21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Design of simultaneous compensators for a segment of systems by using an interpolation approach with stable polynomial interpolants.

Christophe FONTE[1,2], Houda MEDDEB[1,2] and Michel ZASADZINSKI[1,2]

Abstract— **In this paper, we present the design of a simultaneous compensator for a segment of systems based on an interpolation method with stable polynomial interpolants. This problem leads to formulate conditions of polynomial divisibility in the case of the simultaneous control as a polynomial interpolation issue. The feasibility and infeasibility of the approach is also analyzed. Finally, an algorithm permitting to compute a simultaneous controller stabilizing a segment of systems is given.**

Index Terms— **simultaneous stabilization, polynomial interpolation, interlacing property, segment of systems.**

Notations : *The degree of a real polynomial A is denoted $\delta(A)$. The set of Hurwitz polynomials is denoted H.*

I. Introduction

The simultaneous control was essentially raised in the ring of stable rational functions RH_∞ but some works have shown the limitations of this approach, see [1], [2]. Hence, the need to develop other approaches to synthesize simultaneous compensators for example in the space of stable polynomials in order to consider problems that we can not deal with rational functions in RH_∞. That is the case of the issue of the stabilization of a segment of systems with a simultaneous compensator.

The question of the simultaneous stabilization in the polynomial space has been formulated as an optimization problem see [3]. In this context, [3] gives bilinear matrix inequality (BMI) conditions for stabilizing simultaneously a finite family of n systems when the order of the controller is fixed. That is this method that we have initially adopted to deal with the case of the simultaneous stabilization of a segment of systems, see [4]. Unfortunately, there does not exist off-the-shelf algorithms for solving this non-convex problem formulated as BMIs. However, an algorithm has been developed for approximating the BMI constraints as a special linear matrix inequality (LMI) problem with a rank-one constraint, we can not be pleased of this close solution. This is the reason why we have looked for another approach using no approximation methods.

Similarly to the simultaneous control formulated in the ring of stable rational functions as an unit interpolation issue in RH_∞, see [5], [6], [7] and [8], we can also express the simultaneous control as a polynomial interpolation problem in the space of stable polynomials. The originality of our contribution is to give a tractable method to synthesize simultaneous compensators of a segment of systems by using a polynomial interpolation method producing stable polynomial interpolants when their even (or odd) parts are fixed. The proposed approach highlights results that have been studied in [9] and [10] in regards to the polynomial interpolation and the polynomial stability. Solutions have been adapted to the context of the simultaneous stabilization of a convex family of SISO (single-input single-output) linear systems defined as a segment of systems.

This question of the simultaneous stabilization of a segment of systems was initially tackled by [11], [12] and [13] but no tractable and complete conditions to check the simultaneous stabilizability of such systems have been shown. [14] and [15] have addressed the question of the strong stabilization for this class of systems. These authors have stated existence conditions of stable compensators being able to stabilize strongly each element of this family. That does not imply existence conditions of a single controller stabilizing the whole set of systems belonging to this segment. Conditions for the simultaneous stabilizability of such plants have been given in [16]. We can note that for the stabilization of a segment of systems, [17] suggests a parameter-dependent linear controller stabilizing the range of transfer functions corresponding to the interval of parameter values. Some applications using this class of models are treated in [18] and [19].

The paper is organized as follows. In section II, the problem of the simultaneous stabilization of two plants with the same even part for each closed loop characteristic polynomial is stated. Thereafter, an application of this case of simultaneous stabilization is shown for a segment of systems. In section III, some results are studied in the polynomial space to solve this problem. Conditions for the interpolation of stable polynomials with a fixed even part (or odd part) are presented in section IV. The feasibility and infeasibility of the method is also analyzed in section IV-C. Section V contains a brief result giving existence conditions of proper simultaneous compensators. Finally in section VI, a simple algorithm is obtained permitting to design simultaneous controllers for a segment of systems.

II. Problem formulation

A. Preliminaries

***Definition** 1: We say that a proper compensator $C_i = X_i/Y_i$ stabilizes a plant $G_i = N_i/D_i$ iff $\Phi(G_i, C_i) \in$ H where*

$$\Phi(G_i, C_i) = N_i X_i + D_i Y_i$$

[1] Université de Lorraine, CRAN, UMR 7039, 186 rue de Lorraine, Cosnes et Romain, 54400, France

[2] CNRS, CRAN, UMR 7039, 186 rue de Lorraine, Cosnes et Romain, 54400, France

ISBN: 978-90-367-6321-9

and N_i, D_i, X_i, Y_i are real polynomials. □

Consider two proper plants $G_i = N_i/D_i$ and $G_j = N_j/D_j$ and a controller C defined by

$$C = \frac{DQ\, X_i + NQ\, D_i}{DQ\, Y_i - NQ\, N_i} \tag{1}$$

where NQ is a real polynomial, $DQ \in \mathrm{H}$ and $C_i = X_i/Y_i$ is a proper compensator satisfying $\Phi(G_i, C_i) \in \mathrm{H}$.

Let us express a simple condition to stabilize G_j with the compensator C.

***Theorem** 1: The compensator C stabilizes G_j iff there exists a polynomial NQ satisfying (2)*

$$\left(\Phi(G_j, \widetilde{C_i}) + NQ\,\Delta\right) \in \mathrm{H}, \tag{2}$$

where $\Delta = N_j D_i - D_j N_i$ and $\widetilde{C_i} = DQ\, X_i / DQ\, Y_i$.

Proof : According to Definition 1, we write $\Phi(G_j, C)$ as

$$\Phi(G_j, C) = N_j(DQ\, X_i + NQ\, D_i) + D_j(DQ\, Y_i - NQ\, N_i)$$

If the compensator C satisfies $\Phi(G_j, C) \in \mathrm{H}$ then relation (2) holds. Conversely, if relation (2) is true, then there exists a compensator C defined by (1) such that $\Phi(G_j, C) \in \mathrm{H}$. □

***Definition** 2:* Simultaneous stabilization of two plants with a compensator C. *Consider two proper plants G_i and G_j. If there exists a compensator C such that $\Phi(G_i, C) \in \mathrm{H}$ and $\Phi(G_j, C) \in \mathrm{H}$ then G_i and G_j are said simultaneously stabilizable.* □

***Corollary** 1: The compensator C given in (1) is a simultaneous compensator for the systems G_i and G_j iff relation (2) holds.*

Proof : By hypothesis the controller $\widetilde{C_i}$ stabilizes G_i, then we have $\Phi(G_i, C) \in \mathrm{H}$. Moreover if relation (2) is true then $\Phi(G_j, C) \in \mathrm{H}$. Consequently C is a simultaneous compensator for the systems G_i and G_j. Conversely, if C is a simultaneous compensator for the systems G_i and G_j then $\Phi(G_j, C) \in \mathrm{H}$ and relation (2) holds. □

B. Problem statement

In this paper, we study the existence of a simultaneous compensator C for two plants G_i and G_j in the case where the two closed loop characteristic polynomials $\Phi(G_i, C)(s) = \Phi(G_i, C)^e(s^2) + s\Phi(G_i, C)^o(s^2)$ and $\Phi(G_j, C)(s) = \Phi(G_j, C)^e(s^2) + s\Phi(G_j, C)^o(s^2)$ satisfy $\Phi(G_i, C)^e = \Phi(G_j, C)^e$ or $\Phi(G_i, C)^o = \Phi(G_j, C)^o$), i.e. $\Phi(G_i, C)$ and $\Phi(G_j, C)$ have the same even part (or the same odd part).

In the sequel of this paper, the two polynomials $\Phi(G_i, C)$ and $\Phi(G_j, C)$ are assumed to be of same degree. Let us remark that $\Phi(G_i, C)$ and $\Phi(G_i, \widetilde{C_i})$ have the same even parts (and the same odd parts) thus $\Phi(G_i, C) = \Phi(G_i, \widetilde{C_i})$.

***Corollary** 2: There exists a simultaneous compensator C defined by (1) for the two systems G_i and G_j such that the two polynomials $\Phi(G_i, C)$ and $\Phi(G_j, C)$ have the same even part (or odd part) iff there exists a polynomial NQ satisfying the following conditions (3).*

$$\begin{cases} \left(\Phi(G_j, \widetilde{C_i}) + NQ\,\Delta\right) \in \mathrm{H}, \\ \Phi(G_j, C)^e = \Phi(G_i, \widetilde{C_i})^e \end{cases} \tag{3}$$

Proof : This result is a direct consequence of Corollary 1. □

In the following Theorem, we show the interest of dealing with this issue. So, we prove that if there exists a simultaneous compensator C that stabilizes the plants G_i and G_j such that $\Phi(G_i, C)^e = \Phi(G_j, C)^e$ then this compensator stabilizes the segment of systems G_λ given by (4) where $\lambda \in [0, 1]$.

$$G_\lambda = \frac{\lambda N_i + (1-\lambda) N_j}{\lambda D_i + (1-\lambda) D_j} \tag{4}$$

Let us remark that the segment of systems (4) is a continuum of systems when $\lambda \in [0, 1]$ specified by these two endpoints G_i and G_j.

***Theorem** 2: If C is a simultaneous compensator for the plants G_i and G_j such that $\Phi(G_i, C)$ and $\Phi(G_j, C)$ have the same even parts (or odd parts) such that their leading coefficients have the same sign then C stabilizes the segment of systems G_λ given by* (4).
(*Note that the leading coefficient of a polynomial is defined as its coefficient of the highest degree*).

Proof : Consider C a compensator stabilizing simultaneously the systems G_i and G_j such that $\Phi(G_i, C)$ and $\Phi(G_j, C)$ have their leading coefficients of same sign and such that they have the same even part (or same odd part) then the closed-loop characteristic polynomial $\Phi(G_\lambda, C)$ given in (5) is stable for all $\lambda \in [0, 1]$, see [10] corollary 3.

$$\Phi(G_\lambda, C) = \lambda\Phi(G_i, C) + (1-\lambda)\Phi(G_j, C) \tag{5}$$

We deduce that the compensator C stabilizes the segment of systems G_λ given by (4). □

Our contribution in this paper is principally concerned with the theoretical study of existence conditions of a simultaneous linear time invariant controller stabilizing the segment of systems (4) by considering Theorem 2.

III. Existence conditions of a compensator C.

From Corollary 2, the issue of existence of a simultaneous compensator C stabilizing G_i and G_j such that $\Phi(G_i, C)^e = \Phi(G_j, C)^e$ is equivalent to that of existence of a polynomial NQ satisfying relations (3).

It is interesting to note that if $\Delta \in \mathrm{H}$ then an obvious solution at this problem is given by choosing for instance $DQ = \Delta$ and $\Phi(G_j, C) = DQ\, \Phi(G_i, C_i)$ where $\Phi(G_i, C_i)$ is any polynomial in H. These conditions yield to $\Phi(G_i, C)^e = \Phi(G_j, C)^e$. By considering (2), we get a polynomial NQ given by (6)

$$NQ = \Phi(G_i, C_i) - \Phi(G_j, C_i) \tag{6}$$

If $\Delta \notin \mathrm{H}$, the determination of NQ and $\Phi(G_j, C) \in \mathrm{H}$ is not so easy. In this case, there exists a polynomial NQ such that (2) iff there exists a stable polynomial $\Phi(G_j, C)$ such that

$$\Phi(G_j, C) = \Phi(G_j, \widetilde{C_i}) + NQ\,\Delta \tag{7}$$

Consequently $\Phi(G_j, C) - \Phi(G_j, \widetilde{C_i})$ must be divisible by Δ where $\widetilde{C_i} = DQ\, X_i / DQ\, Y_i$. Equivalently, this means that there exists NQ iff there exists a stable polynomial $\Phi(G_j, C)$ that interpolates the values of $\Phi(G_j, \widetilde{C_i})$ at the zeros of Δ with these multiplicities. This provides the following theorem.

***Theorem** 3: There exists a simultaneous compensator C given in* (1) *for the two systems G_i and G_j in the case where $\Phi(G_i, C)$ and $\Phi(G_j, C)$ have the same even part (or odd part) iff there exists a stable polynomial $\Phi(G_j, C)$ with $\Phi(G_j, C)^e = \Phi(G_i, C)^e$ that interpolates the values of $\Phi(G_j, \widetilde{C_i})$ at the zeros of Δ and these multiplicities.*

Proof : To show this result, it suffices to write $\Phi(G_j, C)$ as (7) and to consider the two given polynomials Δ and $\Phi(G_j, \widetilde{C_i})$ as the divisor and the remainder respectively of an euclidean division in the ring of polynomials. Note that the unknown pair $(\Phi(G_j, C), NQ)$ representing the dividend and the quotient of (7) is not necessarily unique. We deduce that Δ must be a factor of $\Phi(G_j, \widetilde{C_i}) - \Phi(G_j, C)$ to assure that NQ is a polynomial where $\Phi(G_j, C) \in H$ such that $\Phi(G_j, C)^e$ is given (i.e $\Phi(G_j, C)^e = \Phi(G_i, \widetilde{C_i})^e$) and $\Phi(G_j, C)^o$ is a polynomial to determine. In this case, there exists a simultaneous compensator C for the two systems G_i and G_j such that $\Phi(G_i, C)$ and $\Phi(G_j, C)$ have the same even part (or odd part). Moreover, $\Phi(G_j, C)(s)$ interpolates the values of $\Phi(G_j, \widetilde{C_i})$ and these multiplicities at the zeros of Δ. Conversely, if $\Phi(G_j, C)$ is a stable polynomial that interpolates the values of $\Phi(G_j, \widetilde{C_i})$ at the zeros of Δ with these multiplicities and satisfies $\Phi(G_i, C)^e = \Phi(G_j, C)^e$ then there exists a polynomial NQ satisfying (7). We deduce that there exists a simultaneous compensator C for the two systems G_i and G_j. □

***Corollary** 3: If there exists a compensator C such that the two following conditions i) and ii) hold then C stabilizes the segment of systems G_λ given by* (4)*.*

i) $\Phi(G_j, C)$ and $\Phi(G_i, C)$ are two stable polynomials with their leading coefficients of same sign,

ii) $\Phi(G_j, C) \in H$ interpolates the values of $\Phi(G_j, \widetilde{C_i})$ at the zeros of Δ with these multiplicities and verifies $\Phi(G_j, C)^e = \Phi(G_i, \widetilde{C_i})^e$.

Proof : It is an immediate consequence of Theorem 2 and Theorem 3. □

From now on, we detail the conditions of Corollary 3.

Let σ_ν be the zeros of Δ and μ_ν the multiplicity of these zeros. Now define $\beta_{\nu,l}$ as

$$\frac{d^l}{ds^l}\Phi(G_j, \widetilde{C_i})(\sigma_\nu) = \beta_{\nu,l}$$

with $l = 0, \ldots \mu_\nu - 1$ and $\nu = 1, \ldots n$ and where the derivative of order zero of $\Phi(G_j, \widetilde{C_i})$ is taken as $\Phi(G_j, \widetilde{C_i})$ itself. The problem amounts to find a stable polynomial $\Phi(G_j, C)$ with $\Phi(G_j, C)^e = \Phi(G_i, \widetilde{C_i})^e$ such that

$$\frac{d^l}{ds^l}\Phi(G_j, C)(\sigma_\nu) = \beta_{\nu,l} \quad (8)$$

In other words, we search to know whether or not there exists a stable polynomial $\Phi(G_j, C)$ with $\Phi(G_j, C)^e = \Phi(G_i, \widetilde{C_i})^e$ such that its functional and derivative values at specified point σ_ν in the complex plane are equal to specified values $\beta_{\nu,l}$. Some existence conditions of stable interpolation polynomial have been shown in [9].

IV. Interpolation conditions of a stable polynomial with a fixed even part.

The purpose of this paper is not to examine the numerous necessary conditions that preserve polynomial stability for this interpolation problem but to derive a general approach to construct stable interpolating polynomials $\Phi(G_j, C)$ such that $\Phi(G_j, C)^e = \Phi(G_i, \widetilde{C_i})^e$. In this sense, it will be given in this section, existence conditions of a stable polynomial $\Phi(G_j, C)$ such that $\Phi(G_j, C)^e = \Phi(G_i, \widetilde{C_i})^e$ and that satisfies the interpolation constraints $\Phi(G_j, C)(\sigma_\nu) = \beta_\nu$ where σ_ν are distinct zeros of Δ with $\nu \in \{1 \ldots n\}$. The feasibility and infeasibility of this problem (3) will be also studied by using a modified version of the Farkas' theorem of alternatives, see [20].

A. Zeros Interlacing Property and stable polynomials

In this first part, a link between roots interlacing and interpolation of stable polynomials is established by using the results of polynomial stability developed by Hermite-Biehler, see [21]. Before to expand this approach, recall the "Zeros Interlacing Property" and define the Cauchy index of a real rational function.

***Definition** 3:* [21], Zeros Interlacing Property.
Let $f^e(u)$ be of degree k and $f^o(u)$ of degree $k-1$ (or k), two real polynomials. Let us assume the roots of these polynomials defined by the following sets

$$\begin{aligned} root(f^e(u)) &= \{a_1, \ldots, a_k\} \\ root(f^o(u)) &= \{b_1, \cdots, b_{k-1}\} \\ (or\ root(f^o(u)) &= \{b_1, \cdots, b_k\}) \end{aligned}$$

Then $f^e(u)$ and $f^o(u)$ interlace iff
- *The roots of $f^e(u)$ and $f^o(u)$ are real and negative and distinct and simple.*
- *The leading coefficients of $f^e(u)$ and $f^o(u)$ have the same sign,*
- *The k roots of $f^e(u)$ alternate with the $k-1$ (or k) roots of $f^o(u)$ as follows*

$$\begin{aligned} & a_1 < b_1 < a_2 < b_2 \ldots a_{k-1} < b_{k-1} < a_k < 0 \\ (or\quad & b_1 < a_1 < b_2 \ldots < b_k < a_k < 0) \end{aligned}$$

□

***Definition** 4:* Cauchy index.
The Cauchy index for a pole u_p of a real rational function R is defined as the number $I_{u_p}(R)$ such that

$$I_{u_p}(R) = \begin{cases} +1, & \text{if } \lim_{u \to {u_p}^-} R(u) = -\infty \ \wedge \ \lim_{u \to {u_p}^+} R(u) = +\infty, \\ -1, & \text{if } \lim_{u \to {u_p}^-} R(u) = +\infty \ \wedge \ \lim_{u \to {u_p}^+} R(u) = -\infty, \\ 0, & \text{otherwise.} \end{cases}$$

□

A generalization over the compact interval (a, b) is direct. We have

$$I_a^b(R) := \sum_{u_p \in (a,b)} I_{u_p}(R)$$

The relationship between Zeros Interlacing Property and Hurwitz stability is emphasized by the Hermite-Biehler's Theorem that we recall below, in the case to the stability study of polynomial $\Phi(G_j, C)$.

***Theorem** 4:* [21], *The three following assertions are equivalent :*

i) *The real polynomial $\Phi(G_j, C)$ is Hurwitz (or stable),*
ii) *All the real parts of the roots of $\Phi(G_j, C)$ are strictly negative,*
iii) *The polynomials $\Phi(G_j, C)^e(u)$ and $\Phi(G_j, C)^o(u)$ verify the Zeros Interlacing Property.*

□

Consequently, by using the Cauchy index previously defined, a rational function is associated to $\Phi(G_j, C)^e(u)$ and $\Phi(G_j, C)^o(u)$ and a simple condition is given for testing the stability of $\Phi(G_j, C)$.

***Theorem** 5: The polynomial $\Phi(G_j, C)$ of degree m ($m = 2k$ or $m = 2k+1$) is stable iff the two following conditions hold*

– *The roots $\{a_1, \ldots, a_k\}$ of $\Phi(G_j, C)^e$ are simple and negative.*
– *All real c_i in (9) are positive with in the case $m = 2k$ $c_0 = 0$.*

$$\Phi(G_j, C)^o(u) = c_0\Phi(G_j, C)^e(u) + \sum_{i=1}^{k} c_i \frac{\Phi(G_j, C)^e(u)}{u - a_i} \tag{9}$$

Proof : According to the Theorem 4, we know that $\Phi(G_j, C)$ is stable iff the roots of $\Phi(G_j, C)^e(u)$ and $\Phi(G_j, C)^o(u)$ interlace as above-noted. Following [21], $\Phi(G_j, C)^e(u)$ and $\Phi(G_j, C)^o(u)$ interlace iff we can write (10)

$$\frac{\Phi(G_j, C)^o(u)}{\Phi(G_j, C)^e(u)} = c_0 + \sum_{i=1}^{k} \frac{c_i}{u - a_i} \tag{10}$$

with

$$c_i = \frac{\Phi(G_j, C)^o(a_i)}{\frac{d\Phi(G_j, C)^e}{du}(a_i)} > 0$$

and with $c_0 = 0$ for $m = 2k$ whether $c_0 > 0$. □

B. Main result

Relation $\Phi(G_j, C)(s) = \Phi(G_j, C)^e(s^2) + s\,\Phi(G_j, C)^o(s^2)$ yields to expression (11) with $c_0 = 0$ in the case $m = 2k$.

$$\beta_\nu = \Phi(G_j, C)^e(\sigma_\nu^2) + \sigma_\nu \left(c_0\Phi(G_j, C)^e(\sigma_\nu^2) + \sum_{i=1}^{k} c_i \frac{\Phi(G_j, C)^e(\sigma_\nu^2)}{\sigma_\nu^2 - a_i} \right) \tag{11}$$

Relation (11) implies (12).

$$\begin{cases} \dfrac{\beta_1}{\prod_{i=1}^{k}(\sigma_1^2 - a_i)} - 1 = c_0\sigma_1 + c_1\dfrac{\sigma_1}{\sigma_1^2 - a_1} + \ldots + c_k\dfrac{\sigma_1}{\sigma_1^2 - a_k} \\ \vdots \qquad\qquad \vdots \\ \dfrac{\beta_n}{\prod_{i=1}^{k}(\sigma_n^2 - a_i)} - 1 = c_0\sigma_n + c_1\dfrac{\sigma_n}{\sigma_n^2 - a_1} + \ldots + c_k\dfrac{\sigma_n}{\sigma_n^2 - a_k} \end{cases} \tag{12}$$

Therefore, the interpolation problem of a stable polynomial $\Phi(G_j, C)$ is expressed as a system of equations (12) to solve where the unknown variables $c_{i,i\in\{0\ldots,k\}}$ are a set of positive parameters. By hypothesis, the set of distinct negative roots $a_{i,i\in\{1\ldots,k\}}$ of $\Phi(G_j, C)^e(u)$ are known and are given by the even part of the stable polynomials $\Phi(G_i, \widetilde{C_i}) = \Phi(G_i, C)$.

Relationship (12) is written equivalently as

$$\Lambda(a_1, \ldots, a_k) = \Psi(a_1, \ldots, a_k)\,\Gamma \tag{13}$$

with

$$\Lambda^T(a_1, \ldots, a_k) = \left[\frac{\beta_1}{\prod_{i=1}^{k}(\sigma_1^2 - a_i)} - 1, \ \ldots \ \frac{\beta_n}{\prod_{i=1}^{k}(\sigma_n^2 - a_i)} - 1 \right]$$

and respectively for $m = 2k$ and $m = 2k+1$, we have

– $\Gamma^T = [c_1 \ \ldots c_k]$,

$$\Psi(a_1, \ldots, a_k) = \begin{bmatrix} \dfrac{\sigma_1}{\sigma_1^2 - a_1} & \cdots & \dfrac{\sigma_1}{\sigma_1^2 - a_k} \\ \vdots & & \vdots \\ \dfrac{\sigma_n}{\sigma_n^2 - a_1} & \cdots & \dfrac{\sigma_n}{\sigma_n^2 - a_k} \end{bmatrix}$$

– $\Gamma^T = [c_0 \ \ldots c_k]$

$$\Psi(a_1, \ldots, a_k) = \begin{bmatrix} \sigma_1 & \dfrac{\sigma_1}{\sigma_1^2 - a_1} & \cdots & \dfrac{\sigma_1}{\sigma_1^2 - a_k} \\ & \vdots & & \vdots \\ \sigma_n & \dfrac{\sigma_n}{\sigma_n^2 - a_1} & \cdots & \dfrac{\sigma_n}{\sigma_n^2 - a_k} \end{bmatrix}$$

By considering equation (13), the next theorem presents a necessary and sufficient condition for interpolation of a stable polynomial $\Phi(G_j, C)$ that satisfies the interpolation constraints $\Phi(G_j, C)(\sigma_\nu) = \beta_\nu$ such that $\Phi(G_j, C)^e = \Phi(G_i, \widetilde{C_i})^e$ in the case where σ_ν are distinct zeros of Δ with $\nu \in \{1 \ldots n\}$.

***Theorem** 6: Let n pairs of numbers be (σ_ν, β_ν), with $\nu \in \{1 \ldots n\}$. Then $\Phi(G_j, C)$ is a stable polynomial that interpolates all (σ_ν, β_ν) iff there exists a set of positive numbers $c_{i,i\in\{0\ldots,k\}}$ such that (13) holds where $a_{i,i\in\{1\ldots,k\}}$ is a set of given negative distinct real numbers representing the roots of $\Phi(G_j, C)^e = \Phi(G_i, \widetilde{C_i})^e$.*

Proof : That is an immediate consequence of Theorem 5 and equation (11). □

C. Feasibility and infeasibility analysis of system (13).

Relationship (13) is a nonlinear system with respect to the negative distinct real numbers $a_{i,i\in\{1\ldots,k\}}$. In this part, conditions are examined affecting existence or non-existence of positive parameters $c_{i,i\in\{0\ldots,k\}}$ in function of $a_{i,i\in\{1\ldots,k\}}$ with the given data (σ_ν, β_ν), $\nu \in \{1, \ldots, n\}$ and verifying $\Lambda = \Psi\Gamma$. If there exists solutions for this algebraic problem, there are precisely in a strictly convex polyhedral cone defined by $C = \{\Lambda = \Psi\Gamma : \Gamma > 0\}$ (*the notation $\Gamma > 0$ means that all $c_{i,i\in\{0\ldots,k\}}$ are positive*). To study a such nonlinear systems with inequality constraints, it is easier to transform equations (13) into another system with only inequalities. It is the sense of the next result, that is a modified version of Farkas' theorem, see [20], that takes into account the strict inequality $\Gamma > 0$.

Theorem *7:* Theorem of alternatives relative to $\Gamma > 0$.

Exactly one of the two following statements is true.

i) *The system* $\Lambda = \Psi\Gamma$ *has a solution* $\Gamma > 0$
ii) *There exists a vector* Y *such that* $\Psi^T Y > 0$ *and* $\Lambda^T Y \leq 0$.

Proof : First, show that both statement i) and ii) cannot be true. Assume that Γ satisfies i) and that Y satisfies ii). This yields to

$$0 \geq \Lambda^T Y = Y^T \Lambda = Y^T(\Psi\Gamma) = (\Psi^T Y)\Gamma > 0$$

that is an obvious contradiction.

Secondly, let us prove that at least one of assertions i) and ii) has a solution. Suppose that statement i) has no solution. Then we have

$$\Lambda \notin \mathcal{C} = \{z = \Psi\Gamma,\ \Gamma > 0\}$$

Moreover, $\mathcal{C}$ is convex and open. Then from the Separating Hyperplane Theorem, we deduce that there exists an hyperplane that separates properly Λ and $\mathcal{C}$, i.e. there exists a vector $Y \neq 0$ such that

$$\forall z \in \mathcal{C},\ \Lambda^T Y < z^T Y \tag{14}$$

As $0 \in \overline{\mathcal{C}}$, then we deduce from (14) that

$$\Lambda^T Y \leq 0^T Y = 0$$

Now let us show that $\Psi^T Y > 0$. Suppose that $\Psi^T Y \leq 0$. Then for any $\lambda > 0$ and for any $\Gamma > 0$, we have

$$\lambda \Gamma^T \Psi^T Y \leq 0.$$

As $\Lambda^T Y \leq 0$, we can choose λ great enough such that

$$\lambda \Gamma^T \Psi^T Y = (\Psi\lambda\Gamma)^T Y \leq \Lambda^T Y$$

which is in contradiction with (14). □

The previous problem described by relation (13) where the unknown variables are $c_{i,i\in\{0\ldots,k\}}$ function of the negative distinct real numbers $a_{i,i\in\{1\ldots,k\}}$, has been transformed in a new problem where the unknown variables are the vector coordinates Y, see Theorem 7. In order to determine if there exists infeasible solutions to the original problem (13), we can now study the existence conditions of Y given in Theorem 7 function of $a_{i,i\in\{1\ldots,k\}}$ with

$$Y^T = [y_1\ \ldots\ y_n]$$

This yields to write $\Psi^T Y > 0$ and $\Lambda^T Y \leq 0$ as it follows

$$\begin{cases} \left(\dfrac{\sigma_1}{\sigma_1^2 - a_1}\right) y_1 + \quad \ldots \quad + \left(\dfrac{\sigma_n}{\sigma_n^2 - a_1}\right) y_n > 0 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ \left(\dfrac{\sigma_1}{\sigma_1^2 - a_k}\right) y_1 + \quad \ldots \quad + \left(\dfrac{\sigma_n}{\sigma_n^2 - a_k}\right) y_n > 0 \\ \left(\dfrac{\beta_1}{\prod_{i=1}^k (\sigma_1^2 - a_i)} - 1\right) y_1 + \ldots \\ \qquad\qquad + \left(\dfrac{\beta_n}{\prod_{i=1}^k (\sigma_n^2 - a_i)} - 1\right) y_n \leq 0 \end{cases} \tag{15}$$

V. EXISTENCE CONDITIONS OF A PROPER CONTROLLER.

Let C be the simultaneous compensator defined by (1) then we give sufficient conditions so that C is proper.

Lemma *1: If one of the four relationships* (16) *is satisfied then the simultaneous compensator* C *defined by* (1) *is proper for any degree of the polynomials* NQ *and* DQ *and for any proper compensator* C_i.

$$\begin{array}{ll} 1°) & \delta(D_i) = \delta(D_j) = \delta(N_j) \\ 2°) & \delta(D_i) = \delta(D_j) = \delta(N_i) \\ 3°) & \delta(D_j) \geq \delta(D_i) \text{ and } \delta(D_j) = \delta(N_j) \\ 4°) & \delta(D_i) \geq \delta(D_j) \text{ and } \delta(D_i) = \delta(N_i) \end{array} \tag{16}$$

Proof : According to relation (1), the compensator C is proper iff we have

$$max(\delta(DQ) + \delta(Y_i), \delta(NQ) + \delta(N_i)) \geq$$
$$max(\delta(DQ) + \delta(X_i), \delta(NQ) + \delta(D_i))$$

Moreover, we can set

$$\begin{array}{rcl} \delta(DQ) + \delta(Y_i) & \geq & \delta(DQ) + \delta(X_i) \\ \delta(NQ) + \delta(D_i) & \geq & \delta(NQ) + \delta(N_i) \end{array}$$

Consequently, C is proper if

$$\delta(DQ) + \delta(Y_i) \geq \delta(NQ) + \delta(D_i) \tag{17}$$

On the other hand, by considering the relation below

$$\Phi(G_j, C) = \Phi(G_j, \widetilde{C_i}) + NQ\,\Delta$$

we have

$$\delta(NQ) = max(\delta(\Phi(G_j, C)), \delta(\Phi(G_j, \widetilde{C_i}))) - \delta(\Delta) \tag{18}$$

Moreover, by hypothesis $\delta(\Phi(G_j, C)) = \delta(\Phi(G_i, C))$ then $\delta(\Phi(G_j, C)) = \delta(\Phi(G_i, \widetilde{C_i}))$. Consequently $\delta(\Phi(G_j, C))$ may be defined as it follows

$$\delta(\Phi(G_j, C)) = \delta(DQ) + \delta(D_i) + \delta(Y_i)$$

and we can state

$$\delta(\Delta) = max(\delta(D_i) + \delta(N_j), \delta(D_j) + \delta(N_i))$$

By considering (18), this yields to

$$\begin{aligned} &\delta(NQ) = \\ &max(\delta(DQ) + \delta(D_i) + \delta(Y_i), \delta(DQ) + \delta(D_j) + \delta(Y_i)) \\ &-max(\delta(D_i) + \delta(N_j), \delta(D_j) + \delta(N_i)) \end{aligned}$$

Condition (17) becomes in this case

$$\begin{aligned} &\delta(DQ) + \delta(Y_i) \geq \\ &max(\delta(DQ) + \delta(D_i) + \delta(Y_i), \delta(DQ) + \delta(D_j) + \delta(Y_i)) \\ &\quad -max(\delta(D_i) + \delta(N_j), \delta(D_j) + \delta(N_i)) + \delta(D_i) \end{aligned} \tag{19}$$

Four cases can be distinguished :

1) If we have

$$\begin{cases} \delta(D_i) + \delta(N_j) \geq \delta(D_j) + \delta(N_i), \\ \delta(D_i) \geq \delta(D_j) \end{cases}$$

then in this case condition (19) is written as

$$\delta(DQ)+\delta(Y_i) \geq \\ \delta(DQ)+\delta(D_i)+\delta(Y_i)-\delta(D_i)-\delta(N_j)+\delta(D_i)$$

We deduce that $\delta(D_i) \leq \delta(N_j)$.
This relation implies that

$$\delta(D_i)=\delta(D_j)=\delta(N_j)$$

2) If we have

$$\begin{cases} \delta(D_j)+\delta(N_i) \geq \delta(D_i)+\delta(N_j), \\ \delta(D_i) \geq \delta(D_j) \end{cases}$$

then condition (19) becomes for this case

$$\delta(DQ)+\delta(Y_i) \geq \\ \delta(DQ)+\delta(D_i)+\delta(Y_i)-\delta(D_j)-\delta(N_i)+\delta(D_i)$$

That implies $2\delta(D_i)-\delta(D_j) \leq \delta(N_i)$.
This condition is verified iff

$$\delta(D_i)=\delta(D_j)=\delta(N_i)$$

3) If we have

$$\begin{cases} \delta(D_i)+\delta(N_j) \geq \delta(D_j)+\delta(N_i), \\ \delta(D_j) \geq \delta(D_i) \end{cases}$$

then condition (19) becomes

$$\delta(DQ)+\delta(Y_i) \geq \\ \delta(DQ)+\delta(D_j)+\delta(Y_i)-\delta(D_i)-\delta(N_j)+\delta(D_i)$$

That implies $\delta(D_j) \leq \delta(N_j)$, then the following equation can be deduced

$$\delta(D_j)=\delta(N_j)$$

4) If we have

$$\begin{cases} \delta(D_j)+\delta(N_i) \geq \delta(D_i)+\delta(N_j), \\ \delta(D_i) \geq \delta(D_j) \end{cases}$$

then relation (19) is written as

$$\delta(DQ)+\delta(Y_i) \geq \\ \delta(DQ)+\delta(D_i)+\delta(Y_i)-\delta(D_j)-\delta(N_i)+\delta(D_i)$$

That implies $\delta(D_i) \leq \delta(N_i)$.
This relation yields to

$$\delta(D_i)=\delta(N_i)$$

□

VI. Simulation results

A. Design of the simultaneous compensator C

In this part, we present the different steps permitting to compute a simultaneous compensator C stabilizing a segment of systems G_λ from the previous results obtained.

1) Calculate $\Delta = N_jD_i - D_jN_i$ and the roots σ_ν of Δ. If $\Delta \in H$ then the solution is obvious, see section III. If $\Delta \notin H$ then we set $\Delta = \Delta^-\Delta^+$ with Δ^- and Δ^+ respectively the stable part and the unstable part of Δ.
2) Find an initial proper controller $C_i = X_i/Y_i$ stabilizing the plant G_i.
3) Let us $DQ = P\Delta^-$ with P any stable polynomial.
4) Compute $\widetilde{C_i} = DQX_i/DQY_i$ and $\Phi(G_i, \widetilde{C_i})$ and extract the set of real distinct roots a_i of $\Phi(G_i, \widetilde{C_i})^e(s^2)$.
5) Calculate c_i from relation $\Lambda = \Psi\Gamma$. If at least one c_i is negative, then repeat the procedure from 2) by changing the regulator $\widetilde{C_i}$.
6) If all c_i are positive, compute $\Phi(G_j, C)$ and determine NQ from relation $\Phi(G_j, \widetilde{C_i}) + NQ\Delta = \Phi(G_j, C)$.
7) Finally, compute C given by (1).

At the moment, we do not know on the way to modify the controller at step 2 if the conditions to be verified fail in step 5. The existence of solutions is dependent on the choice of the initial compensator $\widetilde{C_i}$. Future works are needed to help in the selection of C_i and DQ. An outline would be to study the feasibility given in section IV-C in order to guarantee the existence of negative distinct real numbers a_i.

B. Example

An illustrative example is given hereafter.

Let $G_i(s)$ and $G_j(s)$ be the two endpoints of a segment of systems $G_\lambda(s)$.

$$G_i(s) = \frac{s^2+s+2}{s^2+2s+1}, \qquad G_j(s) = \frac{3s+2}{2s^2-s+2}.$$

We get

$$\Delta(s) = -2s^4+2s^3+3s^2+7s-2$$

The roots σ_ν of $\Delta(s)$ are given by the set E

$$E = \{2.2613,\ 0.2545,\ -0.7579+1.0787i,\ -0.7579-1.0787i\}$$

Let us note that $\Delta(s)$ has two unstable roots. We obtain

$$\Delta^-(s) = s^2+1.5158s+1.7380$$

Let $C_i(s)$ be an initial compensator that stabilizes the system G_i

$$C_i(s) = \frac{s+1}{s^2+s+1}$$

Let $P(s) = s^2+1.02s+0.02$. As $DQ(s) = \Delta^-(s)P(s)$, consequently we have

$$DQ(s) = s^4+2.5358s^3+3.3041s^2+1.8031s+0.0348$$

We deduce $\widetilde{C_i}(s) = DQ(s)X_i(s)/DQ(s)Y_i(s)$. This yields to

$$\Phi(G_i,\widetilde{C_i})(s) = s^8 + 6.5358s^7 + 19.4473s^6 + 36.2344s^5 + 45.2866s^4 + 38.3897s^3 + 20.9394s^2 + 5.6178s + 0.1043$$

$$\Phi(G_j,\widetilde{C_i})(s) = 2s^8 + 6.0716s^7 + 15.1440s^6 + 28.1251s^5 + 40.9121s^4 + 40.8212s^3 + 24.2435s^2 + 7.4209s + 0.139$$

This implies

$$\Phi(G_i,C)^e(s^2) = s^8+19.4473s^6+45.2866s^4+20.9394s^2+0.1043$$

The roots a_i of $\Phi(G_i,C)^e(s^2)$ are given by the set

$$S = \{-16.8304,\ -1.9950,\ -0.6169,\ -0.0050\}$$

Calculate the values $\Phi(G_j,\widetilde{C_i})(\sigma_\nu) = \beta_\nu$

$$\begin{array}{rcl} \Phi(G_j,\widetilde{C_i})(2.2613) & = & 8.5739 * 10^3 \\ \Phi(G_j,\widetilde{C_i})(0.2545) & = & 4.4757 \\ \Phi(G_j,\widetilde{C_i})(-0.7579+1.0787i) & = & 0 \\ \Phi(G_j,\widetilde{C_i})(-0.7579-1.0787i) & = & 0 \end{array}$$

That implies

$$\Lambda^T = [0.8739\ \ 1.7040\ \ -1.0000\ \ -1.0000]$$
$$\Psi =$$

$$\begin{bmatrix} 0.103 & 0.318 & 0.394 & 0.441 \\ 0.0151 & 0.123 & 0.373 & 3.648 \\ -0.052+0.061i & -0.608+0.059i & -0.667-0.452i & -0.438-0.62i \\ -0.052-0.061i & -0.608-0.059i & -0.667+0.452i & -0.438+0.62i \end{bmatrix}$$

We obtain all c_i $(i \in \{1,2,3,4\})$ positive (with $c_0 = 0$) given by

$$\Gamma^T = [3.4640\ \ 1.0091\ \ 0.0315\ \ 0.4154]$$

We get

$$\Phi(G_j,C)(s) = s^8 + 4.92s^7 + 19.4473s^6 + 35.3457s^5 + 45.2866s^4 + 34.7062s^3 + 20.9394s^2 + 8.6827s + 0.1043$$

We deduce $NQ(s)$

$$NQ(s) = 0.5s^4 + 1.0758s^3 - 0.3258s^2 - 0.5725s + 0.0168$$

The simultaneous compensator $C(s) = X(s)/Y(s)$ is the following

$$X(s) = 0.5s^6 + 3.0758s^5 + 5.8616s^4 + 5.6916s^3 + 3.6532s^2 + 1.2989s + 0.0516$$

$$Y(s) = 0.5s^6 + 1.9600s^5 + 5.0899s^4 + 6.3897s^3 + 6.3493s^2 + 2.9660s + 0.0012$$

We can see that the compensator C stabilizes simultaneously $G_i(s)$ and $G_j(s)$ and that the characteristic polynomials $\Phi(G_i,C)(s)$ and $\Phi(G_j,C)(s)$ have the same even parts.

$$\Phi(G_i,C)(s) = s^8 + 6.5358s^7 + 19.4473s^6 + 36.2344s^5 + 45.2866s^4 + 38.3897s^3 + 20.9394s^2 + 5.6178s + 0.1043$$

$$\Phi(G_j,C)(s) = s^8 + 4.9200s^7 + 19.4472s^6 + 35.3459s^5 + 45.2867s^4 + 34.7051s^3 + 20.9380s^2 + 8.6835s + 0.1054$$

Moreover, $\Phi(G_i,C)(s)$ and $\Phi(G_j,C)(s)$ are two polynomials of same degree with their leading coefficients of same sign. Consequently, the segment of systems $G_\lambda(s)$ defined in (4) by these two endpoints $G_i(s)$ and $G_j(s)$ is stable with the proper simultaneous controller $C(s)$.

For this example, we can compare the results obtained with these got with others literature methods. By considering the approach described in [4] which is inspired of [3] and adapted to the case of the simultaneous stabilization of a segment of systems, then this control problem is formulated as a BMI and approximated as a rank-one LMI optimization constraint. Unfortunately, if we apply this method to this example, we have to manage a computational complexity that does not allow to process and to look for solutions. This complexity is due to the number and to the size of the Hermite-Fujiwara matrices to take into account. In our case, their size is 8 and their number is $2 * 14^2$. Let us recall that the size of the Hermite-Fujiwara matrices is equal to the degree of the polynomial $\Phi(G_i,\widetilde{C_i})$ and their number is $2 * r^2$ where r is the number of controller coefficients (i.e 14). This formulation gives a problem numerically difficult to formulate and unlikely to be solved in polynomial time with no guarantee to the convergence of the heuristic algorithm. Moreover, by assuming that the algorithm would converge, the solution would be given with an amount of conservatism.

VII. CONCLUSION

In this paper, an interpolation approach with stable polynomial interpolants has been shown. This method has been applied to the simultaneous stabilization of a segment of systems. The feasibility and infeasibility of the method has been analyzed. The proposed framework has permitted to design simultaneous controllers without conservatism for this family of systems. A prospect would be to extend these results to the case of the simultaneous stabilization of several segments of systems.

REFERENCES

[1] O. Toker, “On the order of the simultaneous stabilizing compensators,” *IEEE Trans. Aut. Contr.*, vol. 41, pp. 430–433, 1996.

[2] V. Blondel, *Simultaneous Stabilization of Linear Systems*. Berlin : Springer-Verlag, 1994.

[3] D. Henrion, S. Tarbouriech, and M. Šebek, “Rank-one LMI approach to simultaneous stabilization of linear systems,” *Syst. & Contr. Letters*, vol. 38, pp. 79–89, 1999.

[4] H. Meddeb, C. Fonte, and M. Zasadzinski, “Simultaneous stabilization of a segment of systems by a parameterized compensator,” in *MED’13*, Chania, Greece, 2013.

[5] V. Patel and K. Datta, “A modification of theorem for the least bound of the minimum degree of interpolating unit,” *Syst. & Contr. Letters*, vol. 25, pp. 235–236, 1995.

[6] ——, “A note on direct interpolation algorithm for a strictly positive real function,” *IEEE Trans. Aut. Contr.*, vol. 40, pp. 1960–1962, 1995.

[7] P. Dorato, H. Park, and Y. Li, “An algorithm for interpolation with units in H_∞, with application to feedback stabilisation,” *Automatica*, vol. 25, pp. 427–430, 1989.

[8] M. Vidyasagar, *Control System Synthesis : A Factorization Approach*. Cambridge, USA : MIT Press, 1985.

[9] C. Fonte and C. Delattre, “Conditions for interpolation of stable polynomials,” in *MTNS-2010*, Hungry, Budapest, Juillet 2010.

[10] ——, “Wronskian-based tests for stability of polynomial combinations,” *Syst. & Contr. Letters*, vol. 60, pp. 590–595, 2011.

[11] B. Ghosh, “Some new results on the simultaneous stabilizability of single input, single output systems,” *Syst. & Contr. Letters*, vol. 6, pp. 39–45, 1985.

[12] ——, “Simultaneous partial pole-placement : a new approach to multimode system design,” *IEEE Trans. Aut. Contr.*, vol. 31, pp. 440–443, 1986.

[13] C. Abdallah, P. Dorato, F. Prez, and D. Docampo, “Controller synthesis for a class of interval plants,” *Automatica*, vol. 31, pp. 341–343, 1995.

[14] G. Chockalingam and S. Dasgupta, “Minilality, stabilizability and strong stabilizability of uncertain plants,” *EEE Trans. Aut. Contr.*, vol. 38, pp. 1651–1661, 1993.

[15] ——, “Strong stabilizability of systems with multiaffine uncertainties and numerator denominator coupling,” *EEE Trans. Aut. Contr.*, vol. 39, pp. 1955–1958, 1994.

[16] C. Fonte, “Conditions for the simultaneous stabilizability of a segment of polynomials,” in *ACC*, Seattle, USA, 2008.

[17] H. Prochazka, A. Lanzon, and B. Anderson, “Syanthesis of parameter-dependent controllers yielding affine-in-parameters characteristic polynomials,” in *Proc. IEEE Conf. Decision & Contr.*, San Diego, USA, 2006.

[18] C.-C. Hsu, W.-Y. Shieh, and C.-H. Gao, “Digital redesign of uncertain interval systems based on extremal gain/phase margins via a hybrid particle swarm optimizer,” *Applied Soft Computing*, vol. 10, pp. 602–612, 2010.

[19] L. Grman and V. Vesely, “Extremal transfer functions in robust control system design,” *Journal of electrical engineering*, vol. 55, pp. 11–17, 2004.

[20] J. Farkas, “Über die Theorie der Einfachen Ungleichungen,” *Journal für die Reine und Angewandte Mathematik*, vol. 359, pp. 1–27, 1902.

[21] F. Gantmacher, *The Theory of Matrices*. New York : Chelsea Publishing Company, 1959.