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Existence for the Neumann stochastic semilinear equations via an optimal control approach

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This work is concerned with the semilinear stochastic equation with Neumann boundary condition

$$\begin{cases} dX(t) - \Delta X(t)\,dt = \sqrt{Q}dW(t), & (0,T)\times\mathcal{O}, \\ \dfrac{\partial X}{\partial n} + \Phi(X) \ni 0, & (0,T)\times\partial\mathcal{O}, \\ X(0) = x, & \mathcal{O}, \end{cases} \tag{1}$$

where $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$, $\dfrac{\partial}{\partial n}$ is the outward normal derivative on the boundary of $\mathcal{O}$, Φ is a maximal monotone graph (possibly multivalued). We assume that $W(t)$ is a cylindrical Wiener process on a stochastic basis $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\right)$ taking values in the Hilbert space $L^2(\mathcal{O})$, defined by $W(t) = \sum\limits_{j=1}^{\infty}\beta_j(t)\,e_j$, for all $t \geq 0$, where $\{e_j\}$ is an orthonormal basis in $L^2(\mathcal{O})$ and $\{\beta_j\}_{j=1}^{\infty}$ is a sequence of mutually independent Brownian motions on the probability space. We also assume that $x \in L^2(\mathcal{O})$.

The operator Q is considered to be linear, continuous, self-adjoint and positive on $L^2(\mathcal{O})$, with finite trace such that $\mathrm{Ker}Q = \{0\}$.

Existence and uniqueness of the solution for this equation was studied so far by standard existence theory only under very restrictive hypotheses, i.e. Φ is assumed to be a maximal monotone operator defined everywhere on $\mathbb{R}$, monotonically increasing and satisfying

$$\begin{aligned} &\Phi(0) = 0, \\ &|\Phi(r)| \leq C_1|r| + C_2, \text{ for all } r\in\mathbb{R}, \\ &(\Phi(r)-\Phi(s))(r-s) \geq C_3(r-s)^2, \text{ for all } r,s\in\mathbb{R}, \end{aligned} \tag{2}$$

for some constants $C_i > 0$, $i = \overline{1,3}$.

Indeed, we can consider the operator

$$A : H^1(\mathcal{O}) \to \left(H^1(\mathcal{O})\right)^*$$

defined by

$$(A(y),\psi) = \int_{\mathcal{O}} \langle\nabla y,\nabla\psi\rangle\,dx + \int_{\partial\mathcal{O}} \Phi(\gamma_0(y))\,\gamma_0(\psi)\,d\sigma,$$

for all $\psi \in H^1(\mathcal{O})$, where

$$\gamma_0 : H^1(\mathcal{O}) \to H^{1/2}(\partial\mathcal{O}) \tag{3}$$

is the trace function.

Consequently, we can rewrite equation (1) as

$$\begin{cases} dX(t) + AX(t)\,dt = \sqrt{Q}dW(t), \\ X(0) = x \end{cases}$$

and check that, under assumptions (2), the operator A satisfies hypotheses from [14]. See also [12] and [13].

In the present work we prove existence and uniqueness of the solution for equation (1) under more general assumptions, motivated by physical applications. To this purpose we shall use an optimal control approach based on the method formulated by Brezis and Ekeland in [6].

An approach based on a similar idea was used in [11] for deterministic equations.

Concerning stochastic differential equations, similar results were already proved for the semilinear equation with Dirichlet boundary conditions and for the porous media equation in [1] and [2], but the case with Neumann boundary conditions is still an open problem.

Physical motivation

An important model which is not covered by assumptions (2) is the temperature control regulated by the temperature flux at the boundary. In this case the operator Φ is multivalued and the standard form is

$$\Phi(x) = \begin{cases} g_1, & \text{if } x < h_1, \\ [g_1,0], & \text{if } x = h_1, \\ 0, & \text{if } h_1 < x < h_2, \\ [0,g_2], & \text{if } x = h_2, \\ g_2 & \text{if } h_2 < x. \end{cases}$$

Here $[g_1, g_2]$, with $0 \in [g_1, g_2]$, is the closed interval confining the flux of injected heat which can be measured by $\frac{\partial X}{\partial n}$ and h_1 and h_2 are the reference temperature. For details see Example 3.6 from page 31 of [9]. A similar model describes a diffusion process through semi-permeable walls.

Hypotheses and definition of the solution

We assume that

H$_1$ The operator $\Phi : \mathbb{R} \to 2^{\mathbb{R}}$, is a maximal monotone operator such that $D(\Phi) = \mathbb{R}$.

H$_2$ The potential g of the operator Φ (i.e. $\partial g = \Phi$) verifies

$$g(-r) \leq C_1 g(r) + C_2, \quad \forall r \in \mathbb{R},$$

where $C_1 > 0$. Without loss of generality we may also consider that $g \geq 0$ and therefore we have also that the conjugate $g^* \geq 0$. Recall that

$$g^*(z) = \sup\{zx - g(x);\quad x \in \mathbb{R}\}, \quad z \in \mathbb{R}.$$

H$_3$ The operator Q from the noise is such that the stochastic convolution $W_Q \in C\left([0,T] \times \overline{\mathcal{O}}\right)$.

Here we considered

$$W_Q(t) = \int_0^t S(t-s)\sqrt{Q}dW(s), \quad t \geq 0,$$

and $S(t-s)$ is the C_0-semigroup generated on $L^2(\mathcal{O})$ by the Laplace operator with Neumann boundary condition.

For sufficient assumptions on Q under which the condition above holds, see Theorem 2.13, page 29 from [8].

Note that the assumptions above hold for the operators presented in the physical motivation.

Definition 1 *A mild solution to equation (1) is a stochastic adapted process* $X \in C_W\left([0,T];\ \left(H^1(\mathcal{O})\right)^*\right)$ *which satisfies*

$$X(t) = e^{-A_0 t}x - \int_0^t e^{-A_0(t-s)}B(Z)\,ds + W_Q(t),$$

$\mathbb{P}$-*a.s.*, $t \in [0,T]$, *where*

$$Z \in L^1([0,T] \times \partial\mathcal{O} \times \Omega) \cap L^2_W\left([0,T];\ H^{-1/2}(\partial\mathcal{O})\right)$$

is such that $Z \in \Phi(X)$ *a.e. on* $[0,T] \times \partial\mathcal{O} \times \Omega$,

$$A_0 : H^1(\mathcal{O}) \to \left(H^1(\mathcal{O})\right)^*$$

is the Laplace operator with Neumann boundary condition, $e^{-A_0 t}$ *is the* C_0 *semigroup generated by* A_0 *on* $\left(H^1(\mathcal{O})\right)^*$ *and*

$$B : H^{-1/2}(\partial\mathcal{O}) \to \left(H^1(\mathcal{O})\right)^*$$

is the adjoint of the trace operator γ_0.

The optimal control formulation

Equation (1) is equivalent for $Y = X - W_Q$ to the following optimal control problem

Minimize

$$\int_0^T \int_{\partial\mathcal{O}} \{g(\gamma_0(Y + W_Q)) + g^*(Z) - \gamma_0(Y + W_Q)Z\}\,d\sigma dt$$

where

$$\gamma_0(Y) \in L^1(\Sigma_T), \quad Y \in L^2\left((0,T);\ H^1(\mathcal{O})\right)$$

and

$$Z \in L^1([0,T] \times \partial\mathcal{O}) \cap L^2\left([0,T];\ H^{-1/2}(\partial\mathcal{O})\right),$$

are subject to

$$\begin{cases} \dfrac{\partial Y}{\partial t} - \Delta Y dt = 0, & (0,T) \times \mathcal{O}, \\ \dfrac{\partial Y}{\partial n} + Z = 0, & \Sigma_T, \\ Y(0) = x, & \mathcal{O}. \end{cases}$$

Theorem 2 *For each* $x \in L^2(\mathcal{O})$, *there is a unique mild solution to equation (1) in the sense of Definition 1, such that* $X \in L^2_W\left([0,T];\ H^1(\mathcal{O})\right)$ *and*

$$g(\gamma_0(X)),\ g^*(Z) \in L^1(\Omega \times [0,T] \times \partial\mathcal{O}),$$

where $Z \in \Phi(\gamma_0(X))$ *a.e. on* $[0,T] \times \partial\mathcal{O} \times \Omega$.

Proof. The proof is available in the preprint [7] ■

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