21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Estimation of the Suspicious Observations Center*

Krzysztof J. Szajowski[1]

***Abstract*— The paper extends the results on the problem of change point detection for Markov processes generalizing the results contained in the publications [11], [15] and [8]. The short description are as follows. A random sequence having segments being the homogeneous Markov processes is registered. Each segment has his own transition probability law and the length of the segment is unknown and random. The transition probabilities of each process are known and joint *a priori* distribution of the disorder moments are given. The detection of the disorder rarely is precise. The decision maker accepts some deviation in estimation of the disorder moments. In this note the aim is to indicate the segment of given length between disorders with maximal probabilities. The case with various precision for over and under estimation of the middle point is analyzed including situation when the disorders do not appear with positive probability is also included. The observed sequence, when the change point is known, has the Markov properties. The results explain the structure of optimal detector in various circumstances and shows new details of the solution construction as well insignificantly extends range of application. The motivation for this investigation is the modeling of selection the suspicious observations in the experiments. Such observation can be treated as outliers or disturbed. The objectives is to detect such inaccuracy immediately or in very short time before or after it appearance with highest probability. The problem is reformulated to optimal stopping of the observed sequences. The detailed analysis of the problem is presented to show the form of optimal decision function. The application of the results to analysis of piecewise deterministic processes with change points appearing at the moment of jumps is shown (see [6], [10], [5]).**

***Index Terms*— Bayesian approach, disorder problem, sequential detection, optimal stopping, Markov process, change point**

AMS Subject Classification: ***Primary:* 60G40; *Secondary:* 60K99; 90D60**

I. INTRODUCTION

Suppose that the process $X = \{X_n, n \in \mathbb{N}\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, is observed sequentially. It is obtained from Markov processes by switching between them at a random moments θ_1 and θ_2 in such a way that the process after θ_1 starts from the state X_{θ_1-1} and after θ_2 starts from the state X_{θ_2-1}. It means that the state at moment $n \in \mathbb{N}$ has conditional distribution given the state at moment $n-1$, where the formulae describing these distributions have the different form: one for $n < \theta_1$, the second for $\theta_1 \le n < \theta_2$, and another for $n \ge \theta_2$. Our objective is to indicate the segment of given length between disorders with maximal probability based on observation of X. Such model of data appears in many practical problems of the quality control (see [12], [13] and in the collection of the papers [2]), traffic anomalies in networks (see [16]), epidemiology models (see [1]). The model considered here generalizes the basic problem stated in [14] (see also [4], [18], [15]).

The classical disorder problem is limited to the case of switching between sequences of independent random variables (see [4]). Some developments of the basic model can be found in [17] where the optimal detection rule of the switching moment has been obtained when the finite state-space Markov chains is disordered. [7] formulates conditions which help to reduce the problem of the quickest detection for dependent sequences before and after the change to the case for independent random variables. Our result admits Markovian dependence structure for switched sequences. We obtain an optimal rule under probability maximizing criterion.

Formulation of the problem can be found in Section II. The main result is presented in Section III.

II. FORMULATION OF THE PROBLEM

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space which supports sequence of observable random variables $\{X_n\}_{n\in\mathbb{N}}$ generating filtration $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$. Random variables X_n take values in $(\mathbb{E}, \mathcal{B})$, where $\mathbb{E}$ is a subset of $\mathbb{R}$. Space $(\Omega, \mathcal{F}, \mathbf{P})$ supports also unobservable random variables θ_1, θ_2 with values in $\mathbb{N}$ and the following distributions:

$$\begin{aligned}\mathbf{P}(\theta_1 = j) &= \mathbb{I}_{\{j=0\}}(j)\pi \\ &\quad +\mathbb{I}_{\{j>0\}}(j)\bar{\pi}p_1^{j-1}q_1, \end{aligned} \tag{II.1}$$

$$\begin{aligned}\mathbf{P}(\theta_2 = k \mid \theta_1 = j) &= \mathbb{I}_{\{k=j\}}(k)\rho \\ &\quad +\mathbb{I}_{\{k>j\}}(k)\bar{\rho}p_2^{k-j-1}q_2 \end{aligned} \tag{II.2}$$

where $j = 0, 1, 2, ...$, $k = j, j+1, j+2, ...$, $\bar{\pi} = 1-\pi$, $\bar{\rho} = 1-\rho$. Additionally we consider Markov processes $(X_n^i, \mathcal{G}_n^i, \mathbf{P}_x^i)$ on $(\Omega, \mathcal{F}, \mathbf{P})$, $i = 0, 1, 2$, where σ-fields $\mathcal{G}_n^i$ are the smallest σ-fields for which $(X_n^i)_{n=0}^{\infty}$, $i = 0, 1, 2$, are adapted, respectively. Let us define process $(X_n)_{n\in\mathbb{N}}$ in the following way:

$$\begin{aligned}X_n &= X_n^0\mathbb{I}_{\{\theta_1>n\}} + X_{n-\theta_1+1}^1\mathbb{I}_{\{X_0^1=x_{\theta_1-1}^0,\theta_1\le n<\theta_2\}} \\ &\quad +X_{n-\theta_2+1}^2\mathbb{I}_{\{X_0^2=x_{\theta_2-\theta_1}^1,\theta_2\le n\}}. \end{aligned} \tag{II.3}$$

We make inference on θ_1 and θ_2 from the observable sequence $(X_n,\, n \in \mathbb{N})$ only. Our aim is to stop the observed sequence between the two disorders.This can be interpreted as a strategy for protecting against a second failure when the

*The research has been supported by grant S20096/I-18

[1]Krzysztof J. Szajowsk is with Faculty of Fundamental Problems of Technology, Institute of Mathematics and Computer Science, Wrocław University of Technology, PL-50-370 Wrocław, Poland Krzysztof.Szajowski@pwr.wroc.pl

first has already happened. The mathematical model of this is to control the probability $\mathbf{P}_x(\tau<\infty, \theta_1+d_1\le\tau<\theta_2-d_2)$ by choosing the stopping time $\tau^*\in\mathcal{S}$ for which

$$\begin{aligned}&\mathbf{P}_x(\theta_1+d_1\le\tau^*<\theta_2-d_2) \qquad \text{(II.4)}\\ &=\sup_{\tau\in\mathcal{T}}\mathbf{P}_x(\tau<\infty,\theta_1+d_1\le\tau<\theta_2-d_2).\end{aligned}$$

III. SOLUTION OF THE PROBLEM

Let us denote for $n=0,1,2,\ldots$:

$$\begin{aligned}Z_n^{(d_1,d_2)} &= \mathbf{P}_x(\theta_1+d_1\le\tau<\theta_2-d_2\mid\mathcal{F}_n),\\ V_n^{(d_1,d_2)} &= \operatorname*{ess\,sup}_{\{\tau\in\mathfrak{S}^X,\ \tau\ge n\}}\mathbf{P}_x(\theta_1+d_1\le\tau<\theta_2-d_2\mid\mathcal{F}_n),\\ \tau_0 &= \inf\{n: Z_n^{(d_1,d_2)}=V_n^{(d_1,d_2)}\} \qquad \text{(III.1)}\end{aligned}$$

Notice that, if $Z_\infty^{(d_1,d_2)}=0$, then $Z_\tau^{(d_1,d_2)}=\mathbf{P}_x(\theta_1+d_1\le\tau<\theta_2-d_2\mid\mathcal{F}_\tau)$ for $\tau\in\mathfrak{S}^X$. Since $\mathcal{F}_n\subseteq\mathcal{F}_\tau$ (when $n\le\tau$) we have

$$\begin{aligned}V_n^{(d_1,d_2)} &= \operatorname*{ess\,sup}_{\tau\ge n}\mathbf{P}_x(-d_1\le\theta-\tau\le d_2\mid\mathcal{F}_n)\\ &= \operatorname*{ess\,sup}_{\tau\ge n}\mathbf{E}_x(Z_\tau^{(d_1,d_2)}\mid\mathcal{F}_n).\end{aligned}$$

The following lemma (see [4], [11]) ensures existence of the solution

Lemma III.2: The stopping time τ_0 defined by formula (III.1) is the solution of problem (II.4).

The formulated problems are translated to the optimal stopping problems for some Markov processes. The important part of the reformulation process is choice of the *statistics* describing knowledge of the decision maker. The *a posteriori* probabilities of some events play the crucial role. Let us define the following *a posteriori* processes (cf. [18], [14]).

$$\begin{aligned}\Pi_n^i &= \mathbf{P}_x(\theta_i\le n|\mathcal{F}_n), \qquad \text{(III.3)}\\ \Pi_n^{12} &= \mathbf{P}_x(\theta_1=\theta_2>n|\mathcal{F}_n) \qquad \text{(III.4)}\\ &= P_x(\theta_1=\theta_2>n|\mathcal{F}_{mn}),\\ \Pi_{mn} &= \mathbf{P}_x(\theta_1=m,\theta_2>n|\mathcal{F}_{mn}), \qquad \text{(III.5)}\end{aligned}$$

where $\mathcal{F}_{m\,n}=\mathcal{F}_n$ for $m,n=1,2,\ldots,\ m<n,\ i=1,2$. For recursive representation of (III.3)–(III.5) we need the following functions:

$$\begin{aligned}\Pi^1(x,y,\alpha,\beta,\gamma) &= 1-\frac{p_1(1-\alpha)f_x^0(y)}{\mathbf{H}(x,y,\alpha,\beta,\gamma)}\\ \Pi^2(x,y,\alpha,\beta,\gamma) &= \frac{(q_2\alpha+p_2\beta+q_1\gamma)f_x^2(y)}{\mathbf{H}(x,y,\alpha,\beta,\gamma)}\\ \Pi^{12}(x,y,\alpha,\beta,\gamma) &= \frac{p_1\gamma f_x^0(y)}{\mathbf{H}(x,y,\alpha,\beta,\gamma)}\\ \Pi(x,y,\alpha,\beta,\gamma,\delta) &= \frac{p_2\delta f_x^1(y)}{\mathbf{H}(x,y,\alpha,\beta,\gamma)}\end{aligned}$$

where $\mathbf{H}(x,y,\alpha,\beta,\gamma)=(1-\alpha)p_1f_x^0(y)+[p_2(\alpha-\beta)+q_1(1-\alpha-\gamma)]f_x^1(y)+[q_2\alpha+p_2\beta+q_1\gamma]f_x^2(y)$. In the sequel we adopt the following denotations

$$\begin{aligned}\vec{\alpha} &= (\alpha,\beta,\gamma) \qquad \text{(III.6)}\\ \overrightarrow{\Pi}_n &= (\Pi_n^1,\Pi_n^2,\Pi_n^{12}). \qquad \text{(III.7)}\end{aligned}$$

The basic formulae used in the transformation of the disorder problems to the stopping problems are given in the following

Lemma III.8: For each $x\in\mathbb{E}$ the following formulae, for $m,n=1,2,\ldots,\ m<n$, hold:

$$\begin{aligned}\Pi_{n+1}^1 &= \Pi^1(X_n,X_{n+1},\Pi_n^1,\Pi_n^2,\Pi_n^{12}) \qquad \text{(III.9)}\\ \Pi_{n+1}^2 &= \Pi^2(X_n,X_{n+1},\Pi_n^1,\Pi_n^2,\Pi_n^{12}) \qquad \text{(III.10)}\\ \Pi_{n+1}^{12} &= \Pi^{12}(X_n,X_{n+1},\Pi_n^1,\Pi_n^2,\Pi_n^{12}) \qquad \text{(III.11)}\\ \Pi_{m\,n+1} &= \Pi(X_n,X_{n+1},\Pi_n^1,\Pi_n^2,\Pi_n^{12},\Pi_{m\,n}) \qquad \text{(III.12)}\end{aligned}$$

with boundary condition $\Pi_0^1=\pi$, $\Pi_0^2(x)=\pi\rho$, $\Pi_0^{12}(x)=\bar{\pi}\rho$, and $\Pi_{m\,m}=(1-\rho)\frac{q_1f_{X_{m-1}}^1(X_m)}{p_1f_{X_{m-1}}^0(X_m)}(1-\Pi_m^1)$.

We have

$$\begin{aligned}Z_n^{(d_1,d_2)} &= \mathbf{P}_x(\theta_1+d_1\le\tau<\theta_2-d_2\mid\mathcal{F}_n) \qquad \text{(III.13)}\\ &= \Pi_{n-d_1}^1-\Pi_{n+d_2}^2\\ &= u(\underline{X}_{n-d_1-1,n},\Pi_n^1,\Pi_n^2,\Pi_n^{12}).\end{aligned}$$

By Lemma III.8 the conclusion is that the equivalent optimal stopping problem will be dependent on the segment of observations and the posterior distribution (see [8]). The exact solution allows to get the approximate form of optimal detector.

IV. APPLICATIONS

A natural appearance of two disorders can be observed for the renewal process. This is continuous time process having the representation by the sequence of the pairs of random variables. The stream of events is starting at t_0. The time between events are random and each event gives a random effect of some size. Rarely these streams of epochs and effects are homogeneous. It is also difficult to justify that homogeneity of both streams are broken at the same moments. Let us denote

- $\{X_{i,j}\}_{i\in\{0,1\},\ j=0}^{\infty}$ size of effects (two kinds);
- $N_i(t)$, $i\in\{0,1\}$, counting processes related to various stages;
- $\{T_{i,j}\}_{i\in\{0,1\},\ j=0}^{\infty}$ the moment of events related to the point processes;
- change points: θ_1 - for $S_{\cdot,j}$; θ_2 - for $X_{\cdot,j}$;
- $S_{i,j}=T_{i,j}-T_{i,j-1}$ period between successive events;
- $M_{00}(t)$, $M_{10}(t)$, $M_{01}(t)$, $M_{11}(t)$ are related renewal processes, where

$$M_{kr}(t)=\sum_{i=0}^{\tilde{N}_k(t)}\tilde{X}_{ri},\text{for } k,r\in\{1,2\}.$$

In the standard renewal process model, *i.e.* for fixed i, in the above sequences the random variables $X_{\cdot,j}$ are independent. Also the periods between successive events are independent. These assumptions simplify the definition of the related observed process (cf. (II.3)). We have the observed process and random sequences as follows:

- $\tilde{S}_j=S_{0,j}\mathbb{I}_{\{\theta_1\ge j\}}+S_{1,j-\theta_1}\mathbb{I}_{\{\theta_1<j\}}$ – the periods between successive events;
- $\tilde{N}(t)$ – the related counting process with the moments of events $\{T_n\}_{n=0}^{\infty}$;

– $\tilde{X}_j = X_{0,j}\mathbb{I}_{\{\theta_2 \geq j\}} + X_{1,j-\theta_2}\mathbb{I}_{\{\theta_2 < j\}}$ – the size of effects.

The observed renewal process $M(t)$ is represented by the sequence of pairs $\{(T_n, X_n)\}_{n=0}^{\infty}$. At epoch of events we have

$$\begin{aligned} M_n = M(T_n) &= M_{00}(T_n)\mathbb{I}_{\{\theta_1\wedge\theta_2 \geq n\}} \\ &\quad + M_{10}(T_{n-\theta_1})\mathbb{I}_{\{\theta_1=\theta_1\wedge\theta_2<n\leq\theta_1\vee\theta_2\}} \\ &\quad + M_{01}(T_{n-\theta_2})\mathbb{I}_{\{\theta_2=\theta_1\wedge\theta_2<n\leq\theta_1\vee\theta_2\}} \\ &\quad + M_{11}(T_{n-\theta_1\vee\theta_2})\mathbb{I}_{\{\theta_1\vee\theta_2<n\}}. \end{aligned}$$

For any fixed $d_1, d_2 \in \{0, 1, 2, \ldots\}$ (the problem $\mathfrak{D}^M_{d_1 d_2}$) we are looking for the stopping time $\tau^* \in \mathfrak{S}^M$ such that

$$\begin{aligned} &\mathbf{P}_x(-d_1 \leq \theta_1 \wedge \theta_2 - \tau^* \leq d_2) \qquad \text{(IV.1)} \\ = &\sup_{\tau\in\mathfrak{S}^M} \mathbf{P}_x(-d_1 \leq \theta_1 \wedge \theta_2 - \tau \leq d_2) \end{aligned}$$

where $\mathfrak{S}^M$ denotes the set of all stopping times with respect to the filtration $\{\mathcal{F}^M_n\}_{n\in\mathbb{N}}$. Using parameters d_i, $i = 1, 2$, we control the precision level of detection. The problem of the minimum of change points detection for two independent Poisson processes has been tackled by [3], some considerations concerning there same question for the renewal processes was discussed in [9]. The detection of the transition probability change in the case $d_1 = d_2 = d$, when $\pi = 0$ has been studied in [11].

The problem $\mathfrak{D}^M_{d_1 d_2}$ formulated in (IV.1) can be solved according the method described in sections II–III under additional assumption that the statistician knows which disorder is the first one: $\mathfrak{D}^M_{d_1 d_2}(\theta_1 \leq \theta_2)$ or $\mathfrak{D}^M_{d_1 d_2}(\theta_2 \leq \theta_1)$. The solution of $\mathfrak{D}^M_{d_1 d_2}$ is not provided here as well as the problem of disorder moments estimation.

V. FINAL REMARKS

When we admit that the sequence of observations without disorder is possible it is interesting question how to detect not only that we observe the third kind of data but that there were no data of the first and second kind. Lack of a signal disturbance in reasonable time means that such rare event appears. It should be verified by standard testing procedure the homogeneity of observation in such circumstances or when we stop very early with comparison to expected value of the disorder moments.

REFERENCES

[1] M. Baron, "Early detection of epidemics as a sequential change-point problem." in *Longevity, aging and degradation models in reliability, public health, medicine and biology, LAD 2004. Selected papers from the first French-Russian conference, St. Petersburg, Russia, June 7–9, 2004*, ser. IMS Lecture Notes-Monograph Series, V. Antonov, C. Huber, M. Nikulin, and V. Polischook, Eds., vol. 2. St. Petersburg, Russia: St. Petersburg State Politechnical University, 2004, pp. 31–43.

[2] M. Basseville and A. Benveniste, Eds., *Detection of abrupt changes in signals and dynamical systems*, ser. Lecture Notes in Control and Information Sciences. Berlin: Springer-Verlag, 1986, vol. 77, p. 373.

[3] E. Bayraktar and H. Poor, "Quickest detection of a minimum of two Poisson disorder times." *SIAM J. Control Optim.*, vol. 46, no. 1, pp. 308–331, 2007.

[4] T. Bojdecki, Probability maximizing approach to optimal stopping and its application to a disorder problem, *Stochastics*, vol. 3, pp. 61–71, 1979.

[5] E. Z. Ferenstein and A. Pasternak-Winiarski, Optimal stopping of a risk process with disruption and interest rates. in *Advances in dynamic games. Theory, Applications, and Numerical Methods for Differential and Stochastic Games. Dedicated to the memory of Arik A. Melikyan. Selected papers presented at the 13th International Symposium on Dynamic Games and Applications, Wrocław, Poland, Summer 2008.*, M. Breton and K. Szajowski, Ed. Boston, MA: Birkhäuser, 2011, pp. 489–507.

[6] T. Herberts and U. Jensen. Optimal detection of a change point in a Poisson process for different observation schemes. *Scand. J. Statist.* 31 (2004), no. 3, 347–366.

[7] G. Moustakides, Quickest detection of abrupt changes for a class of random processes. *IEEE Trans. Inf. Theory*, vol. 44, no. 5, pp. 1965–1968, 1998.

[8] A. Ochman-Gozdek and K. Szajowski. Detection of a random sequence of disorders. Preprint I-18/2013, Instytut Matematyki i Informatyki, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, August 2013. 59^{th} *ISI World Statistics Congress 25–30 August 2013, Hong Kong Special Administrative Region, China*

[9] A. Pasternak-Winiarski. *Zagadnienia zmian rozkładów a problem optymalnego stopowania w modelowaniu procesów stochastycznych w ubezpieczeniach*, Ph.D. dissertation, Politechnika Warszawska, Wydział Matematyki i Nauk Informacyjnych, Warszawa, 2013.

[10] H.V. Poor and O. Hadjiliadis. Quickest detection. Cambridge University Press, Cambridge, 2009. xiv+229 pp. ISBN: 978-0-521-62104-5

[11] W. Sarnowski and K. Szajowski. Optimal detection of transition probability change in random sequence. *Stochastics*, 83(4-6):569–581, 2011.

[12] W. Shewhart, *Economic control of quality of manufactured products*. Yew York: D. Van Nostrand, 1931.

[13] A. Shiryaev, "The detection of spontaneous effects," *Sov. Math, Dokl.*, vol. 2, pp. 740–743, 1961, translation from Dokl. Akad. Nauk SSSR 138, 799-801 (1961).

[14] K. Szajowski, "Optimal on-line detection of outside observation," *J.Stat. Planning and Inference*, vol. 30, pp. 413–426, 1992.

[15] K. Szajowski. On a random number of disorders. *Probab. Math. Stat.*, 31(1):17–45, 2011.

[16] A. G. Tartakovsky, B. L. Rozovskii, R. B. Blažek, and H. Kim, "Detection of intrusions in information systems by sequential change-point methods," *Stat. Methodol.*, vol. 3, no. 3, pp. 252–293, 2006.

[17] B. Yakir, "Optimal detection of a change in distribution when the observations form a Markov chain with a finite state space." in *Change-point Problems. Papers from the AMS-IMS-SIAM Summer Research Conference held at Mt. Holyoke College, South Hadley, MA, USA, July 11–16, 1992*, ser. IMS Lecture Notes-Monograph Series, E. Carlstein, H.-G. Müller, and D. Siegmund, Eds., vol. 23. Hayward, California: Institute of Mathematical Statistics, 1994, pp. 346–358.

[18] M. Yoshida, "Probability maximizing approach for a quickest detection problem with complicated Markov chain," *J. Inform. Optimization Sci.*, vol. 4, pp. 127–145, 1983.