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Controlled and conditioned invariant varieties for polynomial control systems*

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***Abstract*— We consider polynomially nonlinear state-space systems and given algebraic varieties. A variety V is said to be controlled invariant w.r.t. a given system if we can find a polynomial state feedback law that causes the closed loop system to have V as an invariant set. If this task can be achieved by a polynomial output feedback law, V is called controlled and conditioned invariant. This concept leads to the problem of determining the intersection of a certain (affine) submodule of a free module over a polynomial ring with a free module over a subalgebra of this ring. We suggest various approaches to do this and to decide whether a variety is controlled and conditioned invariant, and if so, to compute all output feedback laws achieving the task.**

I. INTRODUCTION

Let us first define some objects which are frequently used throughout this paper. Let $K \in \{\mathbb{R}, \mathbb{C}\}$ be a field and $n \in \mathbb{N}$. Then we call $R = K[x_1, \ldots, x_n] = K[\underline{x}]$ the polynomial ring in n variables. Further let $I \subseteq R$ be an ideal of R. We intend to study the variety $V = \mathscr{V}(I) \subseteq K^n$, the common zero set of all polyomials in I, and control systems of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t), \quad && f \in R^n,\ g \in R^{n\times m} \\ y(t) &= h(x(t)), && h \in R^p \end{aligned} \tag{1}$$

with $m, p \in \mathbb{N}$. Here $x(t)$ is called the state of the control system at time t, whereas $y(t)$ is the output and $u(t)$ the input at time t. We wish to determine an input function $u(\cdot)$ such that V becomes invariant for (1). Let $S = K[h_1, \ldots, h_p] \subseteq R$ be the subalgebra of R generated by the components h_i of h, $T = K[y_1, \ldots, y_p] = K[\underline{y}]$ be another polynomial ring in p variables and $R_1 = K[x_1, \ldots, x_n, y_1, \ldots, y_p] = K[\underline{x}, \underline{y}]$. If k, l are natural numbers, $m_1, \ldots, m_k \in R_1^l$ and $R_2 \subseteq R_1$ is a subring we write

$$\langle m_1, \ldots, m_k \rangle_{R_2} := \{\sum_{i=1}^{k} a_i m_i \mid a_i \in R_2\}$$

for the R_2-module generated by $m_1, \ldots, m_k$.

This paper is organized as follows: In Section II, we will recall some definitions and known results concerning *controlled invariant varieties*, which have been investigated in [8] and [9]. Section III deals with *controlled and conditioned varieties*, based on a definition given in [4]. We will see that this notion leads to the problem of intersecting *affine submodules* of R^m with S^m. Some solutions and algorithmic approaches will be presented as well as some examples to illustrate these procedures.

II. CONTROLLED INVARIANT VARIETIES

A. Invariant varieties of autonomous systems

For the moment, we will consider autonomous systems of the form

$$\dot{x}(t) = F(x(t)), \qquad x(0) = x_0, \tag{2}$$

where $F \in R^n$ and $x_0 \in K^n$.

Definition 1. Let $\varphi(t, x_0)$ be the solution of (2) at time t, where $t \in J(x_0)$, the maximal existence interval of $\varphi(\cdot, x_0)$. We say that $V \subseteq K^n$ is an *invariant set with respect to F* if $x_0 \in V$ implies $\varphi(t, x_0) \in V$ for all $t \in J(x_0)$.

If V is any variety in K^n, then we define the *vanishing ideal of V*

$$\mathscr{J}(V) = \{p \in R \mid p(v) = 0 \text{ for all } v \in V\}.$$

We recall the following result from [8], [9]:

Theorem 2. *Let a variety $V = \mathscr{V}(I)$ be given, where the ideal I is generated by $p_1, \ldots, p_k \in R$.*

1) *If we have*
$$(\partial_1 p_i)F_1 + \ldots + (\partial_n p_i)F_n \in I \tag{3}$$
for all $i \in \{1, \ldots, k\}$, then V is invariant w.r.t. F.

2) *If V is invariant w.r.t. F, then*
$$(\partial_1 p_i)F_1 + \ldots + (\partial_n p_i)F_n \in \mathscr{J}(V)$$
for all $i \in \{1, \ldots, k\}$.

Thus, if $\mathscr{J}(V) = I$ holds, then condition (3) for all $1 \leq i \leq k$ is necessary and sufficient for V being invariant w.r.t. F.

Now let $p_1, \ldots, p_k \in R$ and I, V be as in the theorem above. For all $i \in \{1, \ldots, k\}$ we define

$$\mathscr{N}_i = \ker(\partial_1 p_i, \ldots, \partial_n p_i, p_1, \ldots, p_k) \subseteq R^{n+k}$$

and set $\mathscr{M}_i := \pi(\mathscr{N}_i)$, where π denotes the projection on the first n components. Finally, let $\mathscr{M} := \bigcap_{i=1}^k \mathscr{M}_i \subseteq R^n$. Again, the next result can be found in [8], [9].

Theorem 3. *We have*

$$\mathscr{M} = \{F \in R^n \mid F \text{ satisfies (3) for all } 1 \leq i \leq k\}.$$

Now assume that $\mathscr{J}(V) = \mathscr{J}(\mathscr{V}(I)) = I$. Then Theorem 3 says that V is invariant w.r.t. $F \in R^n$ if and only if $F \in \mathscr{M}$. For this reason, we call $\mathscr{M}$ the *module of all admissible vector fields of V*.

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B. Controlled invariant varieties

Let us now look at the initial system (1). For the input $u(\cdot)$, we use a state feedback law, that is, $u(t) = \alpha(x(t))$ for an $\alpha \in R^m$. Plugging this into (1) yields the closed loop system

$$\dot{x}(t) = f(x(t)) + g(x(t))\alpha(x(t)) = (f + g\alpha)(x(t)).$$

Definition 4. If $V \subseteq K^n$ is a variety, we call it *controlled invariant w.r.t. system (1)* if there is an $\alpha \in R^m$ such that V is invariant w.r.t. $F := f + g\alpha$.

This definition as well as Theorems 2 and 3 immediately yield the following:

Corollary 5. *Provided that $\mathscr{J}(V) = I$ holds, V is controlled invariant w.r.t. system (1) if and only if there is an $\alpha \in R^m$ with $f + g\alpha \in \mathscr{M}$. This is equivalent to $f \in \mathscr{M} + \mathrm{im}(g)$.*

If we assume that the given ideal I fulfills the assumption of Corollary 5, a state feedback law making V invariant can be obtained as follows: Because R is a noetherian ring, there is a finite system of generators for the R-module $\mathscr{M}$. Collecting these generators in a matrix $\mathbf{m}$, we can test whether V is controlled invariant by answering the following question: Can we write f as an R-linear combination of the columns of $\mathbf{m}$ and g? If the answer is yes, we can find $\alpha \in R^m$ and β with entries in R and of appropriate size such that

$$f = \mathbf{m}\beta - g\alpha,$$

and thus $f + g\alpha \in \mathscr{M}$, that is, α is a feedback law that makes V invariant w.r.t. (1).
Let us determine the non-uniqueness of such an α. For this, let $\alpha_1, \alpha_2 \in R^m$ fulfill $f + g\alpha_1 \in \mathscr{M}$ and $f + g\alpha_2 \in \mathscr{M}$. Then there are β_1, β_2 with entries in R having appropriate sizes such that $f + g\alpha_1 = \mathbf{m}\beta_1$ and $f + g\alpha_2 = \mathbf{m}\beta_2$. Subtraction yields

$$0 = \mathbf{m}(\beta_1 - \beta_2) - g(\alpha_1 - \alpha_2) = (\mathbf{m}, -g)\begin{pmatrix}\beta_1 - \beta_2 \\ \alpha_1 - \alpha_2\end{pmatrix}.$$

We conclude that the set of all state feedback laws making V an invariant variety is given by

$$\alpha + \pi(\ker(\mathbf{m}, -g)), \tag{4}$$

where $\alpha \in R^m$ is a particular solution, i.e. $f + g\alpha \in \mathscr{M}$, and π denotes the projection on the last m components.

Example 6. Let the ring $R = K[x_1, x_2]$ be given. We consider the system

$$\begin{pmatrix}\dot{x_1} \\ \dot{x_2}\end{pmatrix} = \begin{pmatrix} x_1x_2^4 + x_1^2x_2^3 \\ x_1x_2^5 + x_2^2\end{pmatrix} + \begin{pmatrix} x_2^2 & x_1 + x_2 \\ -x_1 & 0\end{pmatrix}\begin{pmatrix}u_1 \\ u_2\end{pmatrix}$$

on the variety $V = \mathscr{V}(p)$, where $p = x_1^2x_2^2 - 1$. The module of all admissible vector fields is given by

$$\begin{aligned}\mathscr{M} &= \pi(\ker(\partial_1 p, \partial_2 p, p)) = \pi(\ker(2x_1x_2^2, 2x_1^2x_2, x_1^2x_2^2 - 1)) \\ &= \langle \begin{pmatrix} x_1 \\ -x_2\end{pmatrix}, \begin{pmatrix} -1 \\ x_1x_2^3\end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2x_2^2 - 1\end{pmatrix}\rangle_R.\end{aligned}$$

Let $\mathbf{m}$ be the matrix which has these three generators of $\mathscr{M}$ as columns. Next, we check that $f \in \mathscr{M} + \mathrm{im}(g)$. Indeed, choosing $\alpha = \begin{pmatrix} -x_1^2x_2 - x_1x_2^2 \\ -x_1^3 - x_2\end{pmatrix}$ yields $f + g\alpha =$

$$\begin{pmatrix} -x_1^4 - x_1^3x_2 - x_1x_2 - x_2^2 \\ x_1x_2^5 + x_1^3x_2 + x_1^2x_2^2 + x_2^2\end{pmatrix} = \mathbf{m}\begin{pmatrix} -x_1^3 - x_1^2x_2 - x_2 \\ x_2^2 \\ 0\end{pmatrix} \in \mathscr{M}.$$

This shows that V is controlled invariant for the given system. Moreover, we compute

$$\begin{aligned}\pi(\ker(\mathbf{m}, -g)) = \langle &\begin{pmatrix} -x_1x_2 - x_2^2 \\ x_2^3 - x_1^2\end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2x_2^2 - 1\end{pmatrix}, \begin{pmatrix} x_1^3x_2 + 1 \\ x_1^4 - x_2\end{pmatrix}, \\ &\begin{pmatrix} x_1^2x_2 - x_1x_2 + x_2 + 1 \\ x_1^3x_2 + x_1^3 - x_1^2x_2 + x_1x_2^2 - x_1^2 + x_1x_2 - x_2^2 + x_1 - x_2 - 1\end{pmatrix}\rangle_R\end{aligned}$$

to get the set of all possible state feedback laws making V an invariant variety according to (4).

III. CONTROLLED AND CONDITIONED INVARIANT VARIETIES

So far, we have seen how to find a state feedback function α such that V is invariant for (1) with $u(t) = \alpha(x(t))$, if this is possible at all. However, in general it is restrictive to assume that the full state is available for the feedback. Instead, we would like to use only the output $y(t) = h(x(t))$ of (1) for the feedback. In view of this, we ask the following question: Is it possible to choose an output feedback law $u(t) = \beta(y(t)) = \beta(h(x(t)))$ making V invariant?

Definition 7. Given a variety V, we call it *conditioned and controlled invariant w.r.t. system (1)* if there is a polynomial state feedback law $\alpha \in R^m$ as in Definition 4 which additionally takes the form $\alpha = \beta(h_1, \ldots, h_p)$ for some $\beta \in T^m$, i.e.

$$\alpha \in S^m = K[h_1, \ldots, h_p]^m. \tag{5}$$

If we wish to decide whether a variety V satisfies Definition 7, we have to determine the set (4) and check whether one of its elements lies in S^m. More generally, we intend to compute the set

$$(v + \mathscr{P}) \cap K[h_1, \ldots, h_p]^m \tag{6}$$

for a given R-module $\mathscr{P} \subseteq R^m, v \in R^m$, and $h_1, \ldots, h_p \in R$. The main goal of this section is to analyze the structure of this set and to give algorithms for its computation. The algebraic methods we present are not limited to $K \in \{\mathbb{R}, \mathbb{C}\}$, so let K be an arbitrary field.

A. Intersection of an ideal and a subalgebra

Our first step will be to give a method to compute (6) in the special case $v = 0$ (or $v \in \mathscr{P}$). For this, we define two K-algebra homomorphisms

$$\begin{aligned}\phi &: T \to R, \quad y_j \mapsto h_j, \ j = 1, \ldots, p, \text{ and} \\ \psi &: R_1 \to R, \quad x_i \mapsto x_i, \quad i = 1, \ldots, n, \\ &\qquad\qquad\quad y_j \mapsto h_j, \ j = 1, \ldots, p.\end{aligned}$$

The next theorem gives some properties of these maps:

Theorem 8. *Let ϕ and ψ be defined as above.*

1) *We have* $\mathrm{im}(\phi) = \mathrm{im}(\psi|_T) = S$. *Hence we may write* $\phi : T \twoheadrightarrow S$ *and thus* $\phi(\phi^{-1}(I)) = I \cap S$ *for all subsets* $I \subseteq R$.
2) $\psi|_R = \mathrm{id}_R$, $\psi|_T = \phi$, *so we have the following commutative diagram:*

$$\begin{array}{ccc} R_1 & \xrightarrow{\psi} & R \\ \uparrow & & \uparrow \\ T & \xrightarrow{\phi} & S \end{array}$$

3) *If $I \subseteq R$ is an ideal, then*

$$\phi^{-1}(I) = (J_\phi + I) \cap T \text{ and } \psi^{-1}(I) = J_\phi + I, \qquad (7)$$

where $J_\phi := \langle y_j - \phi(y_j) \mid j = 1, \ldots, p \rangle_{R_1} = \ker(\psi)$.

Proof. 1) and 2) can be verified easily and 3) follows from [6], Theorem 3.2.1. □

Since an algorithm for the elimination of variables needed in (7) is well known (see e.g. [3, ch.1]), we can use Theorem 8 to find an algorithmic method to compute the S-module $I \cap S$ for any ideal $I \subseteq R$.
For arbitrary $m \in \mathbb{N}$, we define the extension maps to free modules T^m (resp. R_1^m) induced by ϕ (resp. ψ), also denoted by ϕ (resp. ψ):

$$\phi : T^m \to R^m, \quad \sum_{i=1}^m a_i e_i \mapsto \sum_{i=1}^m \phi(a_i) e_i, \qquad (8)$$
$$\psi : R_1^m \to R^m, \quad \sum_{i=1}^m a_i e_i \mapsto \sum_{i=1}^m \psi(a_i) e_i.$$

Here e_i denotes the i-th standard basis vector of the free module T^m (resp. R_1^m or R^m), and $a_i \in T$ (resp. $a_i \in R_1$) are arbitrary polynomials.
After a short computation, we get

$$\phi(\sum_{i=1}^s a_i t_i) = \sum_{i=1}^s \phi(a_i)\phi(t_i), \; s \in \mathbb{N}, \; a_i \in T, \; t_i \in T^m, \qquad (9)$$
$$\psi(\sum_{i=1}^s b_i r_i) = \sum_{i=1}^s \psi(b_i)\psi(r_i), \; s \in \mathbb{N}, \; b_i \in R_1, \; r_i \in R_1^m.$$

If $m_1, \ldots, m_k \in R_1^m$ and R_2 is a subring of R_1, then (9) yields

$$\psi(\langle m_1, \ldots, m_k \rangle_{R_2}) = \langle \psi(m_1), \ldots, \psi(m_k) \rangle_{\psi(R_2)}. \qquad (10)$$

Analogously to Theorem 8, we obtain the following corollary by considering the individual components of the elements of T^m (resp. R_1^m):

Corollary 9. *The extension maps ϕ and ψ have the following properties:*

1) $\mathrm{im}(\phi) = \mathrm{im}(\psi|_{T^m}) = S^m$, *hence* $\phi(\phi^{-1}(\mathscr{P})) = \mathscr{P} \cap S^m$ *for all subsets* $\mathscr{P} \subseteq R^m$.
2) $\psi|_{R^m} = \mathrm{id}_{R^m}$, $\psi|_{T^m} = \phi$.
3) *If $\mathscr{P} \subseteq R^m$ is an R-module, then*

$$\phi^{-1}(\mathscr{P}) = (J_\phi^m + \mathscr{P}) \cap T^m \text{ and } \psi^{-1}(\mathscr{P}) = J_\phi^m + \mathscr{P},$$
$$J_\phi^m := \langle (y_j - \phi(y_j)) \cdot e_i \mid j = 1, \ldots, p, i = 1, \ldots, m \rangle_{R_1}.$$

With this result, we are able to compute the intersection of an R-module $\mathscr{P} \subseteq R^m$ with S^m for a finitely generated subalgebra S of R. It will be the basic and frequently used tool for the determination of (6).

B. Intersection of an affine ideal and a subalgebra

Assume there exists $q \in (v + \mathscr{P}) \cap S^m$. Then $q = v + p \in S^m$ for some $p \in \mathscr{P}$ and for every other $q' \in (v + \mathscr{P}) \cap S^m$, there is $p' \in \mathscr{P}$ with $q' = v + p'$. We conclude

$$q' - q = v + p - v - p' = p - p' \in \mathscr{P} \cap S^m,$$

i.e. $q' \in q + \mathscr{P} \cap S^m$. On the other hand, if $q' \in q + \mathscr{P} \cap S^m$, we see

$$q' = q + p' = v + p + p' \in v + \mathscr{P}.$$

Thus

$$(v + \mathscr{P}) \cap S^m = q + \mathscr{P} \cap S^m. \qquad (11)$$

So our objective is to determine one particular element q of $(v + \mathscr{P}) \cap S^m$ and then we'll get the whole set using (11) and Corollary 9.
First, let $\mathscr{P}_1 \subseteq R_1^m$ be a submodule and $v \in R_1^m$ a vector. Consider a generating set $\{u_1, \ldots, u_l\} \subseteq T^m$ of

$$T^m \cap (\langle v \rangle_{R_1} + \mathscr{P}_1).$$

Define the matrix

$$\mathbf{u} := (-v, u_1, \ldots, u_l) \in R_1^{m \times (l+1)}$$

and the corresponding map

$$\chi : R_1^{l+1} \to R_1^m,$$
$$(z_0, \ldots, z_l) = z \mapsto \mathbf{u}z = -z_0 v + z_1 u_1 + \ldots + z_l u_l.$$

Using methods from the theory of Gröbner bases (which is a frequently used tool for computations concerning (multivariate) polynomials; for an introduction, see e.g. [1, ch.3] or [3, ch.2]) we are able to determine $\chi^{-1}(\mathscr{P}_1)$, hence we can compute

$$\mathscr{P}_2 := T^{l+1} \cap \chi^{-1}(\mathscr{P}_1) \qquad (12)$$

and the ideal

$$\mathscr{L} := \{z_0 \in T \mid z \in \mathscr{P}_2\},$$

which is the projection of $\mathscr{P}_2$ on the first component.

Theorem 10. *The following statements are equivalent:*

1) $1 \in \mathscr{L}$.
2) $(v + \mathscr{P}_1) \cap T^m \neq \emptyset$.

Proof. If $1 \in \mathscr{L}$, then there exists $z = (1, z_1, \ldots, z_l)^{tr} \in \mathscr{P}_2$. This means

$$\chi(z) = -v + \sum_{j=1}^l z_j u_j \in \mathscr{P}_1,$$

hence

$$v + \chi(z) = \sum_{j=1}^l z_j u_j \in (v + \mathscr{P}_1) \cap T^m.$$

Conversely let $w \in \mathscr{P}_1$ be such that

$$v + w \in (v + \mathscr{P}_1) \cap T^m \subseteq (\langle v \rangle_{R_1} + \mathscr{P}_1) \cap T^m = \langle u_1, \ldots, u_l \rangle_T.$$

We can write

$$v + w = \sum_{j=1}^{l} z_j u_j \text{ for some } z_j \in T.$$

Putting $z := (1, z_1, \ldots, z_l) \in T^{l+1}$, this yields

$$\chi(z) = \mathbf{u}z = -v + \sum_{j=1}^{l} z_j u_j = w \in \mathscr{P}_1,$$

so $z \in \mathscr{P}_2$ and therefore $1 \in \mathscr{L}$. □

Now let $\mathscr{P} \subseteq R^m \subseteq R_1^m$ be an R-module and $v \in R^m \subseteq R_1^m$. Using Corollary 9, we define the R_1-module

$$\mathscr{P}_1 := \psi^{-1}(\mathscr{P}) = \mathscr{P} + J_\phi^m \subseteq R_1^m.$$

Theorem 11. *The following statements are equivalent:*

1) $1 \in \mathscr{L}$.
2) $(v + \mathscr{P}_1) \cap T^m \neq \emptyset$.
3) $(v + \mathscr{P}) \cap S^m = \psi((v + \mathscr{P}_1) \cap T^m) \neq \emptyset$.

Proof. The first equivalence has already been proven in Theorem 10. For 2) ⇔ 3) it suffices to show the equality

$$S^m \cap (v + \mathscr{P}) = \psi(T^m \cap (v + \mathscr{P}_1)).$$

First, let $t \in T^m \cap (v + \mathscr{P}_1)$. Corollary 9 yields $\psi(t) \in S^m$ and there is $w \in \mathscr{P}_1 = \psi^{-1}(\mathscr{P})$ with $t = v + w$. This gives us

$$\psi(t) = \psi(\underbrace{v}_{\in R^m}) + \underbrace{\psi(w)}_{\in \mathscr{P}} = v + \psi(w) \in v + \mathscr{P}.$$

For the inclusion $\subseteq$ let $q \in S^m \cap (v + \mathscr{P})$, i.e.

$$q = v + w = \psi(t) \text{ for some } w \in \mathscr{P},\ t \in T^m.$$

Since $v, w \in R^m$ we have $\psi(v) = v$ and $\psi(w) = w$, hence

$$\psi(t - v - w) = q - v - w = 0.$$

Using Corollary 9 we may conclude

$$t - v - w := j \in \ker(\psi) = \psi^{-1}(0) = J_\phi^m$$

and thus

$$t = v + \underbrace{w + j}_{\in \mathscr{P} + J_\phi^m = \mathscr{P}_1} \in T^m \cap (v + \mathscr{P}_1),$$

which shows

$$q = \psi(t) \in \psi(T^m \cap (v + \mathscr{P}_1)).$$

□

The constructions above and the proofs of Theorem 10 and Theorem 11 yield the following algorithm:

Algorithm 12.
Input: R, R_1, S and T as defined in the introduction, $v \in R^m$ and $\mathscr{P} \subseteq R^m$ an R-module.
Output: An element $q \in (v + \mathscr{P}) \cap S^m$ or a message about its non-existence.

Compute $\mathscr{P}_1 = \mathscr{P} + J_\phi^m \subseteq R_1^m$ and a generating set $\{u_1, \ldots, u_l\}$ of $T^m \cap (\langle v \rangle_{R_1} + \mathscr{P}_1)$.
Compute $\mathscr{P}_2 = T^{l+1} \cap \chi^{-1}(\mathscr{P}_1) = \langle z_1, \ldots, z_s \rangle_T$.
Let $\mathscr{L}$ be the ideal in T generated by the first components of the z_i, i.e. $\mathscr{L} = \langle z_{11}, \ldots, z_{1s} \rangle_T$.
if $1 \in \mathscr{L}$ **then**
  derive $c = (c_1, \ldots, c_s) \in T^s$ with $1 = \sum_{i=1}^{s} c_i z_{1i}$
else
  return The set $(v + \mathscr{P}) \cap S^m$ is empty.
end if
Put $w = \phi(\sum_{i=1}^{s} c_i z_i) = (1, w_1, \ldots, w_l) \in \mathscr{P}_2$.
Compute $t = \sum_{j=1}^{s} w_j u_j \in (v + \mathscr{P}_1) \cap T^m$.
return $q := \phi(t) = \psi(t) \in (v + \mathscr{P}) \cap S^m$.

Example 13. Once again we look at the system of Example 6 and assume additionally that there is an output $y(t) = h(x(t))$, where

$$h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} = \begin{pmatrix} x_1 x_2^3 \\ x_1^4 \end{pmatrix}.$$

We have already seen that the variety V is controlled invariant. Now we are interested whether it is conditioned invariant as well. Let $S = K[x_1 x_2^3, x_1^4]$ be the subalgebra of $R = K[x_1, x_2]$ generated by h_1, h_2. Obviously $\alpha \notin S^2$, but we can use Algorithm 12 to determine an element α^* of the intersection of the affine set (4) and S^2 (if it exists). Doing this yields

$$\alpha^* = \begin{pmatrix} -x_1^5 x_2^3 - 1 \\ -x_1^4 - x_1 x_2^3 \end{pmatrix} \in (\alpha + \pi(\ker(\mathbf{m}, -g))) \cap S^2.$$

Because of the existence of α^*, we may conclude that V is indeed controlled and conditioned invariant. Due to (11) the whole set of output feedback laws achieving invariance of V is given by

$$\underbrace{\begin{pmatrix} -h_1 h_2 - 1 \\ -h_2 - h_1 \end{pmatrix}}_{\alpha^*} + \underbrace{\langle \begin{pmatrix} h_1^2 h_2 - 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ h_1^2 h_2 - 1 \end{pmatrix} \rangle_S}_{\pi(\ker(\mathbf{m}, -g)) \cap S^2}.$$

C. Approach via partial elimination

Here, we give a another interpretation of the computation of (6), or more precisely, of a $q \in (v + \mathscr{P}) \cap S^m$. We want to get rid of the blackbox computation of $\mathscr{P}_2$ defined in (12). It turns out that the following considerations will be helpful on our way to do this:
Let $\mathscr{P} \subseteq R^{m+l}$ be an R-module. Then we are interested in the determination of the set

$$\mathscr{N} := \{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathscr{P} \mid b \in S^l \} = \mathscr{P} \cap (R^m \times S^l).$$

Thus the S-module $\mathscr{N}$ contains all elements of $\mathscr{P}$ whose last l components are in the subalgebra $S \subseteq R$.
We want to put this problem in a slightly generalized setting: Let $d \in \mathbb{N}$ and $\mathscr{M} \subseteq R_1^d$ be an R_1-module with a direct decomposition

$$\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2 = \mathscr{M}_1 \times \mathscr{M}_2.$$

Furthermore let $\mathscr{N}_2 \subseteq \mathscr{M}_2$ be a T-submodule and let

$$\pi : \mathscr{M} \to \mathscr{M}_2, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto v_2$$

be the projection of $\mathscr{M}$ on $\mathscr{M}_2$.

Lemma 14. *Let $\mathscr{P}_1 \subseteq \mathscr{M}$ be a submodule, $u_1,\dots,u_s \in \mathscr{N}_2$ such that $\mathscr{N}_2 \cap \pi(\mathscr{P}_1) = \langle u_1,\dots,u_s\rangle_T$ and $v_1,\dots,v_s \in \mathscr{M}_1$ with $(v_j^{tr}, u_j^{tr})^{tr} \in \mathscr{P}_1$ for all $j = 1,\dots,s$.*
Then the following equation holds:

$$\mathscr{P}_1 \cap (\mathscr{M}_1 \times \mathscr{N}_2) = \langle \begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \dots, \begin{pmatrix} v_s \\ u_s \end{pmatrix} \rangle_T + \mathscr{P}_1 \cap (\mathscr{M}_1 \times 0).$$

Proof. The inclusion $\supseteq$ follows immediately by construction. Let $u \in \mathscr{N}_2, v \in \mathscr{M}_1$ such that $\begin{pmatrix} v \\ u \end{pmatrix} \in \mathscr{P}_1$. Then

$$u \in \mathscr{N}_2 \cap \pi(\mathscr{P}_1) = \langle u_1,\dots,u_s\rangle_T,$$

so there are $z_j \in T$ with $u = \sum_{j=1}^s z_j u_j$. This yields

$$\mathscr{P}_1 \ni \begin{pmatrix} v \\ u \end{pmatrix} = \underbrace{\sum_{j=1}^s z_j \begin{pmatrix} v_j \\ u_j \end{pmatrix}}_{\in \mathscr{P}_1 \cap \langle \begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \dots, \begin{pmatrix} v_s \\ u_s \end{pmatrix} \rangle_T} + \underbrace{\begin{pmatrix} v \\ 0 \end{pmatrix} - \sum_{j=1}^s z_j \begin{pmatrix} v_j \\ 0 \end{pmatrix}}_{\in \mathscr{M}_1 \times 0},$$

which shows the assertion. □

With this preparation we can come back to our actual setting, i.e.

$$\mathscr{M} = R_1^{m+l} = R_1^m \times R_1^l, \ \mathscr{N}_2 = T^l \subseteq R_1^l.$$

Theorem 15. *Let $\mathscr{P} \subseteq R^{m+l}$ and $\mathscr{P}_1 = \psi^{-1}(\mathscr{P})$. Choose generators $u_1,\dots,u_l$ of the T-module $T^l \cap \pi(\mathscr{P}_1)$, elements $v_1,\dots,v_s \in R_1^m$ such that $(v_j^{tr}, u_j^{tr})^{tr} \in \mathscr{P}_1$ and $w_1,\dots,w_k \in R_1^m$ satisfying*

$$\mathscr{P}_1 \cap (R_1^m \times 0) = \langle \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} w_k \\ 0 \end{pmatrix} \rangle_{R_1}.$$

Then the following equations hold:

1) $\mathscr{P}_1 \cap (R_1^m \times T^l) = \langle \begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \dots, \begin{pmatrix} v_s \\ u_s \end{pmatrix} \rangle_T + \langle \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} w_k \\ 0 \end{pmatrix} \rangle_{R_1}$

2) $\mathscr{P} \cap (R^m \times S^l)$
$= \langle \begin{pmatrix} \psi(v_1) \\ \phi(u_1) \end{pmatrix}, \dots, \begin{pmatrix} \psi(v_s) \\ \phi(u_s) \end{pmatrix} \rangle_S + \langle \begin{pmatrix} \psi(w_1) \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \psi(w_k) \\ 0 \end{pmatrix} \rangle_R$

Proof. The first assertion follows directly from Lemma 14. To show the second one, consider

$$\begin{aligned}
&\psi(\mathscr{P}_1 \cap (R_1^m \times T^l)) \\
&\overset{1)}{=} \psi(\langle \begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \dots, \begin{pmatrix} v_s \\ u_s \end{pmatrix} \rangle_T + \langle \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} w_k \\ 0 \end{pmatrix} \rangle_{R_1}) \\
&= \psi(\langle \begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \dots, \begin{pmatrix} v_s \\ u_s \end{pmatrix} \rangle_T) + \psi(\langle \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} w_k \\ 0 \end{pmatrix} \rangle_{R_1}) \\
&\overset{(10)}{=} \langle \psi \begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \dots, \psi \begin{pmatrix} v_s \\ u_s \end{pmatrix} \rangle_{\psi(T)} + \langle \psi \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \psi \begin{pmatrix} w_k \\ 0 \end{pmatrix} \rangle_{\psi(R_1)} \\
&= \langle \begin{pmatrix} \psi(v_1) \\ \phi(u_1) \end{pmatrix}, \dots, \begin{pmatrix} \psi(v_s) \\ \phi(u_s) \end{pmatrix} \rangle_S + \langle \begin{pmatrix} \psi(w_1) \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \psi(w_k) \\ 0 \end{pmatrix} \rangle_R.
\end{aligned}$$

So we just have to show that

$$\psi(\mathscr{P}_1 \cap (R_1^m \times T^l)) = \mathscr{P} \cap (R^m \times S^l).$$

The inclusion $\subseteq$ is clear by construction. Take an arbitrary element $\begin{pmatrix} v \\ u \end{pmatrix} \in \mathscr{P} \cap (R^m \times S^l)$. Then there is $t \in T^l$ with $\psi(t) = u$ and we have $u = \psi(u), v = \psi(v)$. This yields

$$0 = \psi(\begin{pmatrix} v \\ t \end{pmatrix} - \begin{pmatrix} v \\ u \end{pmatrix}), \text{ i.e. } \begin{pmatrix} v \\ t \end{pmatrix} - \begin{pmatrix} v \\ u \end{pmatrix} \in \ker(\psi) = J_\phi^{m+l},$$

which means there is a $w \in J_\phi^{m+l}$ satisfying

$$\begin{pmatrix} v \\ t \end{pmatrix} = \begin{pmatrix} v \\ u \end{pmatrix} + w \in \mathscr{P} + J_\phi^{m+l} = \mathscr{P}_1.$$

We conclude

$$\begin{pmatrix} v \\ u \end{pmatrix} = \psi \begin{pmatrix} v \\ t \end{pmatrix} \in \psi(\mathscr{P}_1 \cap (R_1^m \times T^l)),$$

which proves the claim. □

The next algorithm gives us a method to compute the elements u_i, v_i and w_i from the theorem above:

Algorithm 16.
Input: $R_1 = K[\underline{x}, \underline{y}]$ a polynomial ring, $T = K[\underline{y}]$ a subalgebra, $\mathscr{P}_1 \subseteq R_1^{m+l}$ an R_1-module generated by $z_1,\dots,z_r$.
Output: Two sets of generators

$$\{\begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \dots, \begin{pmatrix} v_s \\ u_s \end{pmatrix}\} \text{ and } \{\begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} w_k \\ 0 \end{pmatrix}\},$$

satisfying the first equality of Theorem 15.

1: Compute $\pi(\mathscr{P}_1) = \langle \pi(z_1),\dots,\pi(z_r)\rangle_{R_1} \subseteq R_1^l$ and set
$$\mathbf{z_1} := (z_1,\dots,z_r) \text{ and } \mathbf{z_2} := (\pi(z_1),\dots,\pi(z_r)).$$
2: Compute a generating set $\{u_1,\dots,u_s\}$ of $T^l \cap \pi(\mathscr{P}_1)$ (using Corollary 9).
3: **for** $i = 1,\dots,s$ **do**
4: Compute $b_i \in R_1^r$ with $u_i = \mathbf{z_2} b_i$ and set $\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \mathbf{z_1} b_i$.
5: **end for**
6: Compute $\ker_{R_1}(\mathbf{z_2}) = \langle t_1,\dots,t_k\rangle_{R_1}$.
7: **for** $i = 1,\dots,k$ **do**
8: Set $\begin{pmatrix} w_i \\ 0 \end{pmatrix} = \mathbf{z_1} t_i$.
9: **end for**
10: **return** $\{\begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \dots, \begin{pmatrix} v_s \\ u_s \end{pmatrix}\}, \{\begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} w_k \\ 0 \end{pmatrix}\}$.

Proof. We wish to prove that Algorithm 16 computes the result it is supposed to do. A simple computation shows $\pi(\mathscr{P}_1) = \langle \pi(z_1),\dots,\pi(z_r)\rangle_{R_1}$ and since the columns of $\mathbf{z_1}$ generate $\mathscr{P}_1$ it is clear, that the constructed $\begin{pmatrix} v_i \\ u_i \end{pmatrix}$ are in $\mathscr{P}_1$. The only thing left to prove is the equality

$$\mathscr{P}_1 \cap (R_1^m \times 0) = \langle \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} w_k \\ 0 \end{pmatrix} \rangle_{R_1},$$

where $\supseteq$ is clear by construction. Let $\begin{pmatrix} a \\ 0 \end{pmatrix} \in \mathscr{P}_1 \cap (R_1^m \times 0)$, i.e. there is $b \in R_1^r$ with $\begin{pmatrix} a \\ 0 \end{pmatrix} = \mathbf{z_1} b$. But then we have

$$0 = \pi(\begin{pmatrix} a \\ 0 \end{pmatrix}) = \pi(\mathbf{z_1} b) = \mathbf{z_2} b,$$

so $b \in \ker(\mathbf{z_2})$. Thus we have a presentation $b = \sum_{i=1}^k c_i t_i$ for some $c_i \in R_1$, hence

$$\begin{pmatrix} a \\ 0 \end{pmatrix} = \mathbf{z_1} b = \sum_{i=1}^k c_i \mathbf{z_1} t_i = \sum_{i=1}^k c_i \begin{pmatrix} w_i \\ 0 \end{pmatrix} \in \langle \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} w_k \\ 0 \end{pmatrix} \rangle_{R_1}.$$

□

Example 17. Let $R_1 = K[x,y]$, $T = K[y]$ and the R_1-module $\mathscr{P}_1 = \langle z_1, z_2, z_3 \rangle$ be given, where

$$z_1 = \begin{pmatrix} -x-1 \\ y^2-1 \\ 0 \end{pmatrix}, z_2 = \begin{pmatrix} -x^2 y \\ 0 \\ y^2-1 \end{pmatrix}, z_3 = \begin{pmatrix} x^3+x^2 \\ y^2 \\ -xy-y \end{pmatrix}.$$

We would like to find

$$\mathscr{N} = \{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathscr{P}_1 \mid b \in T \},$$

i.e. $m = 2$ and $l = 1$. We set

$$\mathbf{z_1} = (z_1, z_2, z_3), \ \mathbf{z_2} = (0, y^2-1, -xy-y).$$

Using the elimination of variables we derive

$$\pi(\mathscr{P}_1) \cap T = \langle y^2 - 1 \rangle_T, \text{ so } u_1 = y^2 - 1.$$

Now $b_1 = e_2$ fulfills $\mathbf{z_2} b_1 = u_1$ and we get

$$\begin{pmatrix} v_1 \\ u_1 \end{pmatrix} = \mathbf{z_1} b_1 = \mathbf{z_1} e_2 = z_2.$$

Furthermore, the kernel of $\mathbf{z_2}$ over R_1 is given by

$$\ker_{R_1}(\mathbf{z_2}) = \langle e_1, (xy+y)e_2 + (y^2-1)e_3 \rangle_{R_1}$$

and thus we put $t_1 = e_1$, $t_2 = (xy+y)e_2 + (y^2-1)e_3$ and

$$\begin{pmatrix} w_1 \\ 0 \end{pmatrix} = \mathbf{z_1} t_1 = z_1, \quad \begin{pmatrix} w_2 \\ 0 \end{pmatrix} = \mathbf{z_1} t_2 = \begin{pmatrix} -x^3 - x^2 \\ y^4 - y^2 \\ 0 \end{pmatrix}.$$

Summarized, our result is given by

$$\mathscr{N} = \langle \begin{pmatrix} v_1 \\ u_1 \end{pmatrix} \rangle_T + \langle \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \begin{pmatrix} w_2 \\ 0 \end{pmatrix} \rangle_{R_1}.$$

Now consider $R_1 = K[\underline{x}, \underline{y}]$, $S = T = K[\underline{y}]$ and $\mathscr{P} \subseteq R_1^{m+l}$. In this case we can give another approach to determine $\mathscr{N}$ using Gröbner bases and the elimination of variables.
Let $<_{\text{lex}}$ be a lexicographical order on the set of monomials of $K[\underline{x}, \underline{y}]$ with $\underline{x} > \underline{y}$ and $<_{\text{TOP}}$ be a monomial order TOP ("term over position") on $K[\underline{x}, \underline{y}]^{m+l}$ with respect to $<_{\text{lex}}$ (this means that we first compare the highest occuring terms of two vectors over $K[\underline{x}, \underline{y}]$ with respect to $<_{\text{lex}}$ and only if these are equal we look at their position). Then we can define another order $<$ on the monomials of $K[\underline{x}, \underline{y}]^{m+l}$: If we have monomials $p, q \in K[\underline{x}, \underline{y}]^{m+l}$, then

$$p < q \Leftrightarrow \begin{cases} \pi(p) <_{\text{TOP}} \pi(q) \\ \text{or} \\ \pi(p) = \pi(q) \text{ and } p <_{\text{TOP}} q. \end{cases}$$

A technical check ensures that this indeed defines a monomial order on $K[\underline{x}, \underline{y}]^{m+l}$. Now let $G \subseteq \mathscr{P} \setminus \{0\}$ be a finite subset. We call G a Gröbner basis of $\mathscr{P}$ with respect to $<$ if for all $0 \neq p \in \mathscr{P}$ there is $q \in G$ satisfying $\text{lm}(q) \mid \text{lm}(p)$, where $\text{lm}(\cdot)$ denotes the leading monomial of a polynomial or of an element of $K[\underline{x}, \underline{y}]^{m+l}$.

Lemma 18. *If G is a Gröbner basis of $\mathscr{P}$ with respect to $<$, then $G \cap \mathscr{N}$ is a Gröbner basis of $\mathscr{N}$ as $K[\underline{y}]$-module.*

Proof. Let $0 \neq p \in \mathscr{N} \subseteq \mathscr{P}$. Since G is a Gröbner basis of $\mathscr{P}$, there is a $q \in G$ with $\text{lm}(q) \mid \text{lm}(p)$, so there are monomials $q', p' \in K[\underline{x}, \underline{y}]$ and $1 \leq i \leq m+l$, which fulfill $\text{lm}(q) = q' e_i, \text{lm}(p) = p' e_i$ and $q' \mid p'$. We have to show that $q \in \mathscr{N}$.
Case 1: $1 \leq i \leq m$.
Assume that there is a $m+1 \leq j \leq m+l$ with $q_j \neq 0$. Since the j-th component of $\text{lm}(q) = q' e_i$ is zero, we have $\text{lm}(q_j) e_j > q' e_i$ because of the special choice of $<$. But $q' e_i = \text{lm}(q) \geq \text{lm}(q_j) e_j$, a contradiction. So $q = \begin{pmatrix} q_m \\ 0 \end{pmatrix} \in \mathscr{N}$, where $q_m \in K[\underline{x}, \underline{y}]^m$.
Case 2: $m+1 \leq i \leq m+l$.
Since $q' \mid p'$ and p' only depends on $\underline{y}$, this is also true for q'. We claim that all the last l components of q only depend on $\underline{y}$. Otherwise, if q_j is a component of q depending on $\underline{x}$ and $m+1 \leq j \leq m+l$, we would have $\text{lm}(q_j) e_j > q' e_i$ from the definition of $<$. Again, this is a contradiction to the definition of the leading monomial of q. We conclude $\pi(q) \in K[\underline{y}]^l$, which means $q \in \mathscr{N}$. □

Remark 19. If $\mathscr{P} \subseteq K[\underline{x}, \underline{y}]^{m+l}$ is a $K[\underline{x}, \underline{y}]$-module and a set $G \subseteq K[\underline{x}, \underline{y}]^{m+l}$ is a Gröbner basis of $\mathscr{P}$, then $\langle G \rangle_{K[\underline{x}, \underline{y}]} = \mathscr{P}$ (see e.g. [1]). However, being in the situation of Lemma 18, we have $\mathscr{N} \subset K[\underline{x}, \underline{y}]^{m+l}$, and then $G \cap \mathscr{N} \subseteq K[\underline{x}, \underline{y}]^{m+l}$ is a Gröbner basis of $\mathscr{N}$, which is only a $K[\underline{y}]$-module. So we are not allowed to conclude $\langle G \cap \mathscr{N} \rangle_{K[\underline{y}]} = \mathscr{N}$. To close this gap, define G_1 as the set of all elements of $G \cap \mathscr{N}$ which are zero in the last l components. Then

$$\langle G_1 \rangle_{K[\underline{x}, \underline{y}]} + \langle (G \cap \mathscr{N}) \setminus G_1 \rangle_{K[\underline{y}]} = \mathscr{N}.$$

Algorithm 20.
Input: $R_1 = K[\underline{x}, \underline{y}]$ a polynomial ring, $S = T = K[\underline{y}] \subseteq R_1$ a subalgebra, $\mathscr{P} \subseteq R_1^{m+l}$ an R_1-module.
Output: A Gröbner basis of $\mathscr{N} = \{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathscr{P} \mid b \in S^l \}$.

1: Compute a Gröbner basis G of $\mathscr{P}$ with respect to $<$ defined above.
2: **return** The set of elements of G whose last l components depend only on $\underline{y}$.

Our actual problem was to find a single element of the set $(v+\mathscr{P})\cap S^m = \phi(\phi^{-1}(v+\mathscr{P}))$. Similarly to (11), using some arguments from the proof of Theorem 8, one can show

$$\phi^{-1}(v+\mathscr{P}) = (v+\mathscr{P}+J_\phi^m)\cap T^m = (q+\mathscr{P}_1)\cap T^m,$$

where J_ϕ^m is defined as in Corollary 9 and $q\in(v+\mathscr{P}_1)\cap T^m$. Let $w_1,\dots,w_k$ be generators of $\mathscr{P}_1$ and $u_1,\dots,u_l$ generators of $(\langle v\rangle+\mathscr{P}_1)\cap T^m$. We wish to find all polynomials $c\in T, b_i\in T, a_i\in R_1$ satisfying

$$-cv+\sum_{i=1}^{l} a_i u_i + \sum_{i=1}^{k} b_i w_i = 0.$$

This can be done by the computation of $\mathscr{Q} := \ker_{R_1}(\mathbf{z})$, where

$$\mathbf{z} = (-v, u_1,\dots,u_l,w_1,\dots,w_k).$$

Then we use Algorithm 16 or 20 over the ring $R_1 = K[\underline{x},\underline{y}]$ and the subalgebra $T = K[\underline{y}]$ to find the T-module

$$\mathscr{Q}_1 := \{\left(c,a^{tr},b^{tr}\right)^{tr}\in\mathscr{Q} \mid a\in T^l, c\in T\} \qquad (13)$$

as desired. Finally let

$$\mathscr{Q}_2 := \{\left(c,a^{tr}\right)^{tr} \mid \left(c,a^{tr},b^{tr}\right)^{tr}\in\mathscr{Q}_1\} \subseteq T^{l+1}.$$

Lemma 21. *We have $\mathscr{Q}_2 = \mathscr{P}_2$, where $\mathscr{P}_2$ is defined in (12).*

Proof. $\subseteq$: Let $q = (c,a^{tr})^{tr}\in\mathscr{Q}_2$. Then $q\in T^{l+1}$ and there is $b\in R_1^k$, such that $(c,a^{tr},b^{tr})^{tr}\in\mathscr{Q}$, which means

$$0 = \mathbf{z}\begin{pmatrix} c\\ a\\ b\end{pmatrix} = -cv+\sum_{i=1}^{l} a_i u_i+\sum_{i=1}^{k} b_i w_i = \chi(q)+\sum_{i=1}^{k} b_i w_i,$$

hence

$$\chi(q) = -\sum_{i=1}^{k} b_i w_i \in \mathscr{P}_1,$$

and thus $q\in\chi^{-1}(\mathscr{P}_1)\cap T^{l+1} = \mathscr{P}_2$.
$\supseteq$: If $q = (q_0,\dots,q_l)\in\mathscr{P}_2 = T^{l+1}\cap\chi^{-1}(\mathscr{P}_1)$, then there is $b\in R_1^k$ with

$$\mathscr{P}_1 \ni \chi(q) = -q_0 v+\sum_{i=1}^{l} q_i u_i = \sum_{i=1}^{k} b_i w_i.$$

This yields

$$0 = -q_0 v+\sum_{i=1}^{l} q_i u_i - \sum_{i=1}^{k} b_i w_i = \mathbf{z}\begin{pmatrix} q\\ -b\end{pmatrix},$$

i.e. $(q^{tr},-b^{tr})^{tr}\in\ker_{R_1}(\mathbf{z}) = \mathscr{Q}$ and thus $q\in\mathscr{Q}_2$. □

Now we are able to give an algorithm to compute one element $q\in(v+\mathscr{P}_1)\cap T^m$ (and then $\phi(q)$ is in (6)):

Algorithm 22.
Input: R, R_1, S and T defined as in the introduction, $v\in R^m$ and $\mathscr{P}\subseteq R^m$ an R-module.
Output: A vector $q\in(v+\mathscr{P})\cap S^m$, or a message about its non-existence.

1: Compute generators $w_1,\dots,w_k\in R_1^m$ of $\mathscr{P}_1 = J_\phi^m+\mathscr{P}$.
2: Compute generators $u_1,\dots,u_l\in T^m$ of $(\langle v\rangle+\mathscr{P}_1)\cap T^m$.
3: Compute $\mathscr{Q} := \ker_{R_1}(\mathbf{z})$, where

$$\mathbf{z} = (-v,u_1,\dots,u_l,w_1,\dots,w_k).$$

4: Use Algorithm 16 or 20 to compute generators $z_1,\dots,z_s$ of $\mathscr{Q}_1$ (defined in (13)), whose first $l+1$ components are not all zero.
5: Let $\mathscr{L}$ be the ideal in T generated by the first components of the z_i, i.e. $\mathscr{L} = \langle z_{11},\dots,z_{1s}\rangle$.
6: **if** $1\notin\mathscr{L}$ **then**
7: **return** The set $\phi^{-1}(v+\mathscr{P})$ is empty.
8: **else**
9: Compute $d\in T^s$ satisfying $\sum_{i=1}^{s} d_i z_{1i} = 1$.
10: Compute the vectors $\sum_{i=1}^{s} d_i z_i = (1,a^{tr},b^{tr})^{tr}\in\mathscr{Q}_1$ and $t := v-\sum_{i=1}^{k} b_i w_i\in(v+\mathscr{P}_1)\cap T^m$.
11: **return** $q := \phi(t)\in(v+\mathscr{P})\cap S^m$.
12: **end if**

Example 23. Consider again Example 13. Using Algorithm 22 instead of Algorithm 12 yields

$$\alpha_1^* = \begin{pmatrix} -x_1^7x_2^9 - x_1^5x_2^3 + x_1x_2^3 - 1\\ -2x_1^{10}x_2^6 + x_1^4 - x_1x_2^3\end{pmatrix} = \begin{pmatrix} -h_1^3h_2 - h_1h_2 + h_1 - 1\\ -2h_1^2h_2^2 + h_2 - h_1\end{pmatrix}$$

as an output feedback law making V invariant. But since

$$\alpha^* - \alpha_1^* = \begin{pmatrix} h_1^3h_2 - h_1\\ 2h_1^2h_2^2 - 2h_2\end{pmatrix} \in \langle \begin{pmatrix} h_1^2h_2 - 1\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ h_1^2h_2 - 1\end{pmatrix}\rangle_S$$

these two results are equivalent.

With these tools, we are also able to determine the set

$$\mathscr{G} = \{\begin{pmatrix} a\\ b\end{pmatrix} \in v+\mathscr{P} \mid b\in S^l\},$$

where $v\in R^{m+l}$ and $\mathscr{P}\subseteq R^{m+l}$ is an R-module. Again, if $q\in\mathscr{G}$, we can write

$$\mathscr{G} = q+\{\begin{pmatrix} a\\ b\end{pmatrix}\in\mathscr{P} \mid b\in S^l\}.$$

The second summand can be derived by Algorithm 16, so we have to find one particular $q\in\mathscr{G}$. We can consider an element $q'\in(\pi(v)+\pi(\mathscr{P}))\cap S^l$ (if it exists, otherwise $\mathscr{G}=\emptyset$), where π is the projection on the last l components. So there is a $p\in\mathscr{P}$ satisfying $q' = \pi(v)+\pi(p)$. Setting $q = p+v$ yields $\pi(q) = \pi(p)+\pi(v) = q'\in S^l$, so $q\in\mathscr{G}$.

Algorithm 24.
Input: R, S and T as defined in the introduction, $v\in R^{m+l}$ and $\mathscr{P}\subseteq R^{m+l}$ an R-module.
Output: An element $q\in\{\begin{pmatrix} a\\ b\end{pmatrix}\in v+\mathscr{P} \mid b\in S^l\}$ or a message about its non-existence.

1: Compute $\pi(v)$ and $\pi(\mathscr{P})$.
2: Use Algorithm 12 or 22 to find an element

$$q'\in(\pi(v)+\pi(\mathscr{P}))\cap S^l$$

or return a message about its non-existence.
3: Compute a $p\in\mathscr{P}$ satisfying $q' = \pi(p)+\pi(v)$.
4: **return** $q := v+p$.

Example 25. Taking $\mathscr{P}$ from Example 17 above and $v = \begin{pmatrix} x_1^3 \\ x_1^2 - x_1 + 1 \\ -x_1 x_2 \end{pmatrix}$, Algorithm 24 yields

$$q = \begin{pmatrix} x_1^3 \\ -x_1^2 x_2^2 + x_1^2 - x_1 + 1 \\ x_2 \end{pmatrix} \in \underbrace{\{ \begin{pmatrix} a \\ b \end{pmatrix} \in v + \mathscr{P} \mid b \in K[x_2] \}}_{=\mathscr{G}},$$

and we may conclude

$$\mathscr{G} = q + \mathscr{N},$$

where $\mathscr{N}$ is as defined in Example 17.

IV. CONCLUSION

With the techniques described in this paper, we are able to decide whether a variety is controlled and conditioned invariant for a polynomial state-space system and, if this is the case, to derive an output feedback law such that the variety becomes invariant for the closed loop system.

The methods we have developed deal with the constructive intersection of affine ideals and subalgebras in polynomial rings, and with the analogous task in the multivariable (non-scalar) situation. Besides the control theoretic application we have studied here, we expect these tools to be useful in other branches of mathematics as well, e.g., in (computational) invariant theory.

For a specific kind of manifolds, there exists a concept of conditioned invariance on its own, investigated in [4]. It will be an interesting topic for future research to carry over also this concept to the algebraic setting.

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