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The Mathematics of Team Control. Extended Abstract

Alexander B. Kurzhanski, Aleksei I. Mesyats

Short abstract *In this paper we propose a rigorous mathematical treatment of control problems for teams of controlled motions. Each member of the team is described by second-order dynamics. The proposed scheme is directed at solving problems of output feedback target control. For linear-convex systems such problems may be solved completely — from theory to computation. In the nonlinear case this may be reached for fairly broad classes of systems. The solution methods rely on nonlinear analysis, set-valued calculus and Hamiltonian techniques for dynamic processes, including their matrix version applicable to large dimensions.*

I. Introduction

Among problems that are at the focus of present research in mathematics of control are those of designing feedback control strategies for teams of controlled motions. The analysis of coordinated motions for flocks or formations of dynamic agents have been treated within various aspects — collision avoidance, flocking, stability of formation structures, reconfiguration of formations, heterogeneous structures, information coordination, hybrid dynamics etc [1], [2], [3], [4]. All these settings, arising from numerous applied motivations, were mostly treated through investigation of different mathematical models motivated by specific subtasks of more general problems. These were very important contributions explaining the nature and various properties of the new classes of research topics. But at the same time the investigations had not yet reached the level of rigorous mathematical description embracing the whole problem of team control, whose subproblems would be grouped around one ultimate goal — the design of coordinated target-oriented feedback control strategies for the whole team. This is especially true for problems of joint obstacle avoidance. A unified joint treatment of the problem and its subproblems of course requires new approaches and techniques that lie beyond conventional means. This also requires an accurate combination of different mathematical and algorithmic tools within one solution scheme.

Such task also includes related problems of output feedback team control under obstacles, information delays and uncertain input disturbances. This paper covers topics from basic theoretical problems and coordinated subproblems to recommended computational methods. It also indicates routes for dynamic optimization of the proposed control solutions. The achieved scheme and results could then serve as a road map for tackling specific applications, solving them to the end.

The suggested solution schemes rely on combining variational methods of nonlinear analysis and analytical mechanics, including Hamiltonian techniques, with those of set-valued calculus and minimax approaches [9], [10], [23], as well as on progressive modification of earlier developed computational ellipsoidal and polyhedral tools, comparison principles and related procedures [15], [14], [7].

Here we have to deal with the problem of synthesizing closed-loop controls for a group of m controlled systems that are to move together while jointly being a team. We hence begin with the definition of this notion. A group of systems is defined to be a team if throughout the motion its members should be not too close, avoiding collisions, while also not to far from each other, remaining contained within a preassigned moving set of restricted volume — the *container of the team*, an ellipsoid. The three-dimensional motion $x_j(t),\ j = 1,\dots,m$, of each team-mate is governed by second order dynamics with $x_j(t)$ lying at the center of a ball of given constant safety radius. Collisions would then be avoided once the system controls would ensure that the interiors of such balls would never intersect with each other.

The aims of the control would be to steer the container towards a given target set, reaching it within a given interval of time, while avoiding obstacles defined by given static or moving forbidden domains. (These obstacles may also have to be identified on-line, through available measurements). The prescribed limits on the size of the container should be preserved along the motion, however the container itself is allowed to be reconfigured, so as to be able to be squeezed in between the obstacles without changing its size beyond the limits, while also maintaining the property of being a team — the team property.

II. Problem formulation

The team of m members is described within the following framework. Consider the equation of motion with second order dynamics on the given time interval $[t_0, \theta]$

$$\ddot{x}_j(t) = A^j(t)x_j(t) + C^j(t)\dot{x}_j(t) + B^j(t)u_j(t), \qquad (1)$$

$$x_j(t_0) = x_j^0,\ \dot{x}_j(t_0) = \dot{x}_j^0,\ j = 1,\dots,m. \qquad (2)$$

Here $x_j(t) \in \mathbb{R}^n$ is state vector of j-th team member, $u_j(t)$ is control, restricted by geometric bounds $u_j(t) \in \mathscr{P} \subseteq \mathbb{R}^p$ that are assumed compact and convex. Put $x(t) = \{x_1(t),\ \dot{x}_1(t),\ x_2(t), \dot{x}_2(t), \dots, x_m(t), \dot{x}_m(t)\} \in \mathbb{R}^{2nm}$.

The authors are with Moscow State (Lomonosov) University, Faculty of Computational Mathematics & Cybernetics (VMK); GSP-2, Moscow 119992, Russia kurzhans@mail.ru, month_october@mail.ru.

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The dynamics of the container are described with *matrix-valued* system for its shape matrix $Q(t)$,

$$\dot{Q}(t) = T(t)Q(t) + Q(t)T'(t) + B(t)U(t,Q)B'(t), \quad (3)$$

with $Q(t_0) = Q_0 \in \mathbb{R}^{n\times n}$, and with vector system

$$\ddot{q}(t) = v(t),\ q(t_0) = q_0,\ \|v(t)\| \leqslant \mu,\ \mu > 0.$$

for center $q(t)$. Here $v(t) \in \mathbb{R}^n$ and $U \in \mathbb{R}^{m\times m}$ are the control. By $\mathscr{E}(y,Y)$ we will further denote an ellipsoid in $\mathbb{R}^n$,

$$\mathscr{E}(y,Y) = \{x \in \mathbb{R}^n :\ \langle x-y, Y^{-1}(x-y)\rangle \leqslant 1\}.$$

It should be noted that $q(t),\ Q(t)$ are assumed to be controllable and describe a *virtual motion*, whereas $x_j(t)$ (the member of the team trajectories) are *physically realizable*.

We now give a rigorous definition of the team property [13].

Definition 1: The solution $x(t)$ to equation (1) is said to be a *team motion* within time interval $t \in [\tau,\theta]$ if the following constraints are fulfilled:

$$0 < r \leqslant d(x^i(t), x^j(t)), \quad \text{for all } 1 \leqslant i < j \leqslant m, \quad (4)$$

$$x^i(t) \in \mathscr{E}(q(t),Q(t)), \ \text{for all } i = 1,2,\ldots,m. \quad (5)$$

Here $d(\cdot,\cdot)$ stands for Euclidean distance. Requirement (4) ensures collision avoidance of the team members and can be written as $\mathscr{B}_r(x_i(t)) \cap \mathscr{B}_r(x_j(t)) = \varnothing$, where r is a safety radius.

We separate the problem of container control in two: controlling the center, $q(t)$, and controlling the shape matrix $Q(t)$. The outline of the solution can be written then as follows:

1) Design a trajectory $q(t)$ for the container and determine the possible reconfigurations of its shape in order to avoid external obstacles;
2) Solve an optimization problem for the containers shape matrix $Q(t)$;
3) Describe the coordinated controls for the motions of the team within the moving container.

One of the basic tasks will thus be to ensure the team property which depends on the online shape of the container. But the latter will depend on the ultimate goal of the control, which is for the team to reach the target set. The last result would be ensured depending on the ellipsoidal-valued trajectory $\mathscr{E}(q(t),Q(t))$ of the container which will thus depend on the target set and the external obstacles $\mathscr{E}(z_i,Z_i), i = 1,\ldots,N$, to be avoided. The solution will hence be based on designing a virtual ellipsoidal-valued reference motion $\mathscr{E}(q(t),Q(t))$ of the container that defines the external bound on the team, and is being steered to the target set while changing, if necessary, its configuration (at first to satisfy the obstacle constraints, then to fit into the required guaranteed neighborhood of target set). Then we design the realistic closed-loop controls for the team members so that they would keep the team within the moving container while also ensuring collision avoidance.

The solution goals are therefore to present a unified vision and a novel solution approach to the considered problem — a rigorous mathematical model for the problem of designing synchronized goal-oriented team control solutions under conditions of collision and obstacle avoidance, the theoretical analysis and solution of arising subproblems and overall problem in team control and communication, the recommendation of effective related calculation tools.

We further discuss the subproblems of the general problem with emphasis on the design of the virtual motion of the container.

III. Ellipsoidal motion

We begin with the feedback control problem for ellipsoidal container, $\mathscr{E}(q(t),Q(t))$.

Assume the matrix parameters $T \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ to be continuously differentiable. The problem is to find a closed-loop control $U(t,Q)$ that minimizes the functional

$$\Psi(U(\cdot,\cdot)) = \int_{t_0}^{\theta} \langle U(t,Q(t)), U(t,Q(t))\rangle\, dt + \\ + \langle Q(\theta) - M, D(Q(\theta) - M)\rangle \quad (6)$$

over trajectories of system (3). Here $D = D' > 0$. The inner product is treated here as $\langle A,B\rangle = \mathrm{tr} AB'$, inducing a Frobenius matrix norm in $\mathbb{R}^{n\times n}$.

Problem (3),(6) is also considered under the following additional state constraints:

$$\lambda_-^2 I \leqslant Q'(t)Q(t) \leqslant \lambda_+^2 I,\ 0 < \lambda_- < \lambda_+. \quad (7)$$

Here λ_- and λ_+ are known constants, the prime stands for the transpose, and I is the unit matrix. This inequality must hold for all $t \in [t_0,\theta]$. These inequalities restrict possible size of an ellipsoid with configuration matrix Q thus allowing the container to move between the external obstacles while holding the team inside.

To solve this problem, we will use *representations* of matrix operators, i.e. type $(2,2)$ tensors [21], using approach from tensor analysis. Let $\{E^{ij}\}_{i,j=1}^n$, $E^{ij} \in \mathbb{R}^{n\times n}$, be an arbitrary basis in $\mathbb{R}^{n\times n}$. For an arbitrary linear operator over matrix spaces $\mathscr{A} \in \mathscr{L}(\mathbb{R}^{n\times n}, \mathbb{R}^{m\times m})$ its action on an arbitrary $X \in \mathbb{R}^{n\times n} = \sum_{i,j=1}^n X_{ij}E^{ij}$ can be expressed as $\mathscr{A}X = = \mathscr{A}\sum_{i,j=1}^n X_{ij}E^{ij} = \sum_{i,j=1}^n X_{ij}\mathscr{A}E^{ij}$.

Definition 2: A set of matrices $A = \{A^{ij} = \mathscr{A}E^{ij}\}_{i,j=1}^n \in \in \mathbb{R}^{n\times n} \times \mathbb{R}^{m\times m}$ is called a *representation* of operator $\mathscr{A}$.

It can be directly established that a superposition of $\mathscr{A} \in \mathscr{L}\left(\mathbb{R}^{m\times m}, \mathbb{R}^{k\times k}\right)$, $\mathscr{A} = \{A^{ij}\}_{i,j=1}^m$, and $\mathscr{B} \in \mathscr{L}(\mathbb{R}^{n\times n}, \mathbb{R}^{m\times m})$, $\mathscr{B} = \{B^{ij}\}_{i,j=1}^n$, will have the following representation:

$$AB = C,\ C^{ij} = \sum_{k,l=1}^m A^{kl}B^{ij}_{kl},\ i,j = 1,\ldots,n,$$

and the adjoint operator $\mathscr{A}^*$ will have representation $\tilde{A} = \{\tilde{A}^{ij}\}_{i,j=1}^m$, where $\tilde{A}^{kl}_{ij} = A^{ij}_{kl}$.

Let us consider the original problem in the absence of the state constraints (7) first. We will obtain an operator form of the problem using operator representations, solve in the problems in terms of operators, and then return to matrices using formulas for superposition and adjoint operator.

Introduce two operators, $\mathscr{A}(t)X = T(t)X + XT'(t)$, $\mathscr{B}(t)X = B(t)XB'(t)$ with representations

$$A_{ab}^{ij} = \left(\mathscr{A}E^{ij}\right)_{ab} = \begin{cases} T_{ii}+T_{jj}, & a=i \text{ and } b=j, \\ T_{bj}, & a=i \text{ and } b\neq j, \\ T_{ai}, & a\neq i \text{ and } b=j, \\ 0, & a\neq i \text{ and } b\neq j, \end{cases} \quad (8)$$

$$B_{ab}^{ij} = \left(\mathscr{B}E^{ij}\right)_{ab} = B_{ai}B_{bj}. \quad (9)$$

We now obtain on operator form of the problem:

$$\dot{Q} = \mathscr{A}Q + \mathscr{B}U,\ Q(t_0) = Q_0,$$

$$\int_{t_0}^{\theta} \langle U,U\rangle\, dt + \langle Q(\theta)-M, D(Q(\theta)-M)\rangle \to \min.$$

We will carry on with the solution using dynamic programming. Define *value function* as $V(t_0,Q_0) = = \min_{U(\cdot,\cdot)}\{\Psi(U(\cdot,\cdot)) \mid Q(t_0)=Q_0\}$. The corresponding Hamilton–Jacobi–Bellman (HJB) equation [9] for V is as follows:

$$\frac{\partial V}{\partial t} + \min_U \left\{ \left\langle \frac{\partial V}{\partial Q}, \mathscr{A}Q + \mathscr{B}U \right\rangle + \langle U,U\rangle \right\} = 0, \quad (10)$$
$$V(\theta,Q) = \langle Q-M, D(Q-M)\rangle.$$

The value function can found as quadratic form, $V(t,Q) = \langle Q, \mathscr{P}(t)Q\rangle + \langle Q,K(t)\rangle + \gamma(t) = \sum_{i,j,k,l=1}^n Q_{ij}Q_{kl}P_{ij}^{kl}(t) + \langle Q,K(t)\rangle + \gamma(t)$, the parameters of which are described by the following set of differential equations in backward time:

$$\dot{P}^{ij} + \sum_{k,l=1}^n \left(P^{kl}A_{kl}^{ij} + \tilde{A}^{kl}P_{kl}^{ij}\right) +$$
$$+ \sum_{k,l,p,q=1}^n P^{kl}(BB')_{kp}(BB')_{lq}P_{pq}^{ij} = 0, \quad (11)$$
$$P^{ij}(\theta) = DE^{ij},\ i,j=1,\ldots,n,$$

$$\dot{K} + \sum_{i,j=1}^n K_{ij}\tilde{A}^{ij} + \sum_{k,l=1}^n P^{kl}(BB'KBB')_{kl} = 0, \quad (12)$$
$$K(\theta) = -2DM,$$

$$\dot{\gamma} - \frac{1}{4}\left\langle K, BB'KBB'\right\rangle = 0,\ \gamma(\theta) = \langle M, DM\rangle. \quad (13)$$

The optimal control will be given by $U = = -B'\left(\sum_{k,l=1}^n P^{kl}Q_{kl} + \frac{1}{2}K\right)B$.

A. Internal and external constraints

In presence of state constraints, a modified value function is defined using penalty methods. Define

$$W_{\alpha,\beta}(t,Q) = \alpha(t)(\langle Q,Q\rangle - \lambda_+^2)_+ + \beta(t)(\lambda_-^2 - \langle Q,Q\rangle)_+,$$

$$V_\gamma(t_0,Q_0) = \min_{U(\cdot)} \max_{\alpha(\cdot),\beta(\cdot)} \int_{t_0}^{\theta} \Big[\langle U(t,Q(t)), U(t,Q(t))\rangle +$$
$$+ W_{\alpha,\beta}(t,Q(t)) \Big]\, dt + \langle Q(\theta)-M, D(Q(\theta)-M)\rangle,$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ are nonnegative scalar functions, $\alpha(t)+\beta(t)=L$ for all $t\in[t_0,\theta]$. Here $(a)_+ = \max\{a,0\},\ a\in\mathbb{R}$. We can interchange maximum and minimum [22] and come to a series of HJB equations,

$$\frac{\partial V_{\alpha,\beta}}{\partial t} + \left\langle \frac{\partial V_{\alpha,\beta}}{\partial Q}, \mathscr{A}Q \right\rangle - \frac{1}{4}\left\| \mathscr{B}^* \frac{\partial V_{\alpha,\beta}}{\partial Q} \right\|^2 + W_{\alpha,\beta} = 0,$$

with the same terminal condition. We will seek the solution as piecewise–quadratic form [16], $V_{\alpha,\beta}(t,Q) = \langle Q, \mathscr{P}_i(t)Q\rangle + \langle Q, K_i(t)\rangle + \gamma_i(t),\ \ Q \in L_i$, where $L_i, i = 1,\ldots,4$, are regions of $\mathbb{R}^{n\times n}$ defined by combinations of signs of differences in $W_{\alpha,\beta}(t,Q)$. The original value function is given by $V_L(t_0,Q_0) = \max_{\alpha(\cdot),\beta(\cdot)} V_{\alpha,\beta}(t_0,Q_0)$. The solution will be obtained as $L\to+\infty$ being achieved at some finite L [20].

B. Obstacle problems

Given external ellipsoidal obstacles $\mathscr{E}_i = \mathscr{E}(z_i,Z_i),\ i = 1,2,\ldots,N$, we can formulate the obstacle avoidance problem: design the trajectory tube $\mathscr{E}(q(t),Q(t))$ such that $d(\mathscr{E}(q(t),Q(t)),\mathscr{E}_i) \geqslant 0,\ i=1,2,\ldots,N$. Here $d(\cdot,\cdot)$ is the Euclid distance between sets. This problem may be solved in both feedforward and feedback modes using generalizations of techniques indicated in [8], [17], [13]. Another approach lies in an advance design of a virtual reference motion which is to be tracked on-line.

We now will discuss the construction of a reference motion taking, for brevity, $N=2$. We assume that $d(\mathscr{E}_1,\mathscr{E}_2) = \delta > 2r$ to ensure the existence of an obstacle avoiding solution. Using standard methods of convex calculus, one can write

$$\delta = \max\{\langle l, z_1-z_2\rangle - \sqrt{\langle l, Z_1 l\rangle} - \sqrt{\langle l, Z_2 l\rangle} \mid \langle l,l\rangle \leqslant 1\}.$$

Let l^0 be the unique optimizer. Then $\delta = \left\|d^1-d^2\right\|$, where $\left\langle -l^0, d^1\right\rangle = \left\langle l^0, Z_1 l^0\right\rangle^{1/2}$, $\left\langle l^0, d^2\right\rangle = \left\langle l^0, Z_2 l^0\right\rangle^{1/2}$. Denote vectors $d^0 = d^1 - d^2$, $d^* = d^2 + d^0/2$. We now can introduce hyperplanes $\mathscr{H}_z = \{x:\ \left\langle x-d^*, d^0\right\rangle = 0\}$, $\mathscr{H}_1 = \{x:\ \left\langle x-d^1, d^0\right\rangle = 0\}$, $\mathscr{H}_2 = \{x:\ \left\langle x-d^2, d^0\right\rangle = 0\}$.These hyperplanes are parallel and orthogonal to d^0, with $\mathscr{H}_1$ and $\mathscr{H}_2$ being the support hyperplanes for $\mathscr{E}_1$ and $\mathscr{E}_2$, respectively, with $\mathscr{H}_z$ in the middle. Hence we shall demand that the reference curve $q(t)$ that leads from the starting position $q(t_0)$ to the target set should lie when passing the obstacles, exactly on the plane $\mathscr{H}_z$, setting that the semiaxes of $\mathscr{E}(q(t),Q(t))$ should be no greater than $\delta/2$. The time interval when this condition is to be observed may be determined by introducing *barrier hyperplanes*: $\mathscr{H}_b^- = \{q : \langle q,h_b\rangle = c_b^-\}, \mathscr{H}_b^+ = \{q : \langle q,h_b\rangle = c_b^+\},\ c_b^- < c_b^+$, which are such that beyond them, i.e., when $\langle q,h_b\rangle \geqslant c_b^-$, $\langle q,h\rangle \leqslant c_b^+$, we would have $d(\mathscr{E}(q(t),Q(t)),\mathscr{E}_i) > 0,\ i=1,2$ and the obstacle constraints would be irrelevant (passive). Take $\mathscr{H}_{bz}^- = \mathscr{H}_b^- \cap \mathscr{H}_z,\ \ \mathscr{H}_{bz}^+ = \mathscr{H}_b^+ \cap \mathscr{H}_z$. Now we can require that the reference curve $q(t)$ to satisfy the following conditions:

$$q(t_0) = q_0,\ q(t_s^-) \in \mathscr{H}_{zb}^-,\ q(t_s^+) \in \mathscr{H}_{zb}^+, \quad (14)$$
$$q(t) \in \mathscr{H}_z,\ t\in[t_s^-, t_s^+].$$

Time instants t_s^-, t_s^+ are to be calculated within the course of solution, and $q(t_s^-)$, $q(t_s^+)$ may be selected at the final stage, minimizing, for example, the length of the total route $q(t)$. Since the bound on the control μ is at our disposal, it can always be taken such that (14) would be satisfied. Calculating corresponding control $v(\cdot)$ together with values t_s^-, t_s^+ is a standard problem of control theory. The final time θ may also be regulated in order to match the related time intervals for the real motion $x(t)$.

IV. Team control

We will now indicate the sketch of some approaches to the treatment of collision avoidance for team members. The requirements $\mathscr{B}_r(x_i(t)) \cap \mathscr{B}_r(x_j(t)) = \varnothing$ define a *non-convex* bound on the trajectories $x^j(t)$, ensuring that they belong to the union of complements of convex sets. Using duality methods of nonlinear analysis, one can write

$$d(x_i(t), x_j(t)) = \max\left\{ \left\langle \lambda^{(ij)}(t), x_j(t) - x_i(t) \right\rangle - \right.$$
$$\left. - 2r \left\langle \lambda^{(ij)}(t), \lambda^{(ij)}(t) \right\rangle \right\},\ i < j,$$

where the multiplier $\lambda^{(ij)(t)} \in \mathbb{R}^n$ may be confined to the set Ω of Lipschitz-continuous functions. Therefore the total constraint can be expressed as

$$\exists \lambda_{ij}(t) \in \Omega :\ \left\langle \lambda^{(ij)}(t), x_j(t) - x_i(t) \right\rangle -$$
$$- 2r \left\langle \lambda^{(ij)}(t), \lambda^{(ij)}(t) \right\rangle > 0,\ \forall i \neq j,\ \forall t \in [t_0, \theta].$$

This condition may be rewritten in integral form [8],

$$\exists \lambda_{ij}(\cdot) \in \Omega :\ \psi_-(a, b, \alpha_{ij}(\cdot), \lambda^{(ij)}(\cdot)) \geqslant 0,\ \forall \alpha_{ij}(\cdot) \in \mathbf{V}_+[a,b],$$

where $\mathbf{V}_+[a,b]$ is the set of all scalar nonnegative nondecreasing functions with unit variation on $[a.b]$, and

$$\psi_-(a, b, \alpha_{ij}(\cdot), \lambda^{(ij)}(\cdot)) = \int_a^b \left(\left\langle \lambda^{(ij)}(t), x_i(t) - x_j(t) \right\rangle - \right.$$
$$\left. - 2r \left\langle \lambda^{(ij)}(t), \lambda^{(ij)(t)} \right\rangle \right)^{1/2} d\alpha_{ij}(t)$$

The container constraint is a convex state constraint and can be treated in a similar manner using result from [13], [8].

A more detailed description of the team control within the container is reserved for versions longer that this Extended Abstract.

V. Conclusion

This extended abstract emphasizes an array of fairly new problems related to coordinated controlled team dynamics oriented on achieving an ultimate goal while jointly moving under external and internal state constraints (obstacles). The mathematical solutions to these are formulated in terms of matrix-valued HJB equations and set-valued calculus. Indicating respective formulas in detail is a separate important issue. But the necessary relations here require to consider controlled ODE's of dimensions substantially larger than in conventional problems, namely, $6 \times m$ ODE's for a team of m 3D Newtonian motions, having further to deal with matrices of dimensions $36m^2 \times 36m^2$. The treatment of obstacles makes these numbers multiplied by yet more large numbers. It is therefore quite natural that team problems may not be fit for using regularly available computational tools. Hence the computational challenges which follow have to rely on distributed computation based on parallelization of solutions. Available results encourage such further developments.

References

[1] R. Olfati-Saber, "Flocking for multi-agent dynamic systems : algorithms and theory," *IEEE Trans. Automatic Control*, vol.51, No. 3, 2006, pp. 401–420.
[2] K.Somasundaram, J.S.Baras, "Evolutionary dynamics of collaborative environments with heterogeneous agents," *Proceedings of the Mediterranean Conference on Control and Automation (MED 09)*, Thessaloniki, Greece, 2009, pp.121–125.
[3] V.Gazi and K.M. Passino, "Stability analysis of social foraging swarms," *IEEE Transactions on Systems, Man, and Cybernetics*, vol.34, No 1, 2004, pp. 539–557.
[4] O.Junge, S.Ober-Bloebaum, "Optimal reconfiguration of formation flying satellites," *IEEE Conference on Decision and Control and European Control Conference ECC*, Seville, Spain, 2005, pp.66–71.
[5] N. N. Krasovskii and A. I. Subbotin, *Game-Theoretical Control Problems*, Springer, New York, 1988.
[6] A. B. Kurzhanski and T. F. Filippova, *On the theory of trajectory tubes: a mathematical formalism for uncertain dynamics, viability and control*, Advances in Nonlinear Dynamics and Control, Ser. PSCT 17, Birkhäuser, Boston, 122–188, 1993.
[7] A. B. Kurzhanski and I. Vályi, *Ellipsoidal Calculus for Estimation and Control*, Birkhäuser, Boston, 1997.
[8] A. B.Kurzhanski, I. Mitchell, and P. Varaiya, "Optimization problems for state-constrained control and obstacle problems," *Journal of Optimization Theory and Applications*, vol.128, No 3, 2006, pp. 499–521.
[9] A. B. Kurzhanski and P. Varaiya, "Optimization of Output Feedback Control Under Set-Membership Uncertainty," *Journal of Optimization Theory and Applications*, vol. 151, pp. 11–32, 2011.
[10] J.-P. Aubin, *Viability Theory*, Birkhäuser, Boston, 1991.
[11] M. Milanese and G. Belforte, "Estimation theory and uncertainty intervals evaluation in presence of unknown but bounded errors: linear families of models and estimations", *IEEE Transactions on Automatic Control*, vol. 27, pp. 408–414, 1982.
[12] F. C. Schweppe, *Uncertain Dynamic Systems*, Prentice Hall, 1973.
[13] A. B. Kurzhanski and P. Varaiya, "On synthesizing target controls under obstacle and collision avoidance," *Journal of Franklin Institute*, vol. 347, no. 1, pp. 130–145, 2010.
[14] F. L. Chernousko, *State Estimation for Dynamic Systems*, CRC Press, 1994.
[15] A. B Kurzhanski and P. Varaiya, "Ellipsoidal Techniques for Reachability Analysis. Part I: External Approximations. Part II: Internal Approximations. Box-valued Constraints," *Optimization methods and software*, vol. 17, no. 2, pp. 177–237, 2002,
[16] A. B. Kurzhanski and A. I. Mesyats, "Optimal control of ellipsoidal motions," *Differential Equations*, vol. 48, no. 11, pp. 1502–1509, 2012.
[17] I. V. Chebunin, "Controllability Conditions for the Riccati Equation," *Differential Equations*, vol. 39, no. 12, pp. 1741–1749, 2003.
[18] René K. Boel and Jan H. van Schuppen, "Control of the observation matrix for control purposes," in: *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems*, Budapest, Hungary, 2010, pp. 1261–1268.
[19] A. N. Daryin and A. B. Kurzhanski, "Estimation of Reachability Sets for Large-Scale Uncertain Systems: from Theory to Computation," in: *Proceedings of 51st IEEE Conference on Decision and Control*, Maui, Hawaii, 2012, pp. 7401–7406.
[20] V. V. Vasin and I. I. Eremin, *Operators and Iterative Processes of Fejer Type*, Gruyter, Berlin, 2009.
[21] J. L. Synge and A. Schild, *Tensor Calculus*, Dover Publications, New York, 1978.
[22] K. Fan, "Minimax theorems," *Proc. Nat. Acad. Sci.*, vol. 39, no 1, pp. 42–47 (1953).
[23] A. I. Subbotin, *Generalized Solutions of First-Order PDE's. The Dynamic Optimization Perspective*, SCFA, Boston, 1995.
[24] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York, 1993.