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One-dimensional coverage by unreliable sensors

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***Abstract*— This abstract deals with the problem of optimally placing unreliable sensors in a one-dimensional environment. We assume that sensors can fail with a certain probability and we minimize the expected maximum distance from any point in the environment to the closest active sensor. Under these assumptions, we provide a computational method for the optimal placement and we estimate the relative quality of equispaced and random placements, showing that the former achieves optimality when the number of sensors goes to infinity.**

I. Introduction

We consider the problem of placing sensors in a given environment in such a way as to provide the best coverage of it. Several variations of this optimization problem have been studied in the literature (see for instance [1] for a control-theoretic perspective), but little attention has been paid to the negative effects of sensor failures on the coverage performance. Actually, sensor failures may deteriorate the performance of the sensor network and it is not even clear if optimal solutions derived for the case without failures retain good properties in other cases. Indeed, it is shown in [2] that the presence of failures changes the optimization problem and leads to optimal solutions which are qualitatively different. The striking difference between optimal solutions with and without failures is also the topic of [3]. In the latter paper, which has originated our interest in this topic, the author assumes that the number of failures is known a priori and obtains optimal solutions which are shaped by this number.

Contribution

In this work, we extend the well-known problem of optimal disk-coverage (cf. [4]) to allow for unreliable sensors, under a probabilistic failure model which does not assume any *a priori* information about the number or the location of the failures. After formally defining the optimization problemin Section 2, we present our main results. In Section 3 we state that the problem at hand can be cast as a linear optimization problem, albeit with a number of variables growing exponentially with n. This fact allows nevertheless for a computational solution that is tractable if n is not large. In Sections 4 and 5, we provide analytical results in the limit of the number of sensors growing to infinity. Namely, we determine the performance of the equispaced senor placement and show that it is nearly optimal, whereas a random placement is worse.

The proofs of our results are omitted and are available in the report [5]. We note that a closely related and essentially equivalent model of unreliable coverage was recently proposed in [6]: the results that we present here extend and refine those in [6]. Moreover, our results also bear consequences for the failure model in [3]: a detailed discussion can be found in the full-length version.

II. Problem definition

We assume to have a set of sensors indexed in $[n] = \{1, \dots, n\}$. Each sensor can fail with probability p, independently from the others. For each sensor placement $x \in [0,1]^n$ we consider the coverage costs defined as follows. When sensors cannot fail ($p = 0$) we define

$$C_0(x) = \max_{s\in[0,1]} \min_{j\in[n]} |s - x_j|.$$

It is known that the equispaced placement of n sensors, namely

$$x^{\mathrm{eq}} = \frac{1}{2n}(1, 3, \dots, 2n-1),$$

is the optimal solution in this case, achieving a cost $C_0(x^{\mathrm{eq}}) = \frac{1}{2n}$. When instead sensors can fail with positive probability we let A denote the (random) set of active sensors and we define the event $E_A = \{A \text{ is the set of sensors alive}\}$. Then, we consider the expected cost incurred by the placement

$$C(x) = \Pr(E_\emptyset) + \sum_{A\in(2^{[n]}\setminus\emptyset)} \Pr(E_A) \max_{s\in[0,1]} \min_{j\in A} |s - x_j|, \quad (1)$$

where $2^{[n]}$ denotes the set of subsets of $[n]$.

In what follows we assume, for simplicity and without loosing generality, that x is ordered $x_1 \le x_2 \le \dots$. This assumption implies for instance that

$$C_0(x) = \max\{x_1, (1 - x_n), \frac{1}{2} \max_{i=1,\dots,n-1} (x_{i+1} - x_i)\}$$

Moreover, we denote by $|A|$ the cardinality of the set of active sensors A, and by A_k the k^{th} smallest index present in the set A, for $k = 1, \dots, |A|$.

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III. Formulation as a linear program

Theorem 1: Let n be an integer and $p \in [0,1)$. The (ordered) vector $x^* \in [0,1]^n$ minimizes $C(x)$ defined in (1) if and only if there exists a vector $w^* \in \mathbb{R}^{2^n-1}$ such that (x^*, w^*) is an optimal solution to the following linear program:

$$\min \sum_{A \in (2^{[n]} \setminus \emptyset)} \Pr(E_A) w_A$$

$$\begin{aligned}
s.t.\quad & 0 \le x_1 \le x_2 \le \cdots \le x_n \le 1, \\
& \text{and } \forall A \in (2^{[n]} \setminus \emptyset), \\
& w_A \ge \frac{1}{2}(x_{A_{k+1}} - x_{A_k}), \quad \text{for } k = 1, \dots, |A|-1, \\
& w_A \ge x_{A_1}, w_A \ge 1 - x_{A_{|A|}}
\end{aligned}$$

The formulation as a linear program allows us to exactly calculate the optimal placements for different parameters. In Figure 1 we present the evolution of the optimal placement as p changes between 0 and 1. Observe that the evolution of the optimal x with p is piecewise constant. This can actually be explained by the structure of the linear program in Theorem 1. Indeed, one can see that the constraints do not depend on p, which only affects the cost function. For any p, one can thus always find an optimal (x^*, w^*) among the finitely many vertices of the polytope defined by these constraints. It is therefore natural to observe only finitely many different optimal solutions. We can also see that the dependence on p is rather complex and it is not clear how, or if, one could provide a simple exact description of the optimal location of the sensors as a function of n and p.

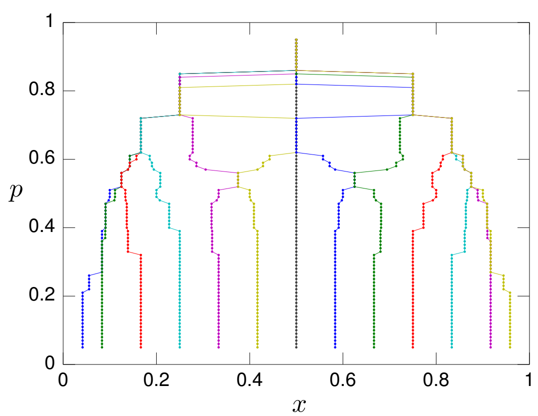

Fig. 1. Optimal sensor placement for $n = 13$ sensors and varying p.

Nevertheless, we can explicitly find the optimal placement when p is close to either 0 or 1. Indeed, we know that when $p = 0$, then the optimal placement is x^{eq}. Since the optimal placement is piecewise constant p, we are able to prove the following result.

Proposition 1: There exists a non-trivial neighborhood of $p = 0$ where x^{eq} is optimal. However, the size of such neighborhood is not larger than c_0/n, for some constant c_0.

Similarly, we also observe that for p close to one, the optimal choice is to place all sensors in a single cluster at $1/2$.

Proposition 2: There exists a non-trivial neighborhood of $p = 1$ where the single-cluster placement is optimal. However, there exists a constant $c_1 > 0$ such that for all $p < 1 - c_1/n$ the single-cluster placement is not optimal.

IV. Performance of the equispaced placement

We have already observed that the equispaced placement is optimal in the case there are no failures, achieving a cost $C_0(x^{\text{eq}}) = \frac{1}{2n}$. In the case of positive failure probability, we can prove that the cost of the equispaced placement is nearly optimal. The proof of this result relies on an alternative version of (1) defined on the circle, for which the equispaced solution is actually optimal.

Theorem 2 (Cost of equispaced): Let $p \in (0,1)$ and let x^{opt} denote the optimal sensor placement for this p. Then,

$$C(x^{\text{eq}}) = \frac{1}{2\log p^{-1}} \frac{\log n}{n} + O\left(\frac{1}{n}\right) \qquad \text{for } n \to +\infty \quad (2)$$

and moreover for every $n \in \mathbb{N}$

$$C(x^{\text{eq}}) \le C(x^{\text{opt}}) + \frac{p}{1-p} \frac{2}{n}.$$

Equation (2) is illustrated via simulations in Figure 2. This result highlights the following two remarkable facts: (i) the order of growth of $C(x^{\text{eq}})$ is only worse than the order of $C_0(x^{\text{eq}})$ by a logarithmic factor; and (ii) x^{eq} asymptotically achieves the optimal performance, since $\frac{C(x^{\text{eq}})}{C(x^{\text{opt}})} \to 1$.

V. Performance of a random placement

We consider x^{rand} a random placement of the sensors. More precisely, the positions $x_1^{\text{rand}}, \dots, x_n^{\text{rand}}$ are i.i.d. random variables, uniformly distributed in the interval $[0,1]$. The following result holds true for the expected value (with respect to the random sensor placement) of the cost defined in (1) (which itself was averaged with respect to sensor failure).

Theorem 3 (Cost of random placement): Let x^{rand} be the above-defined random sensor placement. Then,

$$\mathbb{E}[C(x^{\text{rand}})] = \frac{1}{2(1-p)} \frac{\log n}{n} + O\left(\frac{1}{n}\right) \qquad \text{for } n \to \infty\,.$$

Theorem 3 describes the asymptotic behaviour of $\mathbb{E}[C(x^{\text{rand}})]$. We can argue that $\mathbb{E}[C(x^{\text{rand}})]$ has the same order of growth as $C(x^{\text{eq}})$, but with a larger constant, thus leading to an asymptotically worse performance. This comparison is illustrated via simulations in Figure 2.

VI. Conclusion

In this abstract, we have summarized some of our findings on a new model of coverage by unreliable sensors. This extends the well-known disk-coverage problem to allow for independent sensor failures. It comes out that the resulting optimization problem is a linear program, thus solvable by standard methods. However, since the space of possible solutions grows exponentially with the size of the problem,

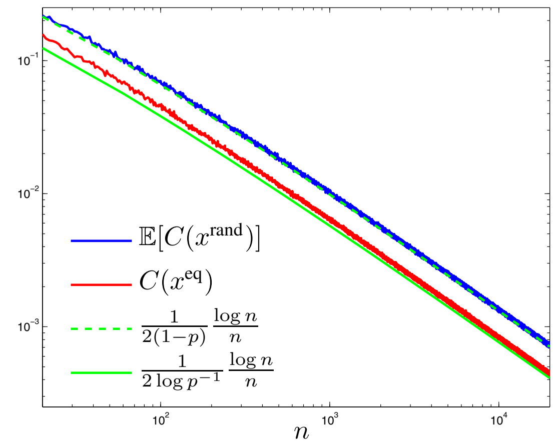

Fig. 2. The plot compares $\mathbb{E}[C(x^{\mathrm{rand}})]$, $C(x^{\mathrm{eq}})$, and their approximations according to Theorems 3 and 2, respectively. The expected costs are simulated as a Monte Carlo average over 100 independent realizations of the placements and of the failures, assuming $p = 0.3$.

we do not know whether a solution can be found in a polynomial time.

Although the optimal solution can possibly be hard to find, and even if its properties are difficult to describe precisely, we have been able to point at a suboptimal solution which asymptotically achieves the optimal performance as the number of sensors grows to infinity. Interestingly, this exemplary solution is just the equally-spaced placement, which is optimal in the case without failures. We have also compared the performance of random sensor placement to the equally-spaced setting to find that there is a constant factor deterioration of the cost. Still, the rate of growth is the same as the number of sensors increases.

Finally, we want to underline that we have defined our problem in a one-dimensional environment: in fact, our results hinge on this assumption. The corresponding analysis in higher dimension is thus a natural and nontrivial open problem.

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