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Model reduction of nonlinear systems in the Loewner framework

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The motivation for model order reduction (MOR) stems from an inevitable fact about modern computing. The need for accurate modeling of physical phenomena often leads to large-scale dynamical systems that require long simulation times and large data storage. For instance, one such example is provided by the discretization of partial differential equations over fine grids, which leads to large-scale systems of ordinary differential equations. In these settings, MOR seeks models of low dimension that accurately capture the input-output behavior of the large-scale system while requiring only a fraction of the large-scale simulation time and storage.

A powerful and versatile approach to MOR is provided by the *Loewner framework* for rational interpolation. This approach was introduced in [3] and a major advance was made in [6]. It has since been successfully applied to two main categories of systems: (a) linear systems with multiple inputs and multiple outputs [6], [5], and (b) linear parametric systems [4], [1], [2]. It is the purpose of this talk to present the most recent extension of the Loewner framework to classes of non-linear systems, namely bilinear systems and quadratic non-linear systems.

The Loewner framework is *data-driven* and starts from empirical data. It employs advanced interpolation techniques, overcoming limitations of standard projection methods The empirical data may be provided by physical experimentation or by direct numerical simulation.

• **A brief overview of the Loewner framework**, as it applies to the problem of (tangential) rational interpolation and rational approximation [3], [6]. Consider data consisting of the *right interpolation data* $\{(\lambda_i, \mathbf{r}_i, \mathbf{w}_i) \mid \lambda_i \in \mathbb{C}, \mathbf{r}_i \in \mathbb{C}^{m\times 1}, \mathbf{w}_i \in \mathbb{C}^{p\times 1}, i = 1, \ldots, k\}$, and of the *left interpolation data* $\{(\mu_j, \ell_j, \mathbf{v}_j) \mid \mu_j \in \mathbb{C}, \ell_j \in \mathbb{C}^{1\times p}, \mathbf{v}_j \in \mathbb{C}^{1\times m}, j = 1, \ldots, q\}$. The quantities λ_i, μ_j, are points where the underlying function is evaluated, $\mathbf{r}_i$, ℓ_j are referred to as tangential directions on the right and on the left, while $\mathbf{w}_i$, $\mathbf{v}_j$ are right and left tangential values.

The rational interpolation problem aims at finding a rational $p \times m$ matrix function $\mathbf{H}(s)$, expressed in terms of a (descriptor) realization $[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}]$, i.e. $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$, such that the *right, left constraints* are satisfied: $\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i$, $\ell_j\mathbf{H}(\mu_j) = \mathbf{v}_j$. The connection of this problem with model reduction lies in the fact that the rational matrix function $\mathbf{H}(s)$, can be considered as the transfer function of the underlying to-be-reduced linear dynamical system.

The key tool for studying this problem is the $q\times k$ *Loewner matrix*, together with the $q \times k$ *shifted Loewner matrix*, associated with the empirical data:

$$(\mathbb{L})_{ij} = \frac{\mathbf{v}_i\mathbf{r}_j - \ell_i\mathbf{w}_j}{\mu_i - \lambda_j},$$

$$(\mathbb{L}_\sigma)_{ij} = \frac{\mu_i\mathbf{v}_i\mathbf{r}_j - \ell_i\mathbf{w}_j\lambda_j}{\mu_i - \lambda_j}.$$

We also define the quantities $\mathbf{W} = [\mathbf{w}_1 \ \cdots \ \mathbf{w}_k] \in \mathbb{C}^{p\times k}$, and $\mathbf{V} = \left[\mathbf{v}_1^* \ \cdots \ \mathbf{v}_q^*\right]^* \in \mathbb{C}^{q\times m}$.

The *solution* to the general tangential interpolation/approximation problem is now as follows.

a. If $k = q$ and $(\mathbb{L}_\sigma, \mathbb{L})$, is a regular pencil, then $\mathbf{E} = -\mathbb{L}$, $\mathbf{A} = -\mathbb{L}_\sigma$, $\mathbf{B} = \mathbf{V}$, $\mathbf{C} = \mathbf{W}$, is a minimal realization of an interpolant of the data. Thus, the associated transfer function $\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_\sigma - s\mathbb{L})^{-1}\mathbf{V}$, satisfies the required interpolation conditions. **b.** In the more common case where there are more data than necessary, $(\mathbb{L}_\sigma, \mathbb{L})$ is a singular pencil. Using the basic fact (see [3]) that the (approximate) rank k of $\mathbb{L}$ is equal to the complexity of the underlying system, rank-revealing SVDs of $\mathbb{L}$ and $\mathbb{L}_\sigma$, yield $\mathbf{Y}, \mathbf{X} \in \mathbb{C}^{N\times k}$. The projection then defined by $\mathbf{X}, \mathbf{Y}$, leads to a realization of degree k, of an (approximate) interpolant of the data: $\mathbf{E} = -\mathbf{Y}^*\mathbb{L}\mathbf{X}$, $\mathbf{A} = -\mathbf{Y}^*\mathbb{L}_\sigma\mathbf{X}$, $\mathbf{B} = \mathbf{Y}^*\mathbf{V}$, $\mathbf{C} = \mathbf{W}\mathbf{X}$, i.e. $\ell_j\mathbf{H}(\mu_j) \approx \mathbf{v}_j$, $\mathbf{H}(\lambda_i)\mathbf{r}_i \approx \mathbf{w}_i$.

• **Further developments**. Recently the Loewner framework was extended to deal with linear parameter-dependent systems [4], [1], [2], and even more recently, it has been extended to certain classes of non-linear systems. *It our purpose to discuss this latter case during the meeting*. We conclude this abstract by summarizing the main aspects of the Loewner framework.

• **Model reduction in the Loewner framework: Summary of features**

1. Given input/output or computed data, we can construct with *no computation* (i.e. no factorizations or matrix solves), a singular high order model in generalized (descriptor) state space form. The key tool is the *Loewner pencil*.

2. The philosophy behind this approach is: *collect data and extract desired information*.

3. In applications the singular pencil must be reduced at some stage. This is a natural way for constructing full and reduced models because it does not *force* inversion of $\mathbf{E}$, and moreover it can deal with many input/output ports.

5. In this framework the *singular values* of $\mathbb{L}, \mathbb{L}_\sigma$, offer a *trade-off between accuracy of fit and complexity of the reduced system*.

6. The approach has been xtended to *parametrized systems*, to *bilinear systems* and to *quadtaric nonlinear systems*.

7. At this early stage, we intend to demonstrate the system theoretic as well as the numerical properties of our approach

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by means of at least the following: reduction of Burgers' equation, reduction of the FitzHugh-Nagumo equations, and reduction of the miscible flow problem as described in [8]. We also intend to make comparisons with the results to [7].

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