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Controllability properties of spin-boson systems

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***Abstract*— In this paper we study the so-called spin-boson system, namely a spin-$1/2$ particle in interaction with a distinguished mode of a quantized bosonic field. We control the system via an external field acting on the bosonic part.**

Applying geometric control techniques to the Galerkin approximation and using perturbation theory to guarantee non-resonance of the spectrum of the drift operator, we prove approximate controllability of the system, for almost every value of the interaction parameter.

I. Introduction

In this presentation we study the so-called Rabi model, which describes the interaction between a bosonic mode and a two-level system. Mathematically, in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}, \mathbf{C}) \otimes \mathbf{C}^2$, we consider the Schrödinger equation

$$\mathrm{i}\,\partial_t \psi = H_{\mathrm{Rabi}} \psi, \tag{1}$$

where

$$H_{\mathrm{Rabi}} = \frac{\omega}{2}(-\partial_x^2 + x^2) \otimes \mathbb{1} + \frac{\Omega}{2} \mathbb{1} \otimes \sigma_3 + g\, x \otimes \sigma_1,$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

is the usual notation for the Pauli matrices.

The physical interpretation of the two factors in the tensor product varies according to the context. For instance, in cavity QED the Hamiltonian H_{Rabi} describes a two-level atom interacting with a single distinguished mode of the quantized electromagnetic field in the cavity. In this context, a simplified version of the operator H_{Rabi} is often called *Jaynes-Cummings Hamiltonian*, after the celebrated work [1] on maser theory (see [2] and references therein).

In a wide variety of experimental situations one can act on the system by an external field. The goal of the controller might be to lead the system from a given initial state to a prescribed final one. For spin-boson models, this amounts to study the control problem

$$\mathrm{i}\,\partial_t \psi(t) = H_{\mathrm{Rabi}} \psi(t) + H_c(u(t))\psi(t), \tag{2}$$

where H_c is a self-adjoint operator describing the coupling between the system and the controlled external field. In general, u takes values in (a subset of) $\mathbf{R}$, or, more generally, in $\mathbf{R}^d$.

In most cases the external field can act on the bosonic mode only, while the spin mode is not directly accessible. This leads to a control Hamiltonian of the form

$$H_c(u(t)) = h_c(u(t)) \otimes \mathbb{1}. \tag{3}$$

where $h_c(u(t))$ is a self-adjoint operator acting in $L^2(\mathbf{R}, \mathbf{C})$.

One of the simplest form for the operator $h_c(u(t))$ is the following

$$h_c(u(t)) = u(t)x. \tag{4}$$

The linearity in u is a consequence of the dipole approximation which is valid in the limit of weak field. The linearity with respect to x of the multiplicative operator h_c represents the action of a force depending on time and constant in x.

We consider here the approximate controllability problem for a system of the form (2), where H_0 is the Rabi Hamiltonian and H_c takes the form (3)-(4). We will also assume the constraints

$$\begin{cases} |\Omega - \omega| \ll \Omega, \\ \omega, \Omega > 0. \end{cases} \tag{5}$$

and that the control set U contains an interval $[0, \delta]$.

We also remark that in physical contexts the constant g appearing in H_{Rabi} is often assumed to satisfy $g \ll \Omega, \omega$. However, in general, this is not the case for applications to circuits QED.

II. Main result

Our main result is the following.

Theorem 1: System (2), with H_c taking the form (3)-(4), and under the assumptions (5) and $u(t) \in U \supset [0, \delta]$, is approximately controllable for almost every $g \in \mathbf{R}$.

For control results on related spin-boson models, see [3], [4], [5], [6], [7], [8], [9].

In more general settings, several controllability results for (infinite dimensional) quantum systems have been obtained in recent years by exploiting different techniques. Methods based on finite dimensional truncations and Lie algebraic properties have been exploited in [10], [11], [12]. These papers provide controllability results under generic assumptions [13]. Lyapunov based techniques were used to prove approximate controllability for instance in [14], [15], while adiabatic perturbation theory have been used for instance in [7], [16]. Exact controllability results were proved for special systems in [17], [18], [19].

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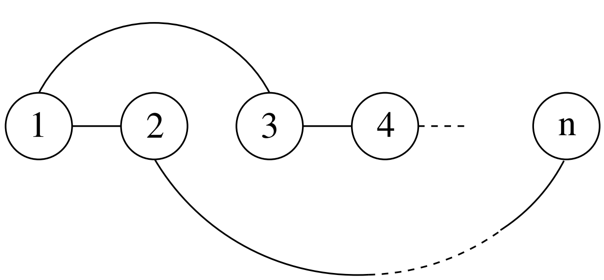

Fig. 1. Each vertex of the graph represents an eigenstate of A (when the spectrum is not simple, several nodes may be attached to the same eigenvalue). An edge links two vertices if and only if B connects the corresponding eigenstates. In this example, $\langle \phi_1, B\phi_2 \rangle$, $\langle \phi_1, B\phi_3 \rangle$ and $\langle \phi_3, B\phi_4 \rangle$ are not zero, while $\langle \phi_1, B\phi_4 \rangle = \langle \phi_2, B\phi_3 \rangle = \langle \phi_2, B\phi_4 \rangle = 0$.

In order to provide a sketch of the proof of our main result we recall a general controllability result for bilinear quantum systems in an abstract setting.

In a separable Hilbert space H, endowed with the Hermitian product $\langle \cdot, \cdot \rangle$, we consider the following control system

$$\frac{d}{dt}\psi = (A + u(t)B)\psi, \quad u(t) \in U, \tag{6}$$

where (A, B, U) satisfies the following assumption.

Assumption 1: U is a subset of $\mathbf{R}$ and (A, B) is a pair of (possibly unbounded) linear operators in H such that

1) A is skew-adjoint on its domain $D(A)$;
2) there exists a Hilbert basis $(\phi_k)_{k \in \mathbf{N}}$ of H consisting of eigenvectors of A: for every k, $A\phi_k = \mathrm{i}\lambda_k \phi_k$ with λ_k in $\mathbf{R}$;
3) for every j in $\mathbf{N}$, ϕ_j is in the domain $D(B)$ of B;
4) $A + uB$ is essentially skew-adjoint for every $u \in U$;
5) $\langle B\phi_j, \phi_k \rangle = 0$ for every j, k in $\mathbf{N}$ such that $\lambda_j = \lambda_k$ and $j \neq k$.

If (A, B, U) satisfies Assumption 1, then $A+uB$ generates a unitary group $t \mapsto e^{t(A+uB)}$. By concatenation, one can define the solution of (6) for every piecewise constant function u taking values in U, for every initial condition ψ_0 given at time t_0. We denote this solution by $t \mapsto \Upsilon^u_{t,t_0}\psi_0$.

A pair (j, k) in $\mathbf{N}^2$ is a *non-resonant transition* of (A, B) if $b_{jk} \neq 0$ and, for every l, m, $|\lambda_j - \lambda_k| = |\lambda_l - \lambda_m|$ implies $\{j, k\} = \{l, m\}$ or $\{l, m\} \cap \{j, k\} = \emptyset$.

A subset S of $\mathbf{N}^2$ is a *chain of connectedness* of (A, B) if for every j, k in $\mathbf{N}$, there exists a finite sequence $p_1 = j, p_2, \ldots, p_r = k$ for which $(p_l, p_{l+1}) \in S$ for every l and $\langle \phi_{p_{l+1}}, B\phi_{p_l} \rangle \neq 0$ for every $l = 1, \ldots, r-1$ (see Figure 1 for a graph representation of this property). A chain of connectedness S of (A, B) is *non-resonant* if every (j, k) in S is a non-resonant transition of (A, B).

Definition 1: Let (A, B, U) satisfy Assumption 1. We say that (6) is *approximately controllable* if for every $\varepsilon > 0$, for every $\psi_0, \psi_1 \in H$, there exists a piecewise constant function $u_\varepsilon : [0, T_\varepsilon] \to U$ such that $\|\Upsilon^u_{T_\varepsilon,0}\psi_0 - \psi_1\| < \varepsilon$.

Theorem 2 ([11]): Assume that $[0, \delta] \subset U$ for some $\delta > 0$ and let (A, B, U) satisfy Assumption 1 and admit a non-resonant chain of connectedness. Then system (6) is approximately controllable.

III. Sketch of the proof

A sketch of the proof of Theorem 1 will be presented separately for the cases $\Omega \neq \omega$ and $\Omega = \omega$.

In both cases, the proof of the theorem is based on a suitable application of Theorem 2.

The strategy of the proof is the following. We first show that, for almost every g in $\mathbf{R}$, some relevant pairs of eigenvalues of H_{Rabi} satisfy the non-resonance condition, see (9). This goal is reached by exploiting the analyticity of the eigenvalues and by using perturbation theory. Then we prove that these pairs of eigenvalues correspond to non-resonant transitions, according to the definition above.

Let $(\varphi_j)_{j \in \mathbf{N}}$ be the standard Hilbert basis of $L^2(\mathbf{R}, \mathbf{C})$ given by real eigenfunctions of $-\partial_x^2 + x^2$, so that $(-\partial_x^2 + x^2)\varphi_j = (2j+1)\varphi_j$ and $\int_{\mathbf{R}} x\varphi_j(x)\varphi_{j+1}(x)dx = \sqrt{(j+1)/2}$ for $j \geq 0$.

Based on $(\varphi_j)_{j \in \mathbf{N}}$, we obtain a Hilbert basis of factorized eigenstates $\Phi_{j,s} = \varphi_j \otimes s$, $j \in \mathbf{N}$, $s \in \{-1, 1\}$, of $H_{\mathrm{Rabi},0}$, the expression of H_{Rabi} at $g = 0$, whose corresponding eigenvalues are

$$E_{j,s} = \omega\left(j + \frac{1}{2}\right) + \frac{s}{2}\Omega. \tag{7}$$

If Ω is not an integer multiple of ω then each eigenvalue $E_{j,s}$ is simple.

In the following, for ease of notations, we write in bold the elements of $\mathbf{N} \times \{-1, 1\}$, and for every $\mathbf{j} \in \mathbf{N} \times \{-1, 1\}$ we define $j(\mathbf{j}), s(\mathbf{j})$ in such a way that $\mathbf{j} = (j(\mathbf{j}), s(\mathbf{j}))$.

For $g \in \mathbf{R}$, denote by $E^g_{\mathbf{j}}$, $\mathbf{j} \in \mathbf{N} \times \{-1, 1\}$, the eigenvalues of H_{Rabi} repeated according to their multiplicities, and by $\Phi^g_{\mathbf{j}}$, $\mathbf{j} \in \mathbf{N} \times \{-1, 1\}$, an orthonormal basis of corresponding eigenstates.

If $E^g_{\mathbf{j}}$ is simple then, by Rellich's theorem [20, Theorem XII.8], up to a reordering of the spectrum of $H_{\mathrm{Rabi}} = H_{\mathrm{Rabi},0} + g'V$, where $V = x \otimes \sigma_1$, for g' in a neighborhood of g, the eigenpair parameterization $g' \mapsto (E^{g'}_{\mathbf{j}}, \Phi^{g'}_{\mathbf{j}})$ is analytic near g.

The previous result can be generalized to the case when $E^g_{\mathbf{j}}$ is of multiplicity $m \geq 1$ ([20, Theorem XII.13]): there exist $g' \mapsto (E^{g'}_{\mathbf{j}_1}, \Phi^{g'}_{\mathbf{j}_1}), \ldots, g' \mapsto (E^{g'}_{\mathbf{j}_m}, \Phi^{g'}_{\mathbf{j}_m})$ analytic near g with $(\mathbf{j}_1), \ldots, (\mathbf{j}_m)$ distinct. Moreover,

$$\langle \Phi^g_{\mathbf{j}_l}, V\Phi^g_{\mathbf{j}_k} \rangle = \delta_{lk} \frac{d}{dg} E^g_{\mathbf{j}_l} \tag{8}$$

(see, for instance, [21]).

In order to describe the spectrum by a countable family of analytic functions defined for $g \in \mathbf{R}$, one should ensure that all such local analytic functions can be extended indefinitely. This is the case if we can exclude the functions to blow up of a bounded interval. This property turns out to be true as a consequence of the fact that the perturbation operator V

is Kato-small with respect to the unperturbed Hamiltonian $H_{\mathrm{Rabi},0}$, that is, $D(H_{\mathrm{Rabi},0}) \subseteq D(V)$ and for every $a > 0$ there exists $b > 0$ such that $\|V\psi\| \le a\|H_{\mathrm{Rabi},0}\psi\| + b\|\psi\|$, for any $\psi \in D(H_{\mathrm{Rabi},0})$.

Let us now explicitly consider the case $\omega \neq \Omega$ and, in particular, because of (5), let us assume that ω is not a multiple of Ω. In this case the proof of the theorem goes as follows.

The first step consists in proving that for almost every $g \in \mathbf{R}$ and every $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l} \in \mathbf{N}\times\{-1,1\}$, with $(\mathbf{i},\mathbf{j}) \neq (\mathbf{k},\mathbf{l})$ and $\mathbf{i} \neq \mathbf{j}$, one has $E^g_{\mathbf{i}} - E^g_{\mathbf{j}} \neq E^g_{\mathbf{k}} - E^g_{\mathbf{l}}$. In order to do so, we observe that it is enough to show that for fixed $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l} \in \mathbf{N}\times\{-1,1\}$ as before, the set

$$S_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} = \{g \mid E^g_{\mathbf{i}} - E^g_{\mathbf{j}} \neq E^g_{\mathbf{k}} - E^g_{\mathbf{l}}\} \qquad (9)$$

is of full measure. By the analytic dependence on g of the eigenvalues of H_{Rabi}, this is equivalent to say that $g \mapsto E^g_{\mathbf{i}} - E^g_{\mathbf{j}}$ and $g \mapsto E^g_{\mathbf{k}} - E^g_{\mathbf{l}}$ have different Taylor expansions at $g = 0$. Based on the Rayleigh–Schrödinger series (see, for instance, [20, Chapter XII]), the coefficients of the Taylor expansion $E^g_{\mathbf{j}}$ with respect to g can be explicitly computed (as functions of the parameters of the problem). Computing Taylor expansions up to order four, and plugging them into the equality $E^g_{\mathbf{i}} - E^g_{\mathbf{j}} = E^g_{\mathbf{k}} - E^g_{\mathbf{l}}$ easily leads to a contradiction, whenever $(\mathbf{i},\mathbf{j}) \neq (\mathbf{k},\mathbf{l})$ and $\mathbf{i} \neq \mathbf{j}$. We remark that the odd terms in the series vanish.

The second step, which concludes the proof of Theorem 1 in the case $\omega \neq \Omega$ (thanks to Theorem 2), consists in showing that the controlled Hamiltonian $x \otimes \mathbb{1}$ couples, directly or indirectly, all the energy levels for almost all $g \in \mathbf{R}$.

More precisely, it is possible to show that $\langle \Phi^g_{\mathbf{j}}, (x \otimes \mathbb{1})\Phi^g_{\mathbf{k}}\rangle \neq 0$ for almost every $g \in \mathbf{R}$ for all $\mathbf{j},\mathbf{k}$ such that $s(\mathbf{j}) = s(\mathbf{k})$ and $|j(\mathbf{j}) - j(\mathbf{k})| = 1$ or $s(\mathbf{j}) = -s(\mathbf{k})$ and $j(\mathbf{j}) = j(\mathbf{k})$. Again, in order to complete this step, one has to show that the Taylor series in g corresponding to the expressions above is nonzero. The explicit computation of the terms of order zero shows that $\langle \Phi^g_{\mathbf{j}}, (x \otimes \mathbb{1})\Phi^g_{\mathbf{k}}\rangle \neq 0$ for almost every g for all $\mathbf{j},\mathbf{k}$ such that $s(\mathbf{j}) = s(\mathbf{k})$ and $|j(\mathbf{j}) - j(\mathbf{k})| = 1$.

On the other hand, one can suitably transform $\frac{d}{dg}\langle \Phi^g_{\mathbf{j}}, (x\otimes \mathbb{1})\Phi^g_{\mathbf{k}}\rangle|_{g=0}$ as a series of terms involving the values of eigenvectors and eigenvalues of $H_{\mathrm{Rabi},0}$, which turns out to have only few nonzero terms. In particular, whenever $j(\mathbf{j}) = j(\mathbf{k})$ and $s(\mathbf{j}) = -s(\mathbf{k})$, one shows that $\frac{d}{dg}\langle \Phi^g_{\mathbf{j}}, (x \otimes \mathbb{1})\Phi^g_{\mathbf{k}}\rangle|_{g=0} = \frac{\omega}{\Omega^2 - \omega^2}$, which leads to the conclusion.

We now briefly sketch the proof in the case $\omega = \Omega$. In this case, except for the lowest one, all the eigenvalues of $H_{\mathrm{Rabi},0}$ are double. The presence of two-dimensional eigenspaces does not allow to directly follow the previous reasoning. Instead, one can compute special eigenvectors $\Phi^0_{\mathbf{j}}$ of $H_{\mathrm{Rabi},0}$ as the limits of the eigenvectors $\Phi^g_{\mathbf{j}}$ of H_{Rabi} as g goes to zero (this can be done exploiting the fact that $\langle \Phi^g_{\mathbf{j}}, V\Phi^g_{\mathbf{k}}\rangle = 0$ for $\mathbf{j} \neq \mathbf{k}$). By using this information one can compute the limit as g goes to zero of the derivative $\langle \Phi^g_{\mathbf{j}}, V\Phi^g_{\mathbf{j}}\rangle$ of each eigenvalue $E^g_{\mathbf{j}}$ of H_{Rabi}. The computation leads to

$$\lim_{g\to 0} \frac{d}{dg} E^g_{\mathbf{j}} = \langle \Phi^0_{\mathbf{j}}, V\Phi^0_{\mathbf{j}}\rangle = s(\mathbf{j})\sqrt{\frac{2j(\mathbf{j}) + 1 + s(\mathbf{j})}{4}}.$$

Based on this explicit expression, some simple estimates ensure that $E^g_{\mathbf{i}} - E^g_{\mathbf{j}} = E^g_{\mathbf{k}} - E^g_{\mathbf{l}}$ holds only when $\mathbf{i} = \mathbf{k}$ and $\mathbf{j} = \mathbf{l}$.

It remains to show that the controlled Hamiltonian $x \otimes \mathbb{1}$ couples, directly or indirectly, all the elements of the basis $(\Phi^0_{\mathbf{j}})_{\mathbf{j}\in\mathbf{N}\times\{-1,1\}}$.

For this purpose it can be shown by direct computations that $\langle \Phi^0_{\mathbf{j}}, (x \otimes \mathbb{1})\Phi^0_{\mathbf{k}}\rangle \neq 0$ whenever $j(\mathbf{j}) = j(\mathbf{k}) + 1$ and $s(\mathbf{j}) = s(\mathbf{k})$, or when $j(\mathbf{j}) = j(\mathbf{k})$ and $s(\mathbf{j}) \neq s(\mathbf{k})$. This completes the proof.

Conclusion

We analyzed the controllability properties of the Rabi model, describing the interaction between a bosonic mode and a two-level system, subject to an external field acting on the bosonic mode only. Namely, approximate controllability has been proved generically with respect to the strength of the interaction term between the two modes. The method relies on perturbation arguments for the spectrum of the Hamiltonian, which allow to apply a general controllability result for the bilinear Schrödinger equation. Future work will address the issue of extending the result to a general class of control terms.

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