

# Synchronization of linear multi-agent systems under input saturation

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**Abstract**—This paper is concerned with state synchronization of linear agents subject to input saturation over a fixed undirected communication graph. We first derive a sufficient condition for achieving the synchronization via relative state feedback control law for any initial condition. Based on this analysis result, we present a linear matrix inequality (LMI) condition for designing the synchronizing state feedback gain. The present LMI condition is scalable as long as we can calculate the eigenvalues of the Laplacian of the communication graph, and is readily solved by an existing convex programming algorithm.

**Keywords**—multi-agent system, synchronization, input saturation, linear matrix inequality

## I. INTRODUCTION

For the last decade, multi-agent coordination has been attracting a great attention in the area of systems and control, since such phenomena can be encountered in many applications in physics, biology, robotics, computer science, etc (see e.g. [1],[2], and the references therein). The feature of multi-agent systems is that coordinative tasks such as synchronization and consensus are achieved by distributed control of individual agents based on their local interactions.

For coordination of linear multi-agent systems, earlier works mainly focused on consensus of simple agents described by single or double integrators [2]. Recently, more attention has been paid to consensus or synchronization of higher-order general linear agents [3]–[7].

On the other hand, most of practical control systems are subject to input saturations due to physical constraints or safety reasons. It thus is important to study the coordination of multi-agent systems under input saturation. There are some related works: Lin, Xiang, and Wei [8] solved the consensus problem for single integrator agents under input saturation, a leader-follower-type cooperative control was studied by Meng, Zhao and Lin [9], and the discrete-time consensus under input saturation was solved for a limited class of high-order linear systems by Yang *et al.*[10]. In [11], Yang *et al.* studied the semi-global output regulation of multi-agent systems subject to input saturation, where the exogenous reference signal is observed by only a subset of agents.

In this paper, we will study synchronization or consensus with respect to the states of linear high-order

agents via relative state feedback control. We will first derive a sufficient condition for globally achieving the state synchronization over a fixed undirected communication graph. Based on this condition, we will develop a method for synthesizing the state feedback gain in terms of linear matrix inequalities (LMIs). It should be noted that, due to the network structure constraint, the development of such an LMI condition for the synchronizing control synthesis is more difficult than the LMI-based stabilization of a single input-saturated linear system, although the latter has been well studied in the literature (e.g. [13], [14]).

## II. PROBLEM FORMULATION

### A. Agent Dynamics

Throughout this paper, we consider the homogeneous multi-agent system consisting of  $N$  linear agents subject to input saturation. The dynamics of each agent is described by

$$\dot{x}_i = Ax_i + B\sigma(u_i), i = 1, \dots, N, \quad (1)$$

where  $x_i : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  and  $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$  are the state and input of the  $i$ -th agent. The memoryless map  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  denotes the saturation nonlinearity defined by

$$\sigma(u) = \begin{cases} \bar{u}, & \text{if } u > \bar{u}, \\ u, & \text{if } -\bar{u} \leq u \leq \bar{u}, \\ -\bar{u}, & \text{if } u < -\bar{u}. \end{cases}$$

$\bar{u}$ : positive constant

*Assumption 1:* The agent dynamics (1) is asymptotically null controllable, namely,  $(A, B)$  is stabilizable, and all the eigenvalues of  $A$  lie in the closed left-half plane.

Recall that global asymptotic stabilization of a linear centralized system with input saturation can be achieved only when the open loop system is null controllable [12].

### B. Communication Graph

Communication between agents can be well described by mathematical graphs. A graph  $\mathcal{G}$  is defined by a couple  $(\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V} = \{1, \dots, N\}$  is the node set, and  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$  is the edge set. Each element in  $\mathcal{V}$  corresponds to the index of an agent. Communication links between agents are defined by edges:  $(i, j) \in \mathcal{E}$  means that there is a communication link from the agent  $j$  to the agent  $i$ . Throughout this paper, communications between any two agents are *bi-directional*, i.e.

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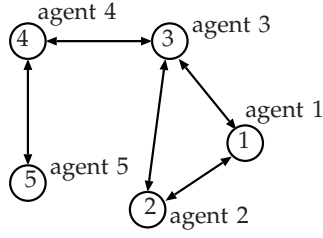


Fig. 1. Communication graph

$(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$ . Then, the graph  $\mathcal{G}$  is identified with an *undirected* graph.

We make the following assumption on the network structure.

*Assumption 2:*

- (i) The topology of the communication graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  is time-invariant.
- (ii)  $\mathcal{G}$  is a connected graph, namely, there always exists an undirected path between any two nodes.

We also define the set of neighbors as

$$\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}, j \neq i\}.$$

The Laplacian matrix  $L \in \mathbb{R}^{N \times N}$  associated with the graph  $\mathcal{G}$  is defined by

$$L = (\ell_{ij}), \quad \ell_{ij} = \begin{cases} |\mathcal{N}_i|, & \text{if } i = j, \\ -1, & \text{if } (i, j) \in \mathcal{E}, \\ 0, & \text{otherwise} \end{cases}$$

It should be noted that  $L$  is non-negative definite for undirected graphs, and it always has a simple zero eigenvalue with an eigenvector  $\mathbb{1} := [1, 1, \dots, 1]^T \in \mathbb{R}^N$ . For ease of later discussion, we denote the eigenvalues of  $L$  by  $\lambda_i, i = 1, \dots, N$  in the ascending order

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{N-1} \leq \lambda_N.$$

It is well known that  $\mathcal{G}$  is a connected graph if and only if  $\lambda_2 > 0$ .

### C. Problem Statement

The state synchronization problem considered in this paper is to find a feedback control law that satisfies

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\| = 0 \quad \forall i, j \in \mathcal{V}$$

for all initial states  $x_1(0), \dots, x_N(0) \in \mathbb{R}^n$ . Such a control law is said to *globally synchronize* the multi-agent system.

One of the typical strategies for distributed coordinative control is the relative state feedback law

$$u_i = F \sum_{j \in \mathcal{N}_i} (x_i - x_j), \quad (2)$$

where  $F$  is the feedback gain to be designed.

## III. STATE SYNCHRONIZATION PROBLEM

### A. Analysis

We here define

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \quad \Sigma(u) = \begin{bmatrix} \sigma(u_1) \\ \sigma(u_2) \\ \vdots \\ \sigma(u_N) \end{bmatrix}.$$

Then the multi-agent system of (1) and (2) is equivalently rewritten as

$$\dot{x} = (I \otimes A)x + (I \otimes B)\Sigma(u), \quad (3a)$$

$$u = (L \otimes F)x, \quad (3b)$$

where  $\otimes$  is the Kronecker product,  $I$  denotes the identity matrix of compatible size, and  $L$  is the Laplacian associated with the communication graph  $\mathcal{G}$ . Moreover, we define

$$\varphi(\mu) = \mu - \sigma(\mu)$$

$$\Phi(u) = u - \Sigma(u) = \begin{bmatrix} \varphi(u_1) \\ \vdots \\ \varphi(u_N) \end{bmatrix}$$

Then, (3) is equivalent to

$$\dot{x} = (I \otimes A + L \otimes BF)x - (I \otimes B)w \quad (4a)$$

$$u = (L \otimes F)x \quad (4b)$$

$$w = \Phi(u) \quad (4c)$$

Since  $\varphi$  satisfies the sector condition  $0 \leq \varphi(\mu)/\mu \leq 1 \quad \forall \mu \in \mathbb{R}$ ,

$$w^\top (w - u) \leq 0. \quad (5)$$

holds for  $w = \Phi(u), u \in \mathbb{R}^{mN}$ . Let  $U$  be the orthogonal matrix such that

$$ULU^\top = \Lambda := \text{diag}\{0, \lambda_2, \dots, \lambda_N\}.$$

We introduce the coordinate transformation

$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix} = (U \otimes I)x. \quad (6)$$

Since  $UU^\top = I$  and  $LU^\top = U^\top \Lambda$ , (4) reduces to

$$\dot{\xi} = (I \otimes A + \Lambda \otimes BF)\xi - (U \otimes B)w \quad (7a)$$

$$u = (U^\top \Lambda \otimes F)\xi \quad (7b)$$

$$w = \Phi(u) \quad (7c)$$

Here, we have used the identity

$$(X \otimes Y)(Z \otimes W) = (XZ) \otimes (YW).$$

Since  $L\mathbf{1} = 0$ , the first row of  $U$  is  $\mathbf{1}^\top/\sqrt{N}$ . It thus follows that  $\xi_1 = \frac{1}{\sqrt{N}} \sum_{i=1}^N x_i$ . It is easily seen from (6) that, if  $\lim_{t \rightarrow \infty} \|\xi_i(t)\| = 0, i = 2, \dots, N$ , then

$$x(t) \rightarrow (U^\top \otimes I) \begin{bmatrix} \xi_1(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{\sqrt{N}} \mathbf{1} \otimes \xi_1(t) \quad (t \rightarrow \infty).$$

This implies that the state synchronization is achieved.

With these preparation, a sufficient condition for state synchronization is stated in the following theorem.

*Theorem 1: Under Assumptions 1 and 2, for a given feedback gain  $F$ , assume that there exists a positive definite matrix  $P$  satisfying*

$$\begin{bmatrix} \left( \begin{array}{c} (A + \lambda_i BF)^\top P \\ +P(A + \lambda_i BF) \end{array} \right) & \lambda_i F^\top - PB \\ \lambda_i F - B^\top P & -2I \end{bmatrix} < 0, \quad (8) \\ i = 2, \dots, N$$

*Then, the multi-agent system of (3) achieves the state synchronization under input saturation.*

(Proof)

Based on the earlier discussion, we shall prove the theorem by showing the convergence of  $\xi_2, \dots, \xi_N$  under the assumption that  $P > 0$  satisfies (8).

Firstly, we define

$$V(\xi) = \xi^\top \mathcal{P} \xi,$$

$$\mathcal{P} = \text{diag}\{0, P, \dots, P\}, \quad J = \text{diag}\{0, I, \dots, I\}.$$

Then,  $V$  is nonnegative definite, and its derivative along the trajectory of the system (7) is given by

$$\dot{V}(\xi) = 2\mathcal{P}[(I \otimes A + \Lambda \otimes BF)\xi - (U \otimes B)w],$$

where  $w$  is given in (7c).

Since (8) is satisfied, there exists a scalar constant  $\epsilon > 0$  such that

$$\begin{bmatrix} \left( \begin{array}{c} (A + \lambda_i BF)^\top P \\ +P(A + \lambda_i BF) + \epsilon I \end{array} \right) & \lambda_i F^\top - PB \\ \lambda_i F - B^\top P & -2I \end{bmatrix} \leq 0, \\ i = 2, \dots, N \quad (9)$$

Stacking and re-arranging the inequalities in (9) yields

$$\begin{bmatrix} \left( \begin{array}{c} (I \otimes A + \Lambda \otimes BF)^\top \mathcal{P} + \\ \mathcal{P}(I \otimes A + \Lambda \otimes BF) + \epsilon J \end{array} \right) & (\Lambda \otimes F)^\top - \mathcal{P}(U \otimes B) \\ (\Lambda \otimes F) - (U \otimes B)^\top \mathcal{P} & -2I \end{bmatrix} \leq 0$$

We apply the congruence transformation with  $\text{diag}\{I, (U \otimes I)\}$  to the above inequality to obtain

$$\begin{bmatrix} \left( \begin{array}{c} (I \otimes A + \Lambda \otimes BF)^\top \mathcal{P} + \\ \mathcal{P}(I \otimes A + \Lambda \otimes BF) + \epsilon J \end{array} \right) & (U^\top \Lambda \otimes F)^\top - \mathcal{P}(U \otimes B) \\ U^\top \Lambda \otimes F - (U \otimes B)^\top \mathcal{P} & -2I \end{bmatrix} \leq 0$$

Pre-multiplying the above inequality with  $[\xi^\top \ w^\top]$  and post-multiplying with  $\begin{bmatrix} \xi \\ w \end{bmatrix}$  yields

$$\begin{aligned} & \xi^\top \mathcal{P} [(I \otimes A + \Lambda \otimes BF)\xi - (U \otimes B)w] \\ & + [(I \otimes A + \Lambda \otimes BF)\xi - (U \otimes B)w]^\top \mathcal{P} \xi \\ & + \xi^\top (U^\top \Lambda \otimes I)^\top w + w^\top (U^\top \Lambda \otimes I) \xi \\ & + \epsilon \xi^\top J \xi - 2w^\top w \\ & \leq 0 \end{aligned}$$

Thus, we have

$$\dot{V}(\xi) + \epsilon \sum_{i=2}^N \|\xi_i\|^2 \leq 2w^\top (w - u)$$

with  $w = \Phi(u)$ ,  $u = (U^\top \Lambda \otimes I)\xi$ . Since (5) holds, this implies

$$\dot{V}(\xi) \leq -\epsilon \sum_{i=2}^N \|\xi_i\|^2 \leq 0. \quad (10)$$

Then, by La Salle's invariance principle, the trajectory of  $\xi$  converges to the largest invariant set contained in  $\{\xi \in \mathbb{R}^{nN} \mid \dot{V}(\xi) = 0\}$  for any initial state  $\xi(0) \in \mathbb{R}^{nN}$ . Hence, from (10),  $\xi_2, \dots, \xi_N$  converge to 0 as  $t$  goes to infinity for any condition. Therefore, we conclude that the state synchronization is globally achieved by the feedback control of (2).  $\square$

The left-hand side of (8) is affine in  $\lambda_i$ . In addition, since  $\lambda_i$ 's are arranged in the ascending order,  $\lambda_3, \dots, \lambda_{N-1}$  can be represented as convex combinations of  $\lambda_2$  and  $\lambda_N$ . From this observation, we can reduce the number of matrix inequalities in Theorem 1. The following result is more suitable for large-scale networks than Theorem 1.

*Corollary 1: Under Assumptions 1 and 2, for a given feedback gain  $F$ , assume that there exists a positive definite matrix  $P$  satisfying*

$$\begin{bmatrix} \left( \begin{array}{c} (A + \lambda_i BF)^\top P \\ +P(A + \lambda_i BF) \end{array} \right) & \lambda_i F^\top - PB \\ \lambda_i F - B^\top P & -2I \end{bmatrix} < 0, \quad i = 2, N \quad (11)$$

*Then, the multi-agent system of (3) globally achieves the state synchronization under input saturation.*

*Remark:* The synchronization conditions in Theorem 1 and Corollary 1 are linear matrix inequalities (LMIs) in  $P$  when  $F$  is fixed. Hence, these conditions can be efficiently checked by using convex programming algorithms [15].

*Remark:* From (7), the closed-loop multi-agent system without input saturation, i.e.  $w = 0$ , is given by

$$\dot{\xi} = (I \otimes A + \Lambda \otimes BF)\xi.$$

Hence, in the absence of input saturation, a necessary and sufficient condition for achieving synchronization

is that  $A + \lambda_i BF$ ,  $i = 2, \dots, N$  are all Hurwitz stable, i.e. all eigenvalues have negative real parts (see Lemma 1 of [4] for a similar result). The Hurwitz stability of these matrices are guaranteed in Theorem 1 and Corollary 1, since the  $(1,1)$ -block element of the conditions (8),(11) implies the Lyapunov inequalities

$$(A + \lambda_i BF)^\top P + P(A + \lambda_i BF) < 0, \quad i = 2, \dots, N.$$

### B. Synthesis of State Feedback Gain

We will present a method to design a synchronizing feedback gain  $F$  based on the results of the previous sub-section.

Based on Corollary 1, we wish to find a positive definite matrix  $P$  and a feedback gain  $F$  satisfying (11). However, since (11) contains a bilinear term between  $P$  and  $F$ , we need to convexify the matrix inequalities in (11) in order to effectively solve the problem.

As a usual technique for the convexification, we perform the change of variables of  $P$  and  $F$  as  $X = P^{-1}$  and  $Y = FP^{-1}$  [15]. Then, by applying the congruence transformation with  $\text{diag}(P^{-1}, I)$  to (11), we end up with the following theorem.

*Theorem 2: Under Assumptions 1 and 2, assume that there exist a positive definite matrix  $X$  and a matrix  $Y$  satisfying*

$$\begin{bmatrix} \left( \begin{array}{c} (AX + \lambda_i BY)^\top \\ +(AX + \lambda_i BY) \end{array} \right) & \lambda_i Y^\top - B \\ \lambda_i Y - B^\top & -2I \end{bmatrix} < 0, \quad i = 2, N. \quad (12)$$

*Then, there exists a feedback gain  $F$  that achieves the global state synchronization under input saturation. One of such feedback gains is given by*

$$F = YX^{-1}. \quad (13)$$

As expected, the inequalities in (12) are LMIs in the variables  $X, Y$ . Moreover, the size of the LMI problem does not depend on the size of the network,  $N$ , as long as the Laplacian eigenvalues  $\lambda_2$  and  $\lambda_N$  are available. Therefore, we can effectively design the synchronizing feedback gain  $F$  by convex programming.

It may also be noted that the feedback gain  $F$  in Theorem 2 depends on the Laplacian eigenvalues  $\lambda_2$  and  $\lambda_N$ , while the feedback gain for the leader-follower problem [9] is independent of the network structure.

## IV. CONCLUSIONS

We have studied the global state synchronization of linear agents subject to input saturation over a fixed undirected communication graph. A sufficient condition for achieving the synchronization via relative state feedback control is derived. Based on this analysis result, we present an LMI condition for designing a state feedback gain which achieves the global state synchronization. The present LMI condition is scalable as long as we can calculate the eigenvalues of the

Laplacian of the communication graph, and is readily solved by an existing convex programming algorithm.

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