

STRUCTURE PRESERVING REDUCTION FOR THERMO-MAGNETO PLASMA CONTROL MODEL

N. M. Trang VU¹, L. LEFÈVRE¹, R. NOUAILLETAS², and S. BRÉMOND²

¹Univ. Grenoble Alpes, LCIS, F-26902, Valence, France

²CEA/DSM/IRFM/STEP, CEA Cadarache, France

Abstract—A geometric spatial reduction method is presented in this paper. It applies to port Hamiltonian models for open systems of balance equations. It is based on projections which make use of the symmetries and on the preservation of the “natural” power pairing for the considered system. The method is applied to a system of two coupled parabolic equations describing the poloidal magnetic flux diffusion and heat radial transport in tokamak reactors. There are two reduction steps first to reduce the model from 3D to 1D, then to reduce the model from 1D to 0D. The assumptions of axial symmetry and quasi-static equilibrium of the plasma are used to perform the reduction from 3D to 1D by using simple integration formulas on toroidal coordinates’ surfaces. A Galerkin-type pseudo-spectral spatial reduction method is used to reduce the 1D model to a 0D one. Both reductions are made symplectic with respect to the power-pairings in the magnetic and thermal domains. Finally, the 0D plasma control model is obtained by reduction of the multi-domains couplings between the two original PDEs. These couplings are the Lorentz force (with non-uniform resistivity) and the bootstrap current.

discretization, distributed parameters system s, plasma con-

trol, port Hamiltonian systems

I. INTRODUCTION

The fusion research connected to the ITER project needs, besides a better understanding of involved physical phenomenon’s, a better optimization/control of these multi-domain in order to improve the plasma confinement and prove the feasibility of energy production with controlled fusion reactions inside tokamak reactors. On the other hand, the port Hamiltonian approach for open systems of conservation and balance equations may provide a multiphysics 3D fluid-like model which captures some of the main macroscopic physical properties and qualitative behavior of the actual system [18]. In turn this model can be reduced to 1D or a 0D control models to which (more or less) classical nonlinear control approaches (such as IDA-PBC control [10]) may be applied. This reduction is the central concern of this paper. It must preserves the main dynamical properties (as for instance the spectrum qualitative properties and quantitative values) and the main energetic properties of the actual model in order to apply successfully passivity or energy based control approaches.

Hamiltonian formulations have already been used in the plasma dynamics context, such as in [7] for ideal fluid

models, in [8] for magnetohydrodynamics models or in [3] to represent the Grad-Shafranov equations using a Lie-Poisson bracket. However some of these earlier works either make use of the “microscopic” (six dimensional) kinetic theory to represent the plasma dynamics, while all of them (to the best of our knowledge) considered only closed systems without any explicit input-output variables. Therefore these previous models, although they bring much more insights into the “geometry” of the magnetohydrodynamic equations, were useless for control purposes. In [15] we proposed to use Stokes-Dirac interconnection structures to embed input and output (distributed and boundary) power conjugated variables in the Hamiltonian formulation of these MHD models. In [18] we extended this port-Hamiltonian model to include the material macroscopic entropy, momentum and mass balance equations. In [16], [17] we considered the simplified case of the so-called resistive diffusion equation for the poloidal magnetic flux where only the magnetic domain (out of five energy domains in the full model) is considered. This simplified model is commonly used for the control of the current density (or security factor) radial profile which is a key issue for confinement problems. We wish now to generalize our previous works to consider simultaneously the coupled models for the Maxwell field equations and the material entropy balance equation. This should allow us to take advantage of the TMHD coupling analysis to improve the estimations for the resistivity η and bootstrap current j_{bs} which depends on the temperature profile and gradient.

The next step after the port-Hamiltonian re-formulation of the coupled system of magnetic flux resistive diffusion and heat diffusion equations is the spatial reduction of the obtained model. Therefore we developed a geometric spatial reduction methodology which is presented in this paper. It applies to port Hamiltonian models for open systems of balance equations and is based on projections which make use of the symmetries and on the preservation of the “natural” power pairing for the considered system.

The method is applied to the system of two coupled parabolic equations describing the poloidal magnetic flux diffusion and heat radial transport in tokamak reactors. The method is applied twice: first to reduce the model from 3D to 1D, then to reduce the model from 1D to 0D. The assumptions of axial symmetry and quasi-static equilibrium of the plasma are used to perform the reduction from 3D to 1D by using simple integration formulas on toroidal coordinates’ surfaces. A Galerkin-type pseudo-spectral spatial reduction

method is used to reduce the 1D model to a 0D one. Both reductions are made symplectic with respect to the power-pairings in the magnetic and thermal domains. Finally, the 0D plasma control model is obtained by reduction of the multi-domains couplings between the two original PDEs. These couplings are the Lorentz force (with non-uniform resistivity) and the bootstrap current.

The paper is organized as follows. In section 2, the Maxwell field balance equations and the entropy material balance equations are stated in the port-Hamiltonian formalism. The general idea of the 3D-1D reduction method, based on symmetries assumptions, is presented and applied to the considered example to derive a 1D plasma model for the coupled system of resistive and thermal diffusion equations. In section 3, a similar reduction idea is applied to derive a symplectic Galerkin-type discretization scheme which leads to a 0D control model and guarantees preservation of the power pairings in the magnetic and thermal domains. In section 4, the developed discrete model is validated against experimental data. The paper ends with a brief conclusion and some perspectives on control applications.

II. 3D-1D GEOMETRIC REDUCTION

A 3D TMHD (thermo-magneto-hydrodynamic) model has been proposed in [18] for the dynamical modelling of a plasma gas in a tokamak's toroidal chamber. It is based on the mass, entropy, momentum and electromagnetic balance equations. With the Dirac and Stokes-Dirac interconnection structures [13], balance and closure (constitutive) equations are organized in a structured port-Hamiltonian model. In this paper we will restrict the TMHD model to the electromagnetic fields and entropy balances equations (and the corresponding closure and coupling equations). In this section we will derive a 1D model more suitable for control issues than the full 3D TMHD model. There are two main reduction assumptions: axial symmetry with respect to the main torus axis (see figure 1) and quasi-static equilibrium for the plasma¹. With these two assumptions, it may be proved that the magnetic flux surfaces form a set of nested toroids which are simultaneously isobaric, isothermal and iso-polooidal flux² [20], [1]. Therefore, after a continuous mapping, these surfaces may be matched into nested regular toroidal surfaces with circular cross-sections and a set of magnetic toric coordinates (ρ, θ, ϕ) (see figure 1) may be defined such that ρ denotes the index of the considered magnetic surface (and the new "radial" coordinate) and such that all the system port variables are independent from θ and ϕ . Therefore the model may be projected onto the 1D domain $\rho \in \Pi = [0, a]$, $a = \rho_{max}$.

¹i.e. the plasma is assumed to have reached a mechanical equilibrium, that is a constant momentum density, at every instant in the evolution of the considered dynamics

²The poloidal flux $\psi(R, z)$ is defined as the flux through a circular horizontal disk determined by the radius R from the torus axis and the height z in cylindrical coordinate

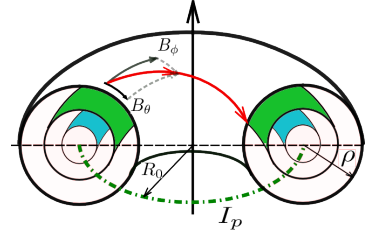


Fig. 1. Magnetic toric coordinate: ρ denotes the magnetic surface index (corresponding to the small radius r), θ the polar angle and ϕ the azimuth angle. R_0 denotes the principal radius of the plasma, I_p the total plasma current and B_θ and B_ϕ the two components of the magnetic field

k -form	new variables	corresponding value
1-form	$\alpha^0 = (\overline{A_\theta}, \overline{A_\phi})$	$(\int_0^{2\pi} \sqrt{g_\theta} A_\theta d\theta, \int_0^{2\pi} \sqrt{g_\phi} A_\phi d\phi)$
2-form	$\alpha^1 = (\overline{A_\theta}, \overline{A_\phi}) d\rho$	$(\int_0^{2\pi} \sqrt{g_\rho g_\theta} A_\theta d\phi, \int_0^{2\pi} \sqrt{g_\rho g_\phi} A_\phi d\theta) d\rho$
3-form	$\alpha^1 = \overline{A} d\rho$	$(\int_0^{2\pi} \int_0^{2\pi} \sqrt{g} A d\theta d\phi) d\rho$

TABLE I

REDUCED VARIABLES DEFINITIONS IN THE 1D DOMAIN

Note that most of the variables "lie" on the magnetic surfaces, so there are no component on ρ "direction".

A. Geometric reduction of the covariant formulation

The TMHD model in [18] is stated in covariant form, that is the state and port variables are not defined as vector fields but rather as differential k -forms corresponding to the integral calculus (cf. [6]). It is assumed in this approach that the total energy in 3D spatial domain Ω may be written $\mathbb{H} = \int_\Omega \mathcal{H}^3$ where the energy density $\mathcal{H}^3 \in \Lambda^3(\Omega)$ is the external product of two k -forms, either $\mathcal{H}^3 = \alpha^1 \wedge \beta^2$ or $\mathcal{H}^3 = \alpha^0 \wedge \beta^3$. The upper indexes (k) denote the degree k of the corresponding k -form. The idea is then to partially integrate \mathcal{H} on 2D coordinate surfaces such that the total energy reads

$$\mathbb{H} = \int_\Omega \mathcal{H}^3 = \int_\Pi \alpha^0 \wedge \beta^1 \quad (1)$$

leading to the definition of the 1D reduced variables α^0 and β^1 and their external product (power pairing) $\alpha^0 \wedge \beta^1$ in Π . We discuss hereafter how to determine these variables.

Let g_ρ , g_θ , g_ϕ , and g^3 denote the transformation coefficients between geometric toric coordinates and magnetic toric coordinates. For instance, the integration in (1) in magnetic toric coordinate reads:

$$\int_\Omega \mathcal{H} = \int_\Omega \alpha \beta \sqrt{g} d\rho d\theta d\phi \quad (2)$$

Therefore, the corresponding reduced variables α and β may be obtained by grouping the variables into two parts and performing integration on suitable integration domains: a curve coordinate for 1-forms and 2-forms and a surface coordinate for 3-forms. These integrations are summarized in table I.

³These coefficients have the form:

$$\begin{cases} g_\rho = (\partial_\rho r)^2 \\ g_\theta = (\partial_\theta r)^2 + r^2 \\ g_\phi = (R_0 + r \cos\theta)^2 \end{cases} ; g = g_\rho g_\theta g_\phi \text{ and } dV = \sqrt{g} d\rho d\theta d\phi$$

B. The resistive diffusion equation

The plasma electromagnetic 3D model was developed in port-Hamiltonian form in [15]. It includes the following covariant formulation for the Maxwell's equations:

$$\begin{pmatrix} -\partial_t D \\ -\partial_t B \end{pmatrix} = \begin{pmatrix} 0 & -d \\ d & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} J \quad (3)$$

with the electric and magnetic field intensities $E, H \in \Lambda^1(\Omega)$, while the field flows and the total current density $D, B, J \in \Lambda^2(\Omega)$. Here d denotes the external spatial derivative. The electromagnetic energy is:

$$\mathbb{H}(D, B) = \frac{1}{2} \int_{\Omega} [E^1 \wedge D^2 + H^1 \wedge B^2] \quad (4)$$

Let us now apply the geometric reduction described in the previous subsection to the magnetic domain to determine the corresponding 1D port-variables:

$$\begin{aligned} \mathbb{H}(B) &= \frac{1}{2} \int_{\Omega} H^1 \wedge B^2 \\ &= \frac{1}{2} \int_{\Omega} (H_{\rho} B_{\rho} + H_{\theta} B_{\theta} + H_{\phi} B_{\phi}) \sqrt{g} d\rho d\theta d\phi \\ &= \frac{1}{2} \int_0^a d\rho \left[\int_0^{2\pi} (\sqrt{g_{\theta}} H_{\theta}) d\theta \int_0^{2\pi} (\sqrt{g_{\rho} g_{\phi}} B_{\theta}) d\phi \right. \\ &\quad \left. + \int_0^{2\pi} (\sqrt{g_{\phi}} H_{\phi}) d\phi \int_0^{2\pi} (\sqrt{g_{\rho} g_{\theta}} B_{\phi}) d\theta \right] \\ &= \frac{1}{2} \int_0^a d\rho [(\bar{H}_{\theta})(\bar{B}_{\theta}) + (\bar{H}_{\phi})(\bar{B}_{\phi})] \\ &= \frac{1}{2} \int_{\Pi} H^0 \wedge B^1 \end{aligned} \quad (5)$$

Thus the 3D model (3) transforms into a 1D model with a similar port-Hamiltonian form, equivalent to the so-called *resistive diffusion equation* for the poloidal magnetic flux (cf. [1], [20]) when the assumption of electric equilibrium ($f_{el\phi} = 0$) holds:

$$\begin{pmatrix} f_{el\phi} \\ f_{mg\theta} \end{pmatrix} = \begin{pmatrix} 0 & -\partial_{\rho} \\ -\partial_{\rho} & 0 \end{pmatrix} \begin{pmatrix} e_{el\phi} \\ e_{mg\theta} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} f_{d\phi} \quad (6)$$

where $f_{el\phi}$, $f_{mg\theta}$, $e_{el\phi}$, $e_{mg\theta}$, $f_{d\phi}$ are flow and effort variables which are here respectively defined by $\partial_t(-\bar{D}_{\phi})$, $\partial_t(-\bar{B}_{\theta})$, \bar{E}_{ϕ} , \bar{H}_{θ} and \bar{J}_{ϕ} . The closure equations (written in toric coordinates) for the balance equations (6) are then:

$$\begin{aligned} e_{el\phi} &= \frac{\eta}{C_3} \bar{J}_{\Omega} && \text{Ohm's law} \\ e_{mg\theta} &= \frac{C_2}{\mu_0} \bar{B}_{\theta} && \text{magnetic constitutive equation} \end{aligned} \quad (7)$$

where $C_2 = \frac{\sqrt{g_{\theta}}}{\sqrt{g_{\rho} g_{\phi}}}$, $C_3 = \frac{\sqrt{g_{\rho} g_{\theta}}}{\sqrt{g_{\phi}}}$. \bar{J}_{Ω} is the 1D ohmic current equal to $(\bar{J}_{\phi} - \bar{J}_{ni})$. The current \bar{J}_{ni} is the 1D non-inductive current equal to the sum of the bootstrap current \bar{J}_{bs} described in [20] (a magnetohydrodynamics coupling effect which produces and extra current density) and external current source \bar{J}_{ext} which is controlled through external heating sources. The magnetic permeability is considered to be the void permeability μ_0 since Tokamaks are operating at very low densities. The boundary conditions will be chosen as:

$$\begin{cases} f_{mg\theta}|_{z=0} = \frac{\partial \psi}{\partial z}|_{z=0} = 0 \\ e_{el\phi}|_{z=a} = \frac{\partial \psi}{\partial t}|_{z=a} = V_{loop} \end{cases} \quad (8)$$

with $z = \frac{\rho}{a}$ the small normalized torus radius. The first one expresses the smoothness and symmetry with respect to the central circular axis (magnetic axis) of the Tokamak torus. The second one defines a boundary control action. Indeed the external magnetic coils is used to produce an equivalent loop voltage V_{loop} at the last closed flux surface.

Remark1. The plasma resistivity η , and the bootstrap current \bar{J}_{bs} are significantly varying with the plasma temperature T (cf. [21]). However, in most existing control design (for the poloidal flux control) the TMHD coupling has been neglected, the temperature T has been considered as an external parameter. Therefore $\eta := \eta(x, t)$ and $\bar{J}_{bs} := \bar{J}_{bs}(x, t)$ have been considered as time and space dependent parameters. Hereafter, adding a diffusion model for the temperature, an explicit dependance of these variables with the new state variable T has been considered.

C. The thermal diffusion equation

In [18], the material domain balance equations for mass, momentum, energy and entropy are written firstly from the Boltzmann equation using the kinetic theory. The connection between the classical macroscopic transport equation and the port-based formulation is made by using the material derivative in covariant form. Then we derive the irreversible entropy source term (from Gibbs-Duhem relation, following the "port-based" approach in [14] or [2]) which contains the heat conduction, the viscous dissipation, the Joule (ohmic) terms, and the external heating source. All of them define the constitutive relations for the heat balance equation, or the thermal diffusion equation.

Let $S \in \Lambda^3$ denote the entropy source term, $s \in \Lambda^3$ the entropy density and $T \in \Lambda^0$ the temperature. The entropy balance equation reads:

$$\begin{pmatrix} T D_t s \\ F \end{pmatrix} = \begin{pmatrix} 0 & -d \\ -d & 0 \end{pmatrix} \begin{pmatrix} T \\ f_q \end{pmatrix} + \begin{pmatrix} S \\ 0 \end{pmatrix} \quad (9)$$

with the heat flux $f_q \in \Lambda^2$ and the thermal force $F \in \Lambda^1$

Remark 1. Note that the above considered port variables are in fact average values since the real plasma consists of different species of ions, electrons and neutrons. Each species have their own temperatures and entropy/energy balance equations should also consider their interactions (see [1], [20]). It will become necessary to separate the species transport equations once the fusion reaction will be considered. Therefore coupled model of ionic and electronic energy transport equations are usually used in the burn control problem (see [11], [12]). However, in our case (no reaction), the average variables may be used and only one energy balance equation may be considered for the sake of simplicity and with no loss of generality.

Applying the proposed 3D-1D reduction, using as well magnetic toric coordinates for thermal domain, the 1D reduced port-conjugated variables in the thermal domain may be defined in the following way:

$$\begin{aligned} \mathbb{H}(T) &= \int_M T^0 \wedge S^3 = \int_{\Omega} T^0 \wedge n S^3 \\ &= \int_{\Omega} T n S \sqrt{g} d\rho d\theta d\phi \\ &= \int_0^a d\rho \left[T \int_0^{2\pi} \int_0^{2\pi} (n \sqrt{g} S) d\theta d\phi \right] \\ &= \int_0^a T \bar{S} d\rho = \int_{\Pi} T^0 \wedge S^1 \end{aligned} \quad (10)$$

Therefore the 3D thermal model in (9) is transformed into the 1D port-Hamiltonian model:

$$\begin{pmatrix} f_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & -\partial_{\rho} \\ -\partial_{\rho} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} \bar{S} \\ 0 \end{pmatrix} \quad (11)$$

where f_1, f_2, e_1, e_2 are the flows and efforts which are respectively defined by $\sqrt{g_n T} (D_t \mathbf{s}), n \sqrt{g_\theta g_\phi} f_q, T,$ and $\sqrt{g_\rho} F$. One of the associated closure relation is the Fourier's law:

$$f_2 = n \chi \frac{\sqrt{g_\theta g_\phi}}{\sqrt{g_\rho}} e_2 \quad (12)$$

where χ is the diffusion coefficient. The perfect gas law is used as the second constitutive equation, relating e_1 and f_1 :

$$\mathbf{s} = \ln \left(\frac{T^{3/2}}{n} \right) \Rightarrow D_t \mathbf{s} = \frac{\partial \mathbf{s}}{\partial T} \frac{DT}{Dt} = \frac{3}{2} \frac{1}{T} \frac{DT}{Dt} \quad (13)$$

One can notice that our port-Hamiltonian model for heat transport is formally equivalent to the usual thermal diffusion equation (c.f [5]) and may be derived from (11) as:

$$\sqrt{g} \frac{3}{2} \frac{\partial n T}{\partial t} = \partial_\rho \left(\frac{\sqrt{g_\theta g_\phi}}{\sqrt{g_\rho}} (n \chi \partial_\rho T) \right) + \bar{S} \quad (14)$$

The boundary conditions will be chosen as:

$$\begin{cases} f_2(0) \sim \frac{\partial T}{\partial z} \Big|_{z=0} = 0 \\ e_1^1 = T \Big|_{z=1} = \text{const} \end{cases} \quad (15)$$

Furthermore, the continuity condition at $z = 0$ gives:

$$f_1(0) = e_2(0) = 0 \quad (16)$$

The smoothness and symmetry of the tokamak torus is also respected with the first condition. The second one defines a boundary control action T_1 . Moreover, a distributed control is represented by the source term \bar{S} , including the ohmic heating, and non-inductive heating source.

D. Thermos-Hydro-Dynamic coupling

The diffusion processes in plasma systems may be subdivided into different time scales (cf.[1]). The particle diffusion τ_n time constant, and heat diffusion τ_e, τ_i for electrons and ions respectively are of the order of a millisecond. The resistive time constant for current density and magnetic field diffusion is of the order of a second. As it will be shown in the next section, the spectral properties of the two diffusion port-Hamiltonian control models still exhibit this time scale separation property.

As a result, the plasma temperature T profile in the thermal diffusion PDE (14) is established thousands times faster than the magnetic field profile in the resistive diffusion PDE (6). This ‘‘separation’’ assumption allows to decouple the two-PDE-solvers. The coupling elements $\eta(T_e)$ and $J_{bs}(T_e, \partial_x T_e)$ in the resistive diffusion model may be computed statically from some ‘‘scale laws’’ (see e.g. [21]) once T_e , the electronic temperature, is determined. The coupling via the thermal diffusion coefficient $\chi(T, \bar{B}_\theta)$ may also be determined by a scale law which depends on T and the state of resistive model \bar{B}_θ .

Remark 1. Furthermore, the electronic temperature T_e is deduced from the average temperature T thanks to the assumptions of the linear dependencies between electronic temperatures T_e, T_i and their densities n_e, n_i :

$$\begin{cases} T_i = \alpha_{T_i} T_e \\ n_i = \alpha_i n_e \\ T = \frac{n_e T_e + n_i T_i}{n_e + n_i} = \frac{1 + \alpha_i \alpha_{T_i}}{1 + \alpha_i} T_e \end{cases}$$

where α_i and α_{T_i} are the ratios defined in [21].

III. 1D-0D REDUCTION, SYMPLECTIC DISCRETIZATION METHOD REVISIT

This section presents two discrete schemes respectively for the resistive and thermal diffusion models. They preserve the corresponding interconnection structures, energetic properties and invariants (first integrals).

Through all of the text f and e will denote respectively the flow and effort variables. It's important to note that in the thermal model (f_1, e_2) are considered as flows and (e_1, f_2) are efforts in the discretization scheme. They are conjugated in the sense that their usual product denotes an energy power. The used symplectic geometric discretization methodology is inspired from [16] where it was applied for the 1D resistive diffusion model (6).

The spatial discretization allows to reduce the infinite dimensional port-Hamiltonian model to a finite dimensional port-controlled Hamiltonian (PCH) system with a similar Dirac interconnection structure, the same energetic behavior and dynamical properties (including a nice convergence result for the finite dimensional spectrum) and invariants. The basic idea of this discretization method is to choose different specific approximation spaces for the variables in (6) in such a way that exact exterior derivation may be performed in the approximation spaces and that the power product between pairs of effort and flow variables may be exactly evaluated [9]. Distributed variables are therefore projected in two different approximation spaces:

$$\begin{aligned} f(t, z) &= \sum_{k=1}^{N-1} w_k^f(z) (\mathbf{f}(t))_k \\ e(t, z) &= \sum_{i=1}^N w_i^e(z) (\mathbf{e}(t))_i \end{aligned} \quad (17)$$

where $\mathbf{f} \in \mathbb{R}^{N-1}$ and $\mathbf{e} \in \mathbb{R}^N$ are the time dependent coefficients; $w^f(z)$ and $w^e(z)$ generate the bases where the structure equations (derivation) in (6) are satisfied, that is:

$$\begin{cases} \bar{\mathcal{E}} &= \text{span}(w^e(z)) \\ \bar{\mathcal{F}} &= \text{span}(w^f(z)) \\ d(\bar{\mathcal{E}}) &= \bar{\mathcal{F}} \end{cases} \quad (18)$$

This exact differentiation condition implies:

$$\begin{aligned} w^f(z) \mathbf{f}(t) &= \partial_z w^e(z) \mathbf{e}(t) \\ \Rightarrow \mathbf{f}(t) &= D \mathbf{e}(t) \end{aligned} \quad (19)$$

where $\mathbf{f}(t), \mathbf{e}(t)$ are respectively the vectors of time variant coefficients of flows and efforts, and D is the derivative matrix $D = (w^f(z))^+ (\partial_z w^e(z))$, with $(w^f(z))^+$ denotes the pseudo-inverse of matrix $(w^f(z))$.

The energy conservation is based on the power conservation. Indeed the supplied power may be exactly represented in the reduced finite dimensional spaces as a bilinear product making use of a rectangular matrix of inner products between effort and flow spaces bases:

$$\begin{aligned} \mathbb{H} &= \int_{\Pi} e \wedge f \\ &= \int_{\Pi} \mathbf{e}(t) w^e(z) w^f(z) \mathbf{f}(t) \\ &= \mathbf{e}(t) \left(\int_{\Pi} w^e(z) w^f(z) \right) \mathbf{f}(t) \\ &= \mathbf{e}(t) M \mathbf{f}(t) \end{aligned} \quad (20)$$

In order to get a minimal Dirac structure (and an explicit PCH model), one still need to project the reduced effort variables because the flow and effort approximation spaces do not have

the same dimension. We choose a projector which doesn't affect the power pairing value. This can be easily done by choosing the projected effort variable $\tilde{\mathbf{e}}$ as

$$\tilde{\mathbf{e}} = M^T \mathbf{e} \quad (21)$$

A. The resistive diffusion equation

Applying the proposed discretization scheme, the following time-dependent dissipative Port Hamiltonian system has been obtained for the electromagnetic domain:

$$\begin{pmatrix} \dot{\mathbf{d}} \\ \dot{\mathbf{b}} \end{pmatrix} = \left[\begin{pmatrix} 0 & -J_1 \\ -J_2 & 0 \end{pmatrix} - \begin{pmatrix} R(t)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \partial_a \mathbb{H}_{EM} \\ \partial_b \mathbb{H}_{EM} \end{pmatrix} - \begin{pmatrix} \mathbf{J}_{bs} + \mathbf{J}_{ext} \\ -J_4 V_{loop} \end{pmatrix} \quad (22)$$

where \mathbf{d} , \mathbf{b} , \mathbf{J}_{bs} and \mathbf{J}_{ext} are respectively the time-varying coefficients of electric and magnetic fields, bootstrap and external current densities. Note that the boundary control V_{loop} is now in the finite dimensional state equation. The matrices J_1 , J_2 with $J_1 = -J_2^T$ represent the spatial derivation ∂_z in the new finite dimension system, J_4 is related to the boundary coefficient, and R is a dissipation matrix which is determined by the resistivity $\eta(z, t)$.

The electromagnetic energy smooth function \mathbb{H}_{EM} is defined as:

$$\mathbb{H}_{EM} = \frac{1}{2} \left(\mathbf{d}^T G_{el} \mathbf{d} + \mathbf{b}^T G_{mg} \mathbf{b} \right)$$

where matrices G_{el} and G_{mg} are simply reduced to the electric and magnetic permeability, $\frac{1}{\epsilon_0 C_3}$ and $\frac{C_2}{\mu_0}$ in the simple anisotropic case.

Theoretical eigenvalues	Numerical eigenvalues		
	N=5	N=8	N=10
-2.301056852	-2.30525310	-2.301848126	-2.301466232
-12.12413006	-12.25659915	-12.14481967	-12.13359575
-29.79659326	-30.92406956	-29.9301737	-29.85455494
-55.32237139	-81.24868031	-55.84563052	-55.53473267
-88.70194524		-90.35712887	-89.30413565
-129.9354296		-135.0263099	-131.4267729
-179.0228628		-297.6242993	-182.523068
-235.9642604			-244.5957644
-300.7596298			-530.509059

TABLE II

EIGENVALUES WITH BESSEL BASE-FUNCTIONS AND WITH $\eta = 5 \cdot 10^{-7}$

The electromagnetic finite dimensional model (22) has real negative eigenvalues as the 1D resistive diffusion model. To illustrate this, we have chosen Bessel functions to span the approximation spaces since they are also the eigenfunctions of the resistive diffusion equation (6) with homogeneous boundary conditions and a uniform resistivity [16]. Results are shown in table II which shows the symplecticity of the proposed discretization scheme.

B. The thermal diffusion equation

Following the same idea, the theoretical eigenfunctions of the thermal diffusion equation (in the simple uniform homogeneous case) are used to span the approximation bases hereafter. The discretization procedure does not differ from the previous case, although the obtained finite dimensional model for the thermal diffusion is given in implicit form only.

1) *The 0D thermal PCH model:* Bessel functions of order 0, J_{B0} , and of order 1, J_{B1} , are used to generate bases functions for efforts and flows in the following way:

$$\begin{aligned} w_k^f(x) &= J_{B1}(\lambda_k x), \text{ with } \lambda_k = \sqrt{\frac{3}{2}} \frac{s}{\chi} \\ w_i^e(x) &= \begin{cases} -\frac{1}{\lambda_i} J_{B0}(\lambda_i x), & i = \overline{1..N-1} \\ 1 & i = N \end{cases} \end{aligned} \quad (23)$$

where λ_k are the zeros of the Bessel function order 0, J_{B0} . The input-output representation of the Dirac interconnection structure in the thermal domain is then:

$$\begin{bmatrix} \begin{pmatrix} f_1 \\ f_2 \\ e_2 \\ -e_1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ -w^e(1) \\ -w^e(0) \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} -D \\ w^e(1) \end{pmatrix} \begin{pmatrix} M^T \\ w^e(0) \end{pmatrix}^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} \begin{pmatrix} \tilde{e}_1 \\ e_1 \\ f_2 \\ f_2 \end{pmatrix} \end{bmatrix} \quad (24)$$

2) *Constitutive relations:* The constitutive relations are obtained from the conservation of the power product (e, f) in the energy storage and dissipation relations.

a) *Energy storage elements:* The thermal energy storage reads:

$$\frac{dH_T}{dt} = \int_{\Pi} e_1 f_1 = \tilde{\mathbf{e}}_1^T \mathbf{f}_1 \quad (25)$$

with

$$f_1 = n\sqrt{g}T(D_t \mathbf{s}) = \frac{3}{2}n\sqrt{g} \frac{DT}{Dt}$$

Denoting

$$\bar{\mathcal{F}} \ni e_{ex} = \frac{3}{2}n\sqrt{g}T = \frac{3}{2}n\sqrt{g}e_1$$

the storage constitutive relation may be written:

$$\begin{aligned} \frac{dH_T}{dt} &= \int_{\Pi} e_1 f_1 = \int_{\Pi} \left(\frac{3}{2}n\sqrt{g} \right)^{-1} e_{ex} f_1 \\ &= \mathbf{e}_{ex}^T G_T \mathbf{f}_1 \end{aligned} \quad (26)$$

with

$$G_{Tij} = \int_0^1 \left(\frac{3}{2}n\sqrt{g} \right)^{-1} w_i^f(x) w_j^f(x), \quad G_T = G_T^T$$

and

$$e_{ex}^T = \sum_{k=1}^{N-1} \left(\mathbf{e}_{ex}^T(t) \right)_k w_k^f(x)$$

We deduce:

$$\tilde{\mathbf{e}}_1 = G_T \mathbf{e}_{ex} \quad (27)$$

In our case, n is time invariant, and $\partial_t e_{ex} = f_1$.

b) *Energy dissipative element:* The dissipated power is:

$$\mathbb{P}_d = \int_{\Pi} e_2 f_2 = \mathbf{e}_2^T \tilde{\mathbf{f}}_2 \quad (28)$$

From the Fourier's law: $f_q = \star \chi F \Rightarrow f_2 = n\chi \frac{\sqrt{g\theta g_\phi}}{\sqrt{g\rho}} e_2$, one gets:

$$\begin{aligned} \mathbb{P}_d &= \int_{\Pi} e_2 f_2 = \int_{\Pi} e_2 n\chi \frac{\sqrt{g\theta g_\phi}}{\sqrt{g\rho}} e_2 \\ &= \mathbf{e}_2^T R_T \mathbf{e}_2 \end{aligned} \quad (29)$$

with $R_{Tij} = \int_0^1 n\chi \frac{\sqrt{g\theta g_\phi}}{\sqrt{g\rho}} w_i^f(x) w_j^f(x)$; $R = R^T > 0$ the dissipative matrix. Hence:

$$\tilde{\mathbf{f}}_2 = R_T \mathbf{e}_2 \quad (30)$$

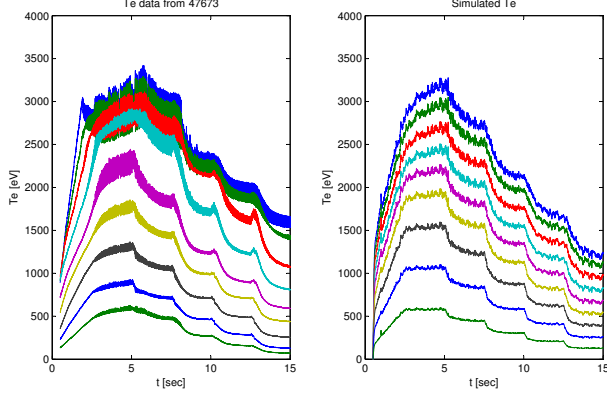


Fig. 2. Electronic temperature profile comparison

Finally the 0D PCH model for thermal domain becomes:

$$\begin{pmatrix} \partial_t \mathbf{e}_{ex} \\ 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & J_{T1} \\ J_{T2} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & R_T^{-1} \end{pmatrix} \right] \begin{pmatrix} G_T \mathbf{e}_{ex} \\ \mathbf{f}_2 \end{pmatrix} + \begin{pmatrix} \bar{S} \\ J_{T4} T_1 \end{pmatrix} \quad (31)$$

The matrices J_{T1} , J_{T2} with $J_{T1} = -J_{T2}^T$ represent the spatial derivation ∂_z in the new finite dimension system, J_{T4} is related to the boundary coefficient, T_1 is the value of the average temperature at the boundary $z = 1$.

The table III here below compares the eigenvalues of the proposed discretization scheme with the theoretical ones, and the ones from a finite difference scheme.

Theoretical eigenvalues	Numerical eigenvalues			FD with N=50
	N=5	N=8	N=10	
				7.234097
-120.483040	-120.702770	-120.524486	-120.504490	-118.141417
-634.817965	-641.754030	-635.901271	-635.313587	-748.190997
-1560.145974	-1619.180495	-1567.140230	-1563.180841	-1827.815893
-2896.672592	-4254.170952	-2924.070376	-2907.791802	-3355.212061
-4644.422992		-4731.088204	-4675.9536082	-5323.638574
-6803.403186		-7069.961051	-6881.489738	-7724.736935
-9373.615177		-15583.5718679	-9556.885491	-10548.547081
-12355.059784			-12807.004273	-13783.474351
-15747.737391			-27777.389374	-17416.282235

TABLE III

EIGENVALUES WITH SYMPLECTIC AND FINITE DIFFERENT DISCRETIZATION SCHEME, WITH $n = 2 \cdot 10^{19} [m^{-3}]$, $\chi = 5 [m^2/s]$

The eigenvalues for the energy transport in table III are indeed hundred times larger than the ones for the magnetic flux transport in table II. The hypothesis of different time scales in plasma transport phenomena is correctly represented by the obtained 0D control model in PCH form. The figure 2 shows the temperature profile comparison between the simulation results (dash line) and the experimental data from the tokamak Tore Supra ohmic shock TS #47673 (solid line). The difference between the two profiles at the center is probably caused by the bad estimation of diffusion coefficient χ , since there's no exact formula but the one proposed in [4].

IV. COUPLED CONTROL MODEL

The coupled control model is made from the electromagnetic model in (22) and the thermal diffusion model (31):

$$\begin{pmatrix} \dot{\mathbf{a}} \\ \dot{\mathbf{b}} \\ \dot{\mathbf{e}}_{ex} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & J_1 & 0 & 0 \\ -J_2 & 0 & 0 & J_{T1} \\ 0 & 0 & J_{T2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{e}_{ex} \\ 0 \end{pmatrix} - \begin{pmatrix} R^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_T^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{e}_{ex} \\ 0 \end{pmatrix} + \begin{pmatrix} G_{el} \mathbf{d} \\ G_{mg} \mathbf{b} \\ G_T \mathbf{e}_{ex} \\ \mathbf{f}_2 \end{pmatrix} + \begin{pmatrix} -\mathbf{J}_{ext} - \mathbf{J}_{bs} \\ J_4 \frac{V_{loop}}{S} \\ J_{T4} T_1 \end{pmatrix} \quad (32)$$

It is written in PCH formulation

$$\dot{x} = [\mathcal{J}(x) - \mathcal{R}(x)] \frac{\partial H}{\partial x}(x) + gu$$

with $\mathcal{J} = -\mathcal{J}^T$, $\mathcal{R} = \mathcal{R}^T \geq 0$. The total Hamiltonian is $\mathbb{H} = \mathbb{H}_{EM} + \mathbb{H}_T$. The couplings between the two submodels are represented implicitly via the thermal source S , the bootstrap current j_{bs} , and the dissipation $R(\eta)$ and $R_T(\chi)$:

- the source term \bar{S} includes the Joule effect $S_{Joule} = \eta J_{total} J_{\Omega}$ and the external heating source term S_{heat} (a control signal which doesn't affect directly the electromagnetic domain).
- the bootstrap current j_{bs} is one of the MHD couplings. Usually it is assumed to be derived from the temperature $T (= e_1)$ and its spatial derivative $\partial_x T (= e_2)$

$$j_{bs} = \frac{1}{b} (C_1 T + C_2 \partial_x T) \quad (33)$$

where C_1 and C_2 are (for instance) the constants determined in [21]. In the case of a constant particle density n , noting that $\mathbf{e}_2 = R_T^{-1} \tilde{\mathbf{f}}_2$, we can deduce:

$$\begin{aligned} \mathbf{J}_{bs} &= \frac{1}{b} (C_1 \mathbf{e}_1 + C_2 \mathbf{e}_2) \\ &= \frac{1}{b} (C_1 + C_2 J_{T2}) \mathbf{e}_1 \\ &= \frac{1}{b} (C_1 + C_2 J_{T2}) (-J_{T1} R_T J_{T2})^{-1} \bar{S} \\ &= \frac{1}{b} (C_1 + C_2 J_{T2}) (-J_{T1} R_T J_{T2})^{-1} (S_{Joule} + S_{heat}) \end{aligned} \quad (34)$$

Once the linearization is taken into account, the bootstrap value varies around the equilibrium $\mathbf{J}_{bs,d}$, one can suppose $\mathbf{J}_{bs} = C_{bs,d} (S_{Joule} + S_{heat})$, where $C_{bs,d}(\mathbf{b}_d)$ is the bootstrap constant depending on the steady state of the magnetic field B_{θ} , and the Joule effect S_{Joule} is considered measurable from the system itself (Ohmic current and external non-inductive current are well-known).

- the resistivity coefficient $\eta(T_e)$ and the thermal diffusion coefficient $\chi(T, \bar{B}_{\theta})$ whose scale laws are given in [21], depend on many factors, including the electronic temperature T_e and the magnetic field B_{θ} . In a first approximation, one can consider the linearization values of these coefficients around the equilibrium to simplify the calculus.

Once the coupled system is defined in PCH form in (32), an IDA-PBC (as in [17]) can be applied. For the simulation, the RAPTOR code with the configuration of TCV (Tokamak of Configuration Variable) is used. The safety factor q is regulated at three positions $z = (0.1, 0.25, 1)$ by three

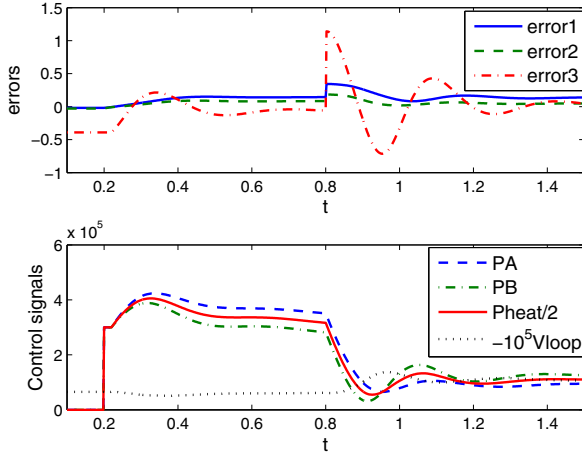


Fig. 3. Feedforward control: errors at three reference points (top) and feedforward control signals (bottom)

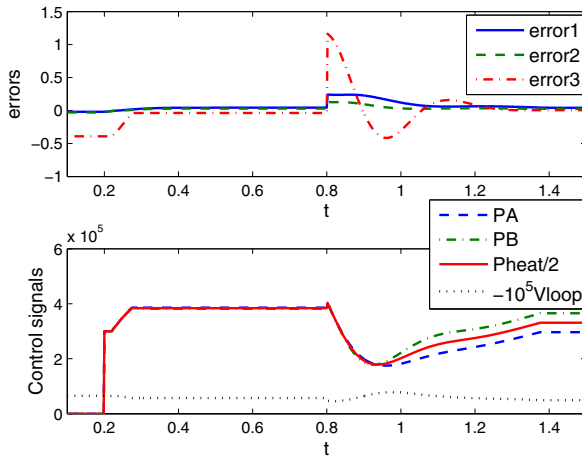


Fig. 4. Feedback control: errors at three reference points (top) and feedback control signals (bottom)

actuators $(P_{ext}, V_{loop}, S_{heat})$. In this case T_1 , a control signal at the boundary, is fixed constant, for instant, $T_1 = 0$. Some simulation results are figured out hereafter, the detail of the IDA-PBC controller design can be found in [19].

Figures 3 and 4 show the results obtained with feedforward and feedback controls. The controller starts at $t = 0.2s$ with the initial values $(P_{ext}, V_{loop}, P_{heat})_{init} = (0, -0.65V, 300kW)$, whereas at $z = (0.1, 0.25, 1)$ the reference q profile is set as $q_a = (0.85, 1, 6.3)$. Then at $t = 0.8s$, the reference is changed to $q_b = (1.05, 1.1, 7.5)$.

The feedforward bring the q profile to the reference values, but the actuator values as well as the q profile oscillate around the equilibrium due to the parameter linearization and approximation discussed in subsection 3.2. The feedback however makes effort to improve the result by continuing to react significantly on P_{heat} .

The profiles of q and T are also showed up in the figure 5 with the q_a target at $t = 0.7s$ and the figure 6 with the q_b

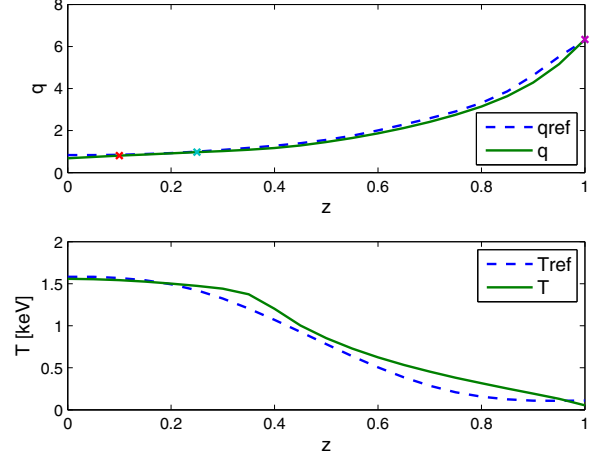


Fig. 5. q_a and T_a profiles at $t = 0.7s$

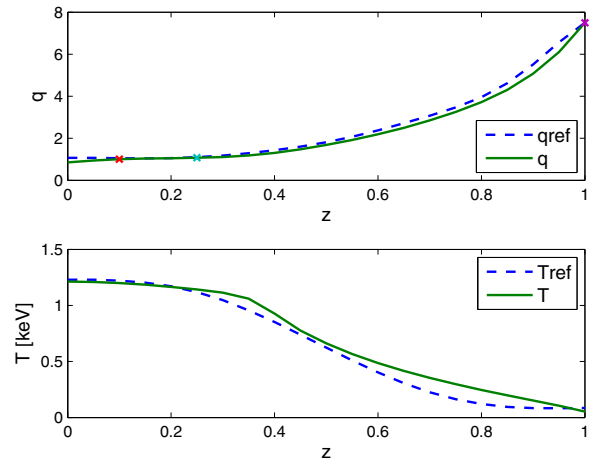


Fig. 6. q_b and T_b profiles at $t = 1.5s$

target at $t = 1.5s$. These profiles between the steady states defined by feedforward control and the one of RAPTOR don't totally match each other but at three reference positions for q profile.

V. CONCLUSION

A symplectic geometric reduction methodology is proposed in this paper. It consists in performing a 3D-1D geometric reduction and then a 1D-0D symplectic discretization. The coupled resistive flux and thermal diffusion equations are used as an illustration of the approach. This example confirms the effectiveness of the reduction methodology in preserving some physical behaviors of the actual 3D model, e.g. spectral (eigenvalues and eigenfunctions) or energetic (conservativeness and passivity) properties. Moreover, the obtained discretized model for the Therms-Magneto-Hydrodynamical model of the plasma is stated in PCH form and, as a consequence, a classical IDA-PBC controller has been designed and preliminary control results presented.

A more detailed model, including separated ionic and electronic thermal transport phenomena as well as nuclear reactions (featuring the burn-control problem) are among the prospects of this work.

- [21] E. Witrant, E. Joffrin, S. Brémond, G. Giruzzi, D. Mazon, O. Barana, and P. Moreau. A control-oriented model of the current profile on tokamak plasma. *Plasma Physics and Controlled Fusion*, 49:1075–1105, 2007.

REFERENCES

- [1] J. Blum. *Numerical Simulation and Optimal Control in Plasma Physics*. Gauthier-Villars, 1989.
- [2] S.R. de Groot and P. Mazur. *Non-equilibrium thermodynamics*. Dover Books on Physics. Dover Publications, 2nd edition edition, 1984. ISBN 0486647412.
- [3] F.L. Waelbroeck E. Tassi, P.J. Morrison and D. Grasso. Hamiltonian formulation and analysis of a collisionless fluid reconnection model. *Plasma Physics and Controlled Fusion*, (50), may 2008.
- [4] M. Erba, T. Aniel, V. Basiuk, A. Becoulet, and X. Litaudon. Validation of a new mixed bohm/gyro-bohm for electron and ion heat transport against the iter, tore supra and start database discharges. *Nuclear Fusion*, 6(7):2835–2839, July 1998.
- [5] F. Felici and O. Sauter. Non-linear model-based optimization of actuator trajectories for tokamak plasma profile control. *Plasma Physics and Controlled Fusion*, 54, 2012.
- [6] T. Frankel. *The Geometry of Physics : an Introduction*. Cambridge University Press, Cambridge, 2nd edition edition, 2004. ISBN 0-521-53927-7.
- [7] P. J. Morrison. Hamiltonian description of the ideal fluid. *Rev. Mod. Phys.*, 70(2):467–521, Apr 1998.
- [8] P. J. Morrison and J. M. Greene. Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics. *Phys. Rev. Lett.*, 45(10):790–794, Sep 1980.
- [9] R. Moulla, L. Lefèvre, and B. Maschke. Pseudo-spectral methods for the spatial symplectic reduction of open systems of conservation laws. *Journal of Computational Physics*, 231(4):1972–1992, December 2012.
- [10] R. Ortega, A.J. van der Schaft, F. Castanos, and A. Astolfi. Control by interconnection and standard passivity-based control of port-hamiltonian systems. *IEEE Trans. on Automatic Control*, 53(11):2527–2542, 2008.
- [11] E. Schuster, M. Krstic, and G. Tynan. Nonlinear lyapunov-based burn control in fusion reactor. *Fusion Engineering and Design*, (63-64):569–575, 2002.
- [12] E. Schuster, M. Krstic, and G. Tynan. Burn control in fusion reactors via nonlinear stabilization techniques. *Fusion Science and Technology*, (43), 2003.
- [13] A.J. van der Schaft. Theory of port hamiltonian systems. *Network Modeling and Control of Physical Systems, DISC*, Chapter 1,2, April 2005.
- [14] S. Stramigioli V.Duindam, A. Macchelli and H. Bruyninckx eds. *Modeling and Control of Complex Physical Systems - The Port-Hamiltonian Approach*. Springer, Sept. 2009. ISBN 978-3-642-03195-3.
- [15] Trang.N.M. Vu, L. Lefèvre, and B. Maschke. Port-hamiltonian formulation for systems of conservation laws: application to plasma dynamics in tokamak reactors. *4th IFAC Workshop on Lagrangian and Hamiltonian Methods for Non Linear Control, Bertinoro, Italy*, August 29-31 2012.
- [16] Trang.N.M. Vu, L. Lefèvre, R. Nouailletas, and S. Brémond. Geometric discretization for a plasma control model. *IFAC Joint conference: 5th Symposium on System Structure and Control; Grenoble, France*.
- [17] Trang.N.M. Vu, L. Lefèvre, R. Nouailletas, and S. Brémond. An ida-pbc approach for the control of 1d plasma profile in tokamaks. *52nd IEEE Conference on Decision and Control, Florence, Italia*, December 10-13 2013.
- [18] Trang.N.M. Vu and L.Lefèvre. Material balance and closure equations for plasmas in tokamaks. *IFAC Workshop on Thermodynamic Foundations of Mathematical Systems Theory*, July 13-16 2013.
- [19] Trang.N.M. Vu, R. Nouailletas, L. Lefèvre, S. Brémond, and F. Felici. Ida-pbc control for the coupled plasma poloidal magnetic flux and heat radial diffusion equations in tokamaks. *19th World Congress of the International Federation of Automatic Control, Cape Town, South Africa*, August 24-29 2014.
- [20] J. Wesson. *Tokamaks. Third edition*. Oxford Science Publications, 2004.