $v_1, v_2 \in \mathbb{R}$ , and note that  $v \in \ker(I - \Pi)$ . Since

$$A \begin{pmatrix} v_1 \\ v_1 \\ v_2 \\ v_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + 2v_2 \\ v_1 + 2v_2 \\ v_1 + v_2 \\ v_1 + v_2 \\ v_1 + 2v_2 \\ v_1 + v_2 \end{pmatrix} \in \ker(I - \Pi),$$

we have  $\ker(I - \Pi)$  an invariant subspace of  $A$ , which implies, now by Lemma 4, that  $\mathcal{P}(\Pi)$  is a partial synchronization manifold. Moreover, it is straightforward to verify that

$$X = \frac{1}{3} \begin{pmatrix} -1 & -1 & -1 & -1 & 2 & 2 \\ 2 & -1 & 0 & 0 & -1 & 0 \\ 2 & -1 & -1 & -1 & -1 & 2 \\ -1 & 2 & 0 & 0 & -1 & 0 \\ -1 & 2 & 1 & 1 & -1 & -2 \\ -1 & -1 & 1 & 1 & 2 & -2 \end{pmatrix}$$

solves the equation  $(I - \Pi)A = X(I - \Pi)$ . Finally, relabeling the nodes,  $\{1, 2, 3, 4, 5, 6\} \mapsto \{1, 2, 5, 3, 4, 6\}$  gives the

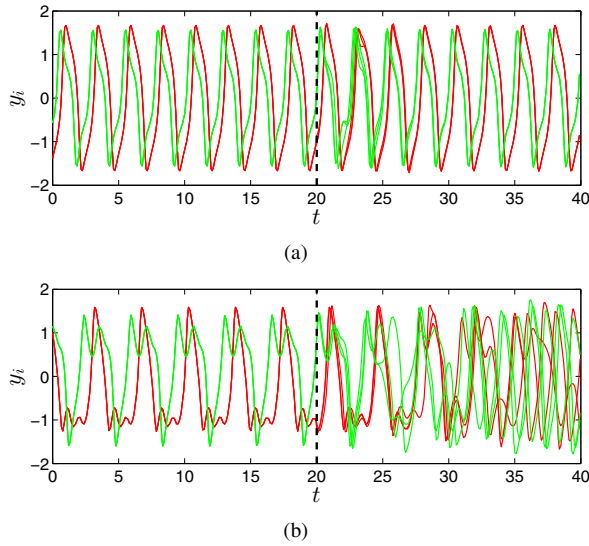


Fig. 5. Results of numerical simulations for (a)  $\tau = 0.5$  and (b)  $\tau = 1$ .

structured adjacency matrix and structured degree matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix}.$$

Note that each block in  $A$  has equal row sums, which implies by Lemma 1 the existence of a partial synchronization manifold corresponding to the partitioning  $\{\{1, 2, 5\}, \{3, 4, 6\}\}$  (using the original labeling) of  $\mathcal{V} = \{1, 2, 3, 4, 5, 6\}$ .

We simulated the network with systems

$$\begin{cases} \dot{x}_{i,1}(t) = 5x_{i,2}(t) + u_i(t) \\ \dot{x}_{i,2}(t) = -x_{i,1}(t) + 5(1 - x_{i,1}^2(t))x_{i,2}(t) \\ y_i(t) = x_{i,1}(t) \end{cases}$$

with initial data on  $\mathcal{P}(\Pi)$ . Results are presented in Figure 5, where the red and green trajectories are of systems assigned color red and green, respectively. Each system receives after 20 time-units a small random perturbation that lasts 0.1 time-units. It can be seen in Figure 5(a) that for  $\tau = 0.5$  the partial synchronization of the coupled systems persists while, as shown in Figure 5(b), in case of  $\tau = 1$  the perturbation destroys this mode of partial synchronization. Thus although  $\mathcal{P}(\Pi)$  is a partial synchronization manifold for any value of the time-delay, the stability properties of  $\mathcal{P}(\Pi)$  may change while varying  $\tau$ .

## VII. CONCLUDING REMARKS

We have presented a number of necessary and sufficient conditions for the existence of partial synchronization manifolds in networks of time-delay coupled systems. Our results are summarized in the next two theorems:

**Theorem 1:** Consider the coupled system (3), (1) and let  $\Pi \in \mathbb{R}^{k \times k}$  be a permutation matrix other than identity. The following statements are equivalent:

- 1)  $\mathcal{P}(\Pi)$  is a partial synchronization manifold for the coupled systems (3), (1);

- 2) All blocks of the structured weighted adjacency matrix (4) have constant row sums;
- 3)  $\Pi D = D\Pi$  and  $\ker(I - \Pi)$  is an invariant subspace of  $A$ ;
- 4)  $\Pi D = D\Pi$  and there exists a matrix  $X \in \mathbb{R}^{k \times k}$  such that

$$(I - \Pi)A = X(I - \Pi);$$

- 5)  $\Pi$  defines balanced equivalence relations  $\bowtie_{\Pi}$  on  $(\tilde{\mathcal{G}}, \sim_V, \sim_E)$ .

**Theorem 2:** Consider the coupled cell system (3), (2) and let  $\Pi \in \mathbb{R}^{k \times k}$  be a permutation matrix other than identity. The following statements are equivalent:

- 1)  $\mathcal{P}(\Pi)$  is a partial synchronization manifold for the coupled systems (3), (2);
- 2) All off-diagonal blocks of the structured weighted adjacency matrix (4) have constant row sums;
- 3)  $\ker(I - \Pi)$  is an invariant subspace of  $L$ ;
- 4) There exists a matrix  $X \in \mathbb{R}^{k \times k}$  such that

$$(I - \Pi)L = X(I - \Pi).$$

For the coupled systems (3), (2) a condition like the last one in Theorem 1 is missing. The reason for this is that balanced equivalence relations induced by a permutation matrix  $\Pi$  on  $(\tilde{\mathcal{G}}, \sim_V, \sim_E)$  are sufficient but not necessary for  $\mathcal{P}(\Pi)$  to be a partial synchronization manifold for (3), (2). Indeed, consider the network shown in Figure 3 and let

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_1 \\ v_1 \\ v_2 \\ v_1 \\ v_2 \end{pmatrix}, v_1, v_2 \in \mathbb{R},$$

then

$$v, Lv = \begin{pmatrix} v_1 - v_2 \\ v_1 - v_2 \\ v_1 - v_2 \\ v_2 - v_1 \\ v_1 - v_2 \\ v_2 - v_1 \end{pmatrix} \in \ker(I - \Pi).$$

Thus according to Theorem 2 (or Lemma 3)  $\mathcal{P}(\Pi)$  is a partial synchronization manifold for the coupled system (3), (2). But if we assign systems 1, 2, 3 and 5 color  $c_1$  (red) and systems 4 and 6 color  $c_2$  (green), we see that system 3 receives input from one system with color  $c_1$  and one system with color  $c_2$ , while the other systems with color  $c_1$  receive inputs from two system with color  $c_1$  and one system with color  $c_2$ . Thus according to Definition 3 the equivalence relations induced by this matrix  $\Pi$  are not balanced. In fact, we were even not allowed to assign color  $c_1$  to system 3 as this system belongs to a different equivalence class. However, if we add the edge (3,3) to the graph the equivalence relations induced by  $\Pi$ , i.e.  $1 \bowtie_{\Pi} 2 \bowtie_{\Pi} 3 \bowtie_{\Pi} 5$  and  $4 \bowtie_{\Pi} 6$ , are balanced. See Figure 6.

Adding edges of the form  $(i, i)$  yields additional coupling terms of the form  $y_i(t-\tau) - y_i(t-\tau)$  which, obviously, do not

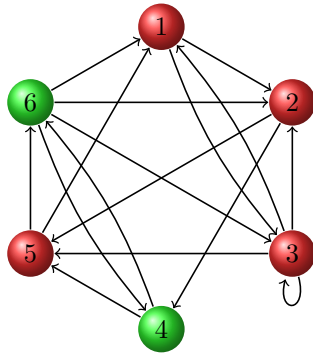


Fig. 6. Augmented network with a balanced coloring

contribute to the dynamics of system  $i$ . This implies that we may define a new multigraph  $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}})$ , where  $\tilde{\mathcal{E}}$  is the set of edges  $\tilde{\mathcal{E}}$  augmented with  $k_{\ell ii}$  edges of the form  $(i, i)$ , such that all sums  $\sum_{j \in \mathcal{N}_i \cup \{i\}} \sum_{\ell=1}^r k_{\ell ij} \bar{a}_\ell$  are constant. It follows readily that  $\mathcal{P}(\Pi)$  is a partial synchronization manifold for the time-delay coupled systems (3), (2) if and only if the equivalence relations  $\sim_\Pi$  on  $(\tilde{\mathcal{G}}, \sim_V, \sim_E)$  are balanced. Note that in this case all systems belong to the same equivalence class.

An important open question is how to find all possible partial synchronization manifold for a given network. As mentioned before, for a given weighted adjacency matrix  $A$  one can in principle determine all possible rearrangements (permutations) of its elements and check if the second condition of Theorem 1 or 2 holds. However, such approach is only feasible for small networks since the number of all possible permutations grows large rapidly for increasing  $k$ . (The number of partitions of a set with  $k$  members is given by Bell's number  $B_k = \sum_{p=0}^{k-1} \binom{k-1}{p} B_p$  with  $B_0 = B_1 = 1$ .) Finding partial synchronization manifolds by computing the eigendecomposition of  $A$  or  $L$  (the third condition of Theorem 1 or 2) is also prohibited for large networks since the computational cost of an eigenvalue decomposition is  $\mathcal{O}(k^3)$ . We hope to provide an efficient method to determine partial synchronization manifolds for a given network in the near future.

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#### REFERENCES

[1] J. Buck and E. Buck, "Synchronous fireflies," *Sci. Am.*, vol. 234, pp. 74–85, 1976.  
 [2] C. S. Peskin, "Mathematical aspects of heart physiology," *Courant Institute of Mathematical Sciences, New York University*, pp. 268 – 278, 1975.  
 [3] C. A. Czeisler, E. D. Weitzman, M. C. Moore-Ede, J. C. Zimmerman, and R. S. Knauer, "Human sleep: Its duration and organization depend on its circadian phase," *Science*, vol. 210, pp. 1264–1267, 1980.  
 [4] G. J. L. Naus, R. P. A. Vugts, J. Ploeg, M. J. G. van de Molengraft, and M. Steinbuch, "String-stable cacc design and experimental validation, a frequency-domain approach," *IEEE trans. Veh. Techn.*, vol. 59, no. 9, pp. 4268–4279, 2010.

[5] K. Y. Pettersen, J. T. Gravdahl, and H. Nijmeijer, *Group Coordination and Cooperative Control*. Springer-Verlag, Berlin, 2006.  
 [6] H. Nijmeijer and A. Rodriguez-Angeles, *Synchronization of mechanical systems*. World Scientific, Singapore, 2003.  
 [7] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.*, vol. 64, no. 8, pp. 821–824, 1990.  
 [8] K. M. Cuomo, A. V. Oppenheim, and S. H. Strogatz, "Synchronization of Lorenz based chaotic circuits with applications to communications," *IEEE trans. Circ. Syst. II*, vol. 40, no. 10, pp. 626–633, 1993.  
 [9] H. J. C. Huijberts, H. Nijmeijer, and R. M. A. Willems, "A control perspective on communication using chaotic systems," *Proc. 37th IEEE Conf. Decision and Control*, vol. 2, pp. 1957–1962, 1998.  
 [10] S. H. Strogatz, *Sync: the emerging science of spontaneous order*, 1st ed. Hyperion, 2003.  
 [11] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization*, 2nd ed. Cambridge University Press, 2003.  
 [12] G. V. Osipov, J. Kurths, and C. S. Zhou, *Synchronization in Oscillatory Networks*. Springer, 2007.  
 [13] C. Koch, *Biophysics of computation*. Oxford University Press, 1999.  
 [14] R. Sipahi, S. Niculescu, C. T. Abdallah, W. Michiels, and K. Gu, "Stability and stabilization of systems with time delay," *IEEE Contr. Syst.*, vol. 31, no. 1, pp. 38–65, 2011.  
 [15] P. J. Uhlhaas and W. Singer, "Neural synchrony in brain disorders: Relevance for cognitive dysfunctions and pathophysiology," *Neuron*, vol. 52, no. 1, pp. 155–168, 2006.  
 [16] M. Bennet and R. Zuckin, "Electrical coupling and neuronal synchronization in the mammalian brain," *Neuron*, vol. 41, pp. 495–511, 2004.  
 [17] S. Coombes, "Neuronal networks with gap junctions: A study of piecewise linear planar neuron models," *SIAM J. Appl. Dyn. Syst.*, vol. 7, no. 3, pp. 1101–1129, 2008.  
 [18] J. G. Mancilla, T. J. Lewis, D. J. Pinto, J. Rinzel, and B. W. Connors, "Synchronization of electrically coupled pairs of inhibitory interneurons in neocortex," *J. Neurosci.*, vol. 27, pp. 2058–2073, 2007.  
 [19] M. Perez-Armendariz, C. Roy, D. C. Spray, and M. V. L. Bennett, "Biophysical properties of gap junctions between freshly dispersed pairs of mouse pancreatic beta cells," *Biophys. J.*, vol. 59, pp. 76–92, 1991.  
 [20] A. Sherman, J. Rinzel, and J. Keizer, "Emergence of organized bursting in clusters of pancreatic beta-cells by channel sharing," *Biophys. J.*, vol. 54, no. 3, pp. 411–425, 1998.  
 [21] A. Rodriguez-Angeles and H. Nijmeijer, "Coordination of two robot manipulators based on position measurements only," *Int. J. Control*, vol. 74, no. 13, 2001.  
 [22] S.-J. Chung and J.-J. E. Slotine, "Cooperative robot control and concurrent synchronization of Lagrangian systems," *IEEE trans. Robotics*, vol. 25, no. 3, pp. 686–700, 2009.  
 [23] M. Zanin, J. M. Buldú, and S. Boccaletti, "Networks of springs: A practical approach," *Int. J. Bif. Chaos*, vol. 20, no. 3, pp. 937–942, 2010.  
 [24] C. W. Wu and L. O. Chua, "Synchronization in an array of linearly coupled dynamical systems," *IEEE trans. Circ. Syst. I*, vol. 42, no. 8, pp. 430–447, 1995.  
 [25] E. Steur, T. Oguchi, C. v. Leeuwen, and H. Nijmeijer, "Partial synchronization in diffusively time-delay coupled oscillator networks," *Chaos*, vol. 22, no. 043144, 2012.  
 [26] M. Golubitsky, I. Stewart, and A. Török, "Patterns of synchrony in coupled cell networks with multiple arrows," *SIAM J. Appl. Dynam. Sys.*, vol. 4, no. 1, pp. 78–100, 2005.  
 [27] I. Belykh and M. Hasler, "Mesoscale and clusters of synchrony in networks of bursting neurons," *Chaos*, vol. 21, no. 016106, 2011.  
 [28] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*. Springer-Verlag, 1993.  
 [29] D. J. Watts and S. H. Strogatz, "Collective dynamics of 'small-world' networks," *Nature*, vol. 393, pp. 440–442, 1998.  
 [30] J. W. Brewer, "Kronecker products and matrix calculus in system theory," *IEEE trans. Circ. Syst.*, vol. 25, no. 9, pp. 772–781, 1978.  
 [31] H. U. Ünal and W. Michiels, "Prediction of partial synchronization in delay-coupled nonlinear oscillators, with application to Hindmarsh-Rose neurons," *Nonlinearity*, vol. 26, no. 12, pp. 3101–3126, 2013.