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H^∞-control model for a partially observed optimal investment problem*

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***Abstract*— Dynamics of assets in financial markets are often modeled by stochastic differential equations driven by Brownian motion, which is uncertain in the sense of probability theory. On the other hand, robust control treats model uncertainty by regarding noises as unknown deterministic functions. In this paper, motivated by risk-averse/small noise limit, we consider an H^∞-control model for an optimal investment problem with logarithmic utility under unobserved economic factor. We point out this model can be a max-plus stochastic control formulation of the investment problem. By using so-called information state in H^∞-control theory, we reduce the problem to a completely observed differential game in infinite-dimensional space. Due to a special affine/quadratic structure of the problem, we can give an optimal investment strategy by solving a linear Isaacs partial differential equation in finite-dimensional space.**

I. INTRODUCTION

Asset prices in financial markets are usually modeled by stochastic processes in mathematical finance. More specifically, dynamics of assets are often described by stochastic differential equations (SDEs) driven by Brownian motion, which is uncertain in the sense of probability theory. There is a vast of literature on stochastic models in mathematical finance. On the other hand, robust control is another way to model system dynamics by ordinary differential equations (ODEs) driven by unknown deterministic functions and it can provide new models in mathematical finance. In option pricing, robust control approaches are found in [12] and [2]. For utility optimization problems, [6] derives a differential game of H^∞-control type by risk-averse/small noise limits of classical utility maximization problems and suggests a new possible model for risk-averse investors (cf. also [11] for some related work). Although the limiting problems are treated in the framework of a differential game theory, they are also understood stochastic control problems under max-plus (Maslov idempotent) probability measures (cf. [4], [7] for general theory of processes under max-plus probability). Max-plus stochastic control may give a natural way for formulations since it can be developed and understood in a similar way to conventional stochastic control.

In this paper, we consider a utility optimization problem of a terminal wealth in a linear factor model in the framework of H^∞-control to seek a possible model for risk-averse investors. The model is an H^∞-control counterpart of [3] and we assume that the economic factor cannot be observed like [13]. Although we might be able to discuss the problems in this paper by developing max-plus stochastic control theory under partial observation, we will follow H^∞-control terminologies because pages are limited and literature on H^∞-control is easily accessible. We will give some remarks on max-plus formulation of our problem.

The paper is organized as follows. In section II, we formulate an optimal investment problem with logarithmic utility function in the sense of H^∞-control under unobserved economic factor. In section III, we will mention a max-plus formulation for our investment problem. In section IV , we reformulate the problem by using so-called information state and reduce the problem to a completely observable differential game in infinite-dimensional space. In section V, by using a special affine/quadratic structure of the problem, we will give an optimal investment strategy by solving a linear Isaacs partial differential equation in finite-dimensional space.

II. MODEL AND PROBLEM FORMULATION

We consider a factor model under a market with one riskless asset and n risky assets. We suppose that the price of the riskless asset s^0_t $(0 \le t \le T)$ satisfies the ODE

$$ds^0_t/s^0_t = r dt, \ s^0_0 > 0, \tag{1}$$

where $r \ge 0$. (1) is understood in the following way:

$$ds^0_t/dt = rs^0_t.$$

We consider the price of the ith risky asset s^i_t $(0 \le t \le T, i = 1, 2, \dots, n)$ governed by the following ODEs:

$$ds^i_t/s^i_t = (a + Ax_t)_i dt + \sum_{j=1}^{m+n} \sigma_{ij} w^j_t dt, \ s^i_0 > 0, \tag{2}$$

where $a \in \mathbb{R}^n$, $A \in M(n,m)$, $\Sigma = [\sigma_{ij}] \in M(n, m+n)$ and $w \in L^2_{n+m}[0,T] = L^2([0,T]; \mathbb{R}^{n+m})$. Here $M(k,l)$ is the set of $k \times l$-matrices. $x_t \in \mathbb{R}^m$ is the economic factor and $(a+Ax_t)_i$ is the ith component of $a+Ax_t$. As opposed to classical SDE formulations, the risky asset dynamics are modeled by the ODEs driven by unknown disturbance $w \in L^2_{m+n}[0,T]$. Instead of (2), we will consider log price $y^i_t = \log s^i_t$ satisfying

$$dy^i_t = (a + Ax_t)_i dt + \sum_{j=1}^{m+n} \sigma_{ij} w^j_t dt. \tag{3}$$

We assume that the factor x_t satisfies the ODE under unknown disturbance w_t

$$dx_t = (b+Bx_t)dt + \Lambda w_t dt, \ 0 \le t \le T, \ x_0 = x \in \mathbb{R}^m, \tag{4}$$

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where $b \in \mathbb{R}^m$, $B \in M(m,m)$, $\Lambda \in M(m, m+n)$. In stochastic models, the unknown deterministic disturbance $w_t = (w_t^1, \ldots, w_t^{m+n})^*$ is replaced by $m+n$-dimensional Brownian motion. Thus prices of risky assets $S_t^1, \ldots, S_t^n$ and a factor X_t are modeled by SDEs (cf. [3], [13]):

$$dS_t^i / S_t^i = (a + AX_t)_i dt + \sum_{j=1}^{m+n} \sigma_{ij} dW_t^i,$$
$$dX_t = (b + BX_t)dt + \Lambda dW_t.$$

Let π_t^i be a proportion of a wealth invested in the ith risky asset at t and v_t be the wealth at t associated with the investor's portfolio $\pi_t = (\pi_t^1, \pi_t^2, \ldots, \pi_t^n)$. Under the self-financing assumption, v_t is determined by

$$\begin{aligned} \frac{dv_t}{v_t} &= \{1 - (\pi_t^1 + \pi_t^2 + \cdots + \pi_t^n)\} \frac{ds_t^0}{s_t^0} + \sum_{i=1}^{n} \pi_t^i \frac{ds_t^i}{s_t^i} \\ &= (1 - \langle \pi_t, \mathbf{1} \rangle) r dt + \langle \pi_t, (a + Ax_t)dt + \Sigma w_t dt \rangle \\ &= \{r + \langle \pi_t, a + Ax_t - r\mathbf{1} \rangle\} dt + \langle \pi_t, \Sigma w_t \rangle dt, \end{aligned} \tag{5}$$

where $\mathbf{1} = (1, 1, \cdots, 1)^* \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is Euclidean inner product. Thus v_T can be described by

$$\begin{aligned} v_T &= v e^{\int_0^T r + \langle \pi_t, a + Ax_t - r\mathbf{1} \rangle dt + \int_0^T \langle \pi_t, \Sigma w_t \rangle dt}, \\ v_0 &= v > 0. \end{aligned} \tag{6}$$

We suppose that the investor is allowed to access the information on only the asset prices. Note that the investor knows the future price of the riskless asset because (1) is not affected by unknown disturbance w. The investor decides his/her portfolio π_t based on the past price history of risky assets s_s^i $(0 \le s \le t)$. In stochastic formulations, a non-anticipative portfolio can be defined by using progressive measurability with respect to the σ-field generated by the risky assets. Since we use the deterministic dynamics, we need to formulate non-anticipative portfolios in a different way. It will be seen later that this can be realized by the notion of Elliott-Kalton strategies in differential games. At this point, we will first formulate an investment problem for a given open-loop strategy $\pi \in L_n^2[0,T]$ and a given observation of log price $y \in AC_n[0,T]$. Here $AC_n[0,T]$ is the set of $\mathbb{R}^n$-valued absolutely continuous functions on $[0,T]$ with $\dot{y} \in L_n^2[0,T]$.

Now let us define our investment problem in the framework of H^∞-control. In connection with risk-averse/small noise limit of power utility functions, we consider log utility of wealth v_T (see Remark 5.7). Let $\phi : \mathbb{R}^m \to (-\infty, 0]$ be a function to measure an uncertainty of an initial state of the factor. For given portfolio $\pi \in L_n^2[0,T]$ and log-price observation $y \in AC_n[0,T]$, a *pre-investment problem under* π *and* y is to find a constant $K = K(\pi, y)$ such that

$$\log v_T \ge -K + \phi(x) - \frac{\gamma^2}{2} \int_0^T |w_t|^2 dt \tag{7}$$

for any $w \in L_{m+n}^2[0,T]$ and $x \in \mathbb{R}^m$ satisfying (3), (4), (6) constrained by π and y. We may call $\gamma > 0$ disturbance attenuation parameter. (7) gives a lower bound for the investor's utility in terms of uncertainty of initial state of the factor $x_0 = x$ and disturbance w, plus constant K which can be controlled by the investor. Since $L > K$ satisfies (7) if (7) holds for K, we are interested in the minimal choice for K. The minimal $\hat{K} = \hat{K}(\pi, y)$ is given by

$$\hat{K}(\pi, y) = \sup_{\substack{w \in L_{m+n}^2[0,T] \\ x \in \mathbb{R}^m}} \left\{ -\log v_T + \phi(x) - \frac{\gamma^2}{2} \int_0^T |w_t|^2 dt \right\},$$

where sup is taken on $w \in L_{m+n}^2[0,T]$ and $x \in \mathbb{R}^m$ satisfying (3), (4), (6) constrained by π and y. By using the explicit form of v_T in (6), we have

$$\begin{aligned} \hat{K}(\pi, y) = &-\log v + \sup_{\substack{w \in L_{m+n}^2[0,T] \\ x \in \mathbb{R}^m}} \left\{ -\int_0^T r + \langle \pi_t, a + Ax_t - r\mathbf{1} \rangle dt \right. \\ &\left. + \phi(x) - \int_0^T \langle \pi_t, \Sigma w_t \rangle dt - \frac{\gamma^2}{2} \int_0^T |w_t|^2 dt \right\} \end{aligned} \tag{8}$$

At this stage, we will state our investment problem in a formal way. If the investor is conservative on the lower bound $-K$ with $K = \hat{K}$ in (7) for all possible observations, he/she is concerned with the smallest $-\hat{K}(\pi, y)$ over observations, *i.e.*,

$$\inf_{y \in AC_n[0,T]} \{-\hat{K}(\pi, y)\}. \tag{9}$$

Next the investor would maximize (9) over π. By reversing the sign, an *investment problem* in H^∞-control sense can be formulated as follows:

(IP) Minimize

$$\sup_{y \in AC_n[0,T]} \hat{K}(\pi, y)$$

on non-anticipative portfolios π with respect to y.

Thus the investor is interested with the value

$$\inf_{\pi} \sup_{y \in AC_n[0,T]} \hat{K}(\pi, y) \tag{10}$$

and a non-anticipative portfolio which attains the infimum. Note that (IP) is a zero-sum game with payoff $\hat{K}(\pi, y)$ where the minimizer is the investor and the maximizer is the observation. We point out that the order of $\inf_\pi \sup_y$ in (10) is crucial because the investor is trying to minimize the worst case outcome of observations.

III. Remarks on max-plus formulation

Our investment problem can be formulated in the framework of max-plus stochastic control (cf. [4], [7]). We set $\mathcal{W}[0,T] = L_{m+n}^2[0,T]$ and $X = \mathbb{R}^m$. $\mathcal{W}[0,T]$ and X correspond to the set of disturbances and the set of initial states of the economic factor, respectively. In a max-plus probability formulation, $X \times \mathcal{W}[0,T]$ is a sample space. For $w \in \mathcal{W}[0,T]$, we consider max-plus probability

$$q(w) = -\frac{\gamma^2}{2} \int_0^T |w_t|^2 dt$$

and we regard $\phi(x)$ as a max-plus probability density for initial state $x \in X$ of the factor. Supposing that the disturbance and the initial state of the factor are independent in max-plus sense, we can define max-plus expectation for random variable $Z : X \times \mathcal{W}[0,T] \to \mathbb{R}$ by

$$E^{\oplus}[Z] = \sup_{w \in \mathcal{W}[0,T], x \in X} \{Z(x,w) + \phi(x) + q(w)\}.$$

Let $y_{[0,t]} = \{y_s\}_{0 \le s \le t}$ be a trajectory of log price on $[0,t]$ and denote by $\mathcal{Y}[0,t]$ the set of all possible $y_{[0,t]}$. For a given $y_{[0,t]} \in \mathcal{Y}[0,t]$, max-plus conditional expectation is defined by

$$\begin{aligned}&E^{\oplus}[Z|y_{[0,t]}]\\ &= \sup\{Z(x,w); x \in X, w \in \mathcal{W}[0,T] \text{ satisfying (3) and (4)}\}.\end{aligned}$$

If we set

$$E^{\oplus}[E^{\oplus}[Z|y_{[0,t]}]] = \sup_{y_{[0,t]} \in \mathcal{Y}[0,t]} E^{\oplus}[Z|y_{[0,t]}],$$

we have tower property

$$E^{\oplus}[Z] = E^{\oplus}[E^{\oplus}[Z|y_{[0,t]}]].$$

Now if we use max-plus notations for our problem, (IP) can be written as follows:

$$\text{Minimize } E^{\oplus}[-\log v_T] = E^{\oplus}[E^{\oplus}[-\log v_T|y_{[0,T]}]]$$
on non-anticipative portfolio π with respect to $y_{[0,T]}$.

This is an investment problem with logarithmic utility under max-plus probability. Here the sign is converse due to max-plus formulation. The representation by the conditional expectation under accessible information $y_{[0,T]}$ is a similar one in stochastic control under partial observation. Thus we need to find dynamics for the conditional expectation as seen in the subsequent sections. Although we might be able to discuss our problem in a max-plus framework, we will follow H^∞-control terminologies.

IV. Information state and its dynamics

A. Information state formulation

From now on, we suppose that an initial wealth $v > 0$ and an initial log-price $y_0 \in \mathbb{R}^m$ are fixed. Noting that the relation via the fundamental theorem of calculus

$$y_t = y_0 + \int_0^t \dot{y}_s ds$$

y is recovered by $\dot{y}$ and vice versa. Thus we may regard $\dot{y}$ as an observation. Under this identification of y with $\dot{y}$, we may write $\hat{K}(\pi, y)$ by $\hat{K}(\pi, \dot{y})$. If we set $\tilde{w}_t = w_t + (1/\gamma^2)\Sigma^*\pi_t$, we can obtain the payoff of the form

$$\begin{aligned}\hat{K}(\pi,\dot{y}) = -\log v + \sup_{\substack{\tilde{w} \in L^2_{m+n}[0,T]\\ x \in \mathbb{R}^m}} \Big\{ -\int_0^T r + \langle \pi_t, a + Ax_t - r\mathbf{1}\rangle dt \\ + \phi(x) + \frac{1}{2\gamma^2}\int_0^T \pi_t^* \Sigma\Sigma^* \pi_t dt - \frac{\gamma^2}{2}\int_0^T |\tilde{w}_t|^2 dt \Big\}. \quad (11)\end{aligned}$$

where sup is taken on $\tilde{w} \in L^2_{m+n}[0,T]$ and $x \in \mathbb{R}^m$ constrained by the ODEs

$$\dot{x}_t = b + Bx_t - \gamma^{-2}\Lambda\Sigma^*\pi_t + \Lambda\tilde{w}_t, \ x_0 = x, \quad (12)$$
$$\dot{y}_t = a + Ax_t - \gamma^{-2}\Sigma\Sigma^*\pi_t + \Sigma\tilde{w}_t. \quad (13)$$

If we consider x_t is a state of the system and y_t is an observation, the payoff (11) with system and observation dynamics (12) and (13) is an H^∞-control problem under partial information.

Now that the investment problem is reduced to a partially observed H^∞-control, we will reformulate the problem in terms of information state as discussed in [9]. We assume that the following conditions hold:
(A1) $\Sigma\Sigma^*$ is positive-definite.
(A2) $\phi(x)$ has the following form:

$$\phi(x) = -\frac{1}{2}(x-\mu)^* Q^{-1}(x-\mu), \ x \in \mathbb{R}^m,$$

where Q is an $m \times m$-symmetric, positive-definite matrix and $\mu \in \mathbb{R}^m$.

Since $\dot{y}$ is a given, we can eliminate a part of w_t from (11), (12), (13).

Proposition 4.1: $\hat{K}(\pi, \dot{y})$ can be written by

$$\begin{aligned}&\hat{K}(\pi,\dot{y})\\ &= -\log v\\ &\quad + \sup_{\substack{\tilde{w}^1 \in L^2([0,T];\mathrm{Ker}\,\Sigma)\\ x \in \mathbb{R}^m}} \Big\{ \phi(x) - \int_0^T r + \langle \pi_t, a + Ax_t - r\mathbf{1}\rangle dt\\ &\quad + \gamma^2 \int_0^T (a + Ax_t - \gamma^{-2}\Sigma\Sigma^*\pi_t)^*(\Sigma\Sigma^*)^{-1}\dot{y}_t dt\\ &\quad - \frac{\gamma^2}{2}\int_0^T |\Sigma^*(\Sigma\Sigma^*)^{-1}(a + Ax_t)|^2 dt + \int_0^T (a + Ax_t)^*\pi_t dt\\ &\quad - \frac{\gamma^2}{2}\int_0^T |\tilde{w}_t^1|^2 dt - \frac{\gamma^2}{2}|\Sigma^*(\Sigma\Sigma^*)^{-1}\dot{y}_t|^2 dt \Big\} \quad (14)\end{aligned}$$

where

$$\begin{aligned}&\dot{x}_t = b + Bx_t - \gamma^{-2}\Lambda\Sigma^*\pi_t + \Lambda\tilde{w}_t^1\\ &\quad + \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}(\dot{y}_t - (a + Ax_t - \gamma^{-2}\Sigma\Sigma^*\pi_t)), \quad (15)\\ &x_0 = x.\end{aligned}$$

Proof of Proposition 4.1. We decompose $\tilde{w} \in L^2_{m+n}[0,T]$ as

$$\tilde{w} = \tilde{w}^1 + \tilde{w}^2,$$

where $\tilde{w}^1 \in L^2([0,T];\mathrm{Ker}\,\Sigma), \tilde{w}^2 \in L^2([0,T];(\mathrm{Ker}\,\Sigma)^\perp)$. Then (11) can be written by

$$\begin{aligned}&\hat{K}(\pi,\dot{y})\\ &= -\log v + \sup_{\tilde{w}^1,\tilde{w}^2,x} \Big\{ -\int_0^T r + \langle \pi_t, a + Ax_t - r\mathbf{1}\rangle dt\\ &+ \phi(x) + \frac{1}{2\gamma^2}\int_0^T \pi_t^*\Sigma\Sigma^*\pi_t dt - \frac{\gamma^2}{2}\int_0^T |\tilde{w}_t^1|^2 dt\\ &- \frac{\gamma^2}{2}\int_0^T |\tilde{w}_t^2|^2 dt \Big\}, \quad (16)\end{aligned}$$

where sup is taken on $\tilde{w}^1 \in L^2([0,T];\mathrm{Ker}\,\Sigma)$, $\tilde{w}^2 \in L^2([0,T];(\mathrm{Ker}\,\Sigma)^\perp)$ and $x \in \mathbb{R}^m$ constrained by the ODEs

$$\dot{x}_t = b + Bx_t - \gamma^{-2}\Lambda\Sigma^*\pi_t + \Lambda\tilde{w}_t^1 + \Lambda\tilde{w}_t^2, x_0 = x, \quad (17)$$

$$\dot{y}_t = a + Ax_t - \gamma^{-2}\Sigma\Sigma^*\pi_t + \Sigma\tilde{w}_t^2. \quad (18)$$

By (18) with $\tilde{w}^2 \in (\mathrm{Ker}\,\Sigma)^\perp$, we can have

$$\tilde{w}_t^2 = \Sigma^*(\Sigma\Sigma^*)^{-1}(\dot{y}_t - (a + Ax_t - \gamma^{-2}\Sigma\Sigma^*\pi_t)).$$

Plugging this into (16) and (17), we obtain

$$\begin{aligned}
&\hat{K}(\pi,\dot{y}) \\
&= -\log v + \sup_{\tilde{w}^1,x}\bigg\{ -\int_0^T r + \langle \pi_t, a + Ax_t - r\mathbf{1}\rangle dt \\
&+ \phi(x) + \frac{1}{2\gamma^2}\int_0^T \pi_t^*\Sigma\Sigma^*\pi_t dt - \frac{\gamma^2}{2}\int_0^T |\tilde{w}_t^1|^2 dt \\
&- \frac{\gamma^2}{2}\int_0^T |\Sigma^*(\Sigma\Sigma^*)^{-1}(\dot{y}_t - (a + Ax_t - \gamma^{-2}\Sigma\Sigma^*\pi_t))|^2 dt\bigg\},
\end{aligned} \quad (19)$$

where sup is taken on $\tilde{w}^1 \in L^2([0,T];\mathrm{Ker}\,\Sigma)$, $x \in \mathbb{R}^m$ and x_t is the solution of

$$\begin{aligned}
&\dot{x}_t = b + Bx_t - \gamma^{-2}\Lambda\Sigma^*\pi_t + \Lambda\tilde{w}_t^1 \\
&+ \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}(\dot{y}_t - (a + Ax_t - \gamma^{-2}\Sigma\Sigma^*\pi_t)), x_0 = x.
\end{aligned}$$

By expanding the last term of (19), we have (14). □

For $(t,x) \in [0,T] \times \mathbb{R}^m$, we define $q(t,x) = q^{\pi,\dot{y}}(t,x)$ (depending on π and $\dot{y}$) by

$$\begin{aligned}
&q(t,x) \\
&= \sup_{\tilde{w}^1 \in L^2([0,t];\mathrm{Ker}\,\Sigma)}\bigg\{\phi(\xi_0) - \int_0^t r + \langle \pi_s, a + A\xi_s - r\mathbf{1}\rangle ds \\
&+ \gamma^2\int_0^t (a + A\xi_s - \gamma^{-2}\Sigma\Sigma^*\pi_s)^*(\Sigma\Sigma^*)^{-1}\dot{y}_s ds \\
&- \frac{\gamma^2}{2}\int_0^t |\Sigma^*(\Sigma\Sigma^*)^{-1}(a + A\xi_s)|^2 ds + \int_0^t (a + A\xi_s)^*\pi_s ds \\
&- \frac{\gamma^2}{2}\int_0^t |\tilde{w}_s^1|^2 ds\bigg\}
\end{aligned} \quad (20)$$

where ξ_s $(0 \le s \le t)$ is the solution of the backward ODE

$$\begin{aligned}
&\dot{\xi}_s = b + B\xi_s - \gamma^{-2}\Lambda\Sigma^*\pi_s + \Lambda\tilde{w}_s^1 \\
&\qquad + \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}(\dot{y}_s - (a + A\xi_s - \gamma^{-2}\Sigma\Sigma^*\pi_s)), \\
&\xi_t = x.
\end{aligned} \quad (21)$$

$q(t,x)$ is called an *information state* (cf. [9]). Proposition 4.1 can be restated in terms of $q(t,x)$.

Proposition 4.2: For $q(t,x)$ defined by (20),

$$\begin{aligned}
&\hat{K}(\pi,\dot{y}) \\
&= -\log v + \sup_{x\in\mathbb{R}^m} q(T,x) - \frac{\gamma^2}{2}\int_0^T |\Sigma^*(\Sigma\Sigma^*)^{-1}\dot{y}_t|^2 dt.
\end{aligned} \quad (22)$$

B. Dynamics of information state

Since $q(t,x)$ is considered a value function of a control problem, we can formally derive the dynamic programming partial differential equation (DP PDE) for $q(t,x)$;

$$\begin{aligned}
\frac{\partial q}{\partial t} &= \sup_{\tilde{w}^1}\bigg\{ -\langle q_x(t,x), b + Bx - \gamma^{-2}\Lambda\Sigma^*\pi_t + \Lambda\tilde{w}^1 \\
&\quad + \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}(\dot{y}_t - (a + Ax - \gamma^{-2}\Sigma\Sigma^*\pi_t))\rangle \\
&\quad - (r + \langle \pi_t, a + Ax - r\mathbf{1}\rangle) \\
&\quad + \gamma^2(a + Ax - \gamma^{-2}\Sigma\Sigma^*\pi_t)^*(\Sigma\Sigma^*)^{-1}\dot{y}_t \\
&\quad - \frac{\gamma^2}{2}|\Sigma^*(\Sigma\Sigma^*)^{-1}(a + Ax)|^2 + (a + Ax)^*\pi_t \\
&\quad - \frac{\gamma^2}{2}|\tilde{w}^1|^2\bigg\},\ (t,x) \in (0,T) \times \mathbb{R}^m, \quad (23)\\
q(0,x) &= \phi(x),\ x \in \mathbb{R}^m, \quad (24)
\end{aligned}$$

where sup is taken over $\tilde{w}^1 \in \mathrm{Ker}\,\Sigma$ and $q_x(t,x) = (q_{x_1}(t,x),\ldots,q_{x_n}(t,x))^*$.

Recall that our investment problem is a zero-sum game with payoff (22). We note that for given portfolio π and log-price observation $\dot{y}$, $q(t,x) = q^{\pi,\dot{y}}(t,x)$ is formally driven by the DP PDE (23) with initial condition (24). If we regard $q(t,x)$ as a new state, the investment problem (IP) becomes a completely observable dynamic game with payoff (22) under infinite-dimensional state dynamics (23), (24). More specifically, (IP) can be reduce to the following problem:

$\widetilde{\text{(IP)}}$ Minimize

$$\begin{aligned}
&\sup_{\dot{y}\in L_n^2[0,T]} \hat{K}(\pi,\dot{y}) \\
&= -\log v + \sup_{\dot{y}\in L_n^2[0,T]}\bigg\{\sup_{x\in\mathbb{R}^m} q(T,x) \\
&\qquad - \frac{\gamma^2}{2}\int_0^T |\Sigma^*(\Sigma\Sigma^*)^{-1}\dot{y}_t|^2 dt\bigg\}
\end{aligned}$$

on non-anticipative portfolio π with respect to $\dot{y}$.

where $q(t,x)$ (formally) satisfies (23) and (24).

To calculate the sup of (23), note that

$$\begin{aligned}
&\sup_{\tilde{w}^1}\left\{-\frac{\gamma^2}{2}|\tilde{w}^1|^2 - \langle q_x(t,x), \Lambda\tilde{w}^1\rangle\right\} \\
&= -\frac{\gamma^2}{2}\inf_{\tilde{w}^1}|\tilde{w}^1 - (-\gamma^{-2}\Lambda^* q_x(t,x))|^2 + \frac{1}{2\gamma^2}|\Lambda^* q_x(t,x)|^2,
\end{aligned}$$

where sup is taken over $\tilde{w}^1 \in \mathrm{Ker}\,\Sigma$. Since the infimum in the above is attained at the orthogonal projection $\tilde{w}^{1,*}$ of $-\gamma^{-2}\Lambda^* q_x(t,x)$ onto $\mathrm{Ker}\,\Sigma$, i.e.,

$$\tilde{w}^{1,*} = -\gamma^{-2}(I - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma)\Lambda^* q_x(t,x). \quad (25)$$

Therefore (23) and (24) become

$$\begin{aligned}\frac{\partial q}{\partial t} &= \frac{1}{2\gamma^2} q_x^*(t,x)\Lambda(I-\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma)\Lambda^* q_x(t,x) \\ &- \langle b+Bx+\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}(\dot{y}_t-(a+Ax)), q_x(t,x)\rangle \\ &- (r+\langle \pi_t, a+Ax-r\mathbf{1}\rangle) \\ &+ \gamma^2(a+Ax-\gamma^{-2}\Sigma\Sigma^*\pi_t)^*(\Sigma\Sigma^*)^{-1}\dot{y}_t \\ &- \frac{\gamma^2}{2}(a+Ax)^*(\Sigma\Sigma^*)^{-1}(a+Ax) + \langle a+Ax, \pi_t\rangle,\end{aligned} \tag{26}$$

$$q(0,x) = \phi(x), \ x \in \mathbb{R}^m. \tag{27}$$

Remark 4.3: We can see a similarity to stochastic control if we use max-plus notations. For an observation y (equivalently $\dot{y} \in L_n^2[0,T]$), we introduce max-plus probability

$$\tilde{q}(y) = -\frac{\gamma^2}{2}\int_0^T |\Sigma^*(\Sigma\Sigma^*)^{-1}\dot{y}_t|^2 dt.$$

A max-plus integral for $g : \mathbb{R}^n \to \mathbb{R}$ is defined by $\int_{\mathbb{R}^n}^{\oplus} g(x)dx = \sup_{x\in\mathbb{R}^n} g(x)$. If we denote by $\tilde{E}^{\oplus}$ the max-plus expectation under $\tilde{q}$, $\widetilde{(\mathrm{IP})}$ can be as follows:

$$\text{Minimize } -\log v + \tilde{E}^{\oplus}\left[\int_{\mathbb{R}^n}^{\oplus} q(T,x)dx\right] \tag{28}$$

on non-anticipative portfolio π with respect to $\dot{y}$.

Note that the observation y is max-plus Brownian motion under max-plus probability $\tilde{q}$. The max-plus representation in (28) under $\tilde{q}$ might be a counterpart of the expression by the unnormalized density after a change of measure as we usually do in the theory of partially observed stochastic control (cf. [10] for this argument for risk-sensitive control). Under this interpretation, (23) might correspond to "Zakai" equation in max-plus sense. We will come back to this issue in the future.

For general $\phi(x)$, $q(t,x)$ has the infinite-dimensional dynamics and moreover $q(t,x)$ is not necessarily smooth. Indeed $q(t,x)$ could be characterized as a viscosity solution of (26) and (27). Under (A2), (26) with (27) is a DP PDE for an affine/quadratic control problem. Thus we can reduce (26) and (27) to dynamics in finite-dimensional spaces.

Proposition 4.4: $q(t,x)$ has the following form:

$$q(t,x) = \alpha_t - \frac{1}{2}(x-m_t)^* P_t^{-1}(x-m_t), \ (t,x) \in [0,T]\times\mathbb{R}^m \tag{29}$$

where P_t ($m\times m$-symmetric positive-definite matrix), $m_t \in \mathbb{R}^m$, $\alpha_t \in \mathbb{R}$ satisfy the system of the ODEs on $[0,T]$:

$$\begin{aligned}\dot{P}_t &= \gamma^{-2}\Lambda\Lambda^* + BP_t + P_tB^* \\ &\quad - (\gamma AP_t + \gamma^{-1}\Sigma\Lambda^*)^*(\Sigma\Sigma^*)^{-1}(\gamma AP_t + \gamma^{-1}\Sigma\Lambda^*), \\ P_0 &= Q,\end{aligned} \tag{30}$$

$$\begin{aligned}\dot{m}_t &= b + Bm_t + (\gamma^2 P_t A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1} \\ &\qquad \times(\dot{y}_t - (a+Am_t)), \\ m_0 &= \mu,\end{aligned} \tag{31}$$

$$\begin{aligned}\dot{\alpha}_t &= \langle Q(m_t,\pi_t), \dot{y}_t\rangle - (1/2\gamma^2)Q(m_t,\pi_t)^*\Sigma\Sigma^*Q(m_t,\pi_t) \\ &\qquad + \Phi(m_t,\pi_t), \\ \alpha_0 &= 0,\end{aligned} \tag{32}$$

where $Q(x,\pi)$ and $\Phi(x,\pi)$ are defined by

$$\begin{aligned}Q(x,\pi) &= (\Sigma\Sigma^*)^{-1}(\gamma^2(a+Ax) - \Sigma\Sigma^*\pi), \\ \Phi(x,\pi) &= (1/2\gamma^2)\pi^*\Sigma\Sigma^*\pi - \langle \pi, a+Ax-r\mathbf{1}\rangle - r.\end{aligned}$$

V. Finite-dimensional differential game under complete observation

A. Reduction to finite-dimensional problem and Isaacs PDEs

In Proposition 4.4, we saw that $q(t,x)$ can be parametrized by P_t, m_t, α_t. This means that the infinite-dimensional dynamics of $q(t,x)$ can be reduced to the finite-dimensional ODEs (30), (31), (32). Thus the investment problem $\widetilde{(\mathrm{IP})}$ becomes a dynamic game in finite-dimensional space.

We will reduce $\widetilde{(\mathrm{IP})}$ to a standard form of a finite-dimensional differential game problem. If we apply the form $q(t,x)$ of (29) to (22), we have

$$\begin{aligned}\hat{K}(\pi,\dot{y}) &= -\log v + \alpha_T - \frac{\gamma^2}{2}\int_0^T |\Sigma^*(\Sigma\Sigma^*)^{-1}\dot{y}_t|^2 dt \\ &= -\log v + \int_0^T \langle Q(m_t,\pi_t), \dot{y}_t\rangle \\ &\quad - \frac{1}{2\gamma^2}Q(m_t,\pi_t)^*\Sigma\Sigma^*Q(m_t,\pi_t) + \Phi(m_t,\pi_t)dt \\ &\quad - \frac{\gamma^2}{2}\int_0^T |\Sigma^*(\Sigma\Sigma^*)^{-1}\dot{y}_t|^2 dt.\end{aligned}$$

Thus we end up with

$$\hat{K}(\pi,\dot{y}) = -\log v + \hat{I}(\pi,\dot{y}) \tag{33}$$

where

$$\hat{I}(\pi,\dot{y}) = \int_0^T L(m_t,\pi_t,\dot{y}_t)dt, \tag{34}$$

$$\begin{aligned}L(x,\pi,\eta) &= \langle Q(x,\pi),\eta\rangle - \frac{1}{2\gamma^2}Q(x,\pi)^*\Sigma\Sigma^*Q(x,\pi) \\ &\quad + \Phi(x,\pi) - \frac{\gamma^2}{2}|\Sigma^*(\Sigma\Sigma^*)^{-1}\eta|^2.\end{aligned}$$

For the dynamics of m_t, we have from (31)

$$\dot{m}_t = F_t + G_t m_t + H_t\dot{y}_t, \ 0 \le t \le T, \ m_0 = m, \tag{35}$$

where

$$\begin{aligned}F_t &= b - (\gamma^2 P_t A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}a, \\ G_t &= B - (\gamma^2 P_t A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A \\ H_t &= (\gamma^2 P_t A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}\end{aligned} \tag{36}$$

Finally $\widetilde{(\mathrm{IP})}$ can be the following problem:

$\widehat{(\mathrm{IP})}$ Minimize

$$-\log v + \sup_{\dot{y}\in L_m^2[0,T]} \hat{I}(\pi,\dot{y})$$

on non-anticipative portfolios π with respect to $\dot{y}$.

Since $-\log v$ is fixed, $\widehat{(\mathrm{IP})}$ is related to a differential game problem with quadratic payoff (34) under affine dynamics (35), where the minimizer is π and the maximizer is $\dot{y}$.

In order to use dynamic programming methods to the differential game problem related to $\widehat{(\mathrm{IP})}$, it is convenient to start at initial time $t \in [0,T]$. For $\pi \in L^2_n[t,T]$ and $\dot{y} \in L^2_n[t,T]$, we consider the system governed by

$$\dot{m}_s = F_s + G_s m_s + H_s \dot{y}_s,\ t \le s \le T,\ m_t = x \in \mathbb{R}^m. \quad (37)$$

The game payoff $\hat{I}(t,x;\pi,\dot{y})$ is given by

$$\hat{I}(t,x;\pi,\dot{y}) = \int_t^T L(m_s,\pi_s,\dot{y}_s)ds. \quad (38)$$

The criterion to be minimized is

$$\begin{aligned} J(t,x;\pi) &= \sup_{\dot{y}\in L^2_n[0,T]} \hat{I}(t,x;\pi,\dot{y}) \\ &= \sup_{\dot{y}\in L^2_n[0,T]} \int_t^T L(m_s,\pi_s,\dot{y}_s)ds. \end{aligned}$$

To give a rigorous treatment on non-anticipativeness, we utilize the notion of Elliott-Kalton strategies (cf. [1]). Set $\Pi[t,T] = L^2_n[t,T]$ and $\dot{\mathcal{Y}}[t,T] = L^2_n[t,T]$, which denote the sets of portfolios and observations, respectively. $\theta : \dot{\mathcal{Y}}[t,T] \to \Pi[t,T]$ is an Elliott-Kalton strategy from $\dot{\mathcal{Y}}[t,T]$ into $\Pi[t,T]$ if the following holds: Let $\dot{y},\ \dot{\tilde{y}} \in \dot{\mathcal{Y}}[t,T]$, $s \in [t,T]$.

$$\text{If } \dot{y}_u = \dot{\tilde{y}}_u \text{ a.e. } u \in [t,s],\ \text{then } \theta[\dot{y}]_u = \theta[\dot{\tilde{y}}]_u \text{ a.e. } u \in [t,s]$$

We denote by $\Gamma[t,T]$ the set of Elliott-Kalton strategies from $\dot{\mathcal{Y}}[t,T]$ into $\Pi[t,T]$. Under the Elliott-Kalton strategies for portfolios, the value function can be

$$\begin{aligned} V(t,x) &= \inf_{\theta\in\Gamma[t,T]} J(t,x;\theta) \\ &= \inf_{\theta\in\Gamma[t,T]} \sup_{\dot{y}\in\dot{\mathcal{Y}}[0,T]} \int_t^T L(m_s,\theta[\dot{y}]_s,\dot{y}_s)ds, \end{aligned} \quad (39)$$

where m_s is the solution of (37).

By the dynamic programming principle, $V(t,x)$ would satisfy the lower Isaacs PDE on $[0,T]\times\mathbb{R}^m$ (cf. [5] for viscosity sense):

$$\frac{\partial V}{\partial t} + \underline{\mathcal{H}}(t,x,V_x(t,x)) = 0,\ (t,x) \in (0,T)\times\mathbb{R}^m, \quad (40)$$

$$V(T,x) = 0,\ x \in \mathbb{R}^m, \quad (41)$$

where

$$\underline{\mathcal{H}}(t,x,p) = \sup_{\eta\in\mathbb{R}^n} \inf_{\pi\in\mathbb{R}^n} \{\langle F_t + G_t x + H_t\eta\rangle + L(x,\pi,\eta)\}. \quad (42)$$

On the other hand, if we change the order of the inf and the sup for $V(t,x)$ and define another value $W(t,x)$ under the Elliott-Kalton strategies of the maximizer, $W(t,x)$ is related with the upper Isaacs PDE:

$$\frac{\partial W}{\partial t} + \overline{\mathcal{H}}(t,x,W_x(t,x)) = 0,\ (t,x) \in (0,T)\times\mathbb{R}^m, \quad (43)$$

$$W(T,x) = 0,\ x \in \mathbb{R}^m, \quad (44)$$

where

$$\overline{\mathcal{H}}(t,x,p) = \inf_{\pi\in\mathbb{R}^n} \sup_{\eta\in\mathbb{R}^n} \{\langle F_t + G_t x + H_t\eta, p\rangle + L(x,\pi,\eta)\}. \quad (45)$$

A min-max condition holds for (42), (45).

Proposition 5.1: For (42) and (45), $\mathcal{H}(t,x,p) = \underline{\mathcal{H}}(t,x,p) = \overline{\mathcal{H}}(t,x,p)$ where

$$\begin{aligned} \mathcal{H}(t,x,p) &= \langle F_t + G_t x + rH_t\mathbf{1}, p\rangle \\ &\quad - \frac{\gamma^2}{2}(a + Ax - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(a + Ax - r\mathbf{1}) - r. \end{aligned}$$

Furthermore the inf in $\overline{\mathcal{H}}(t,x,p)$ and the sup in $\underline{\mathcal{H}}(t,x,p)$ are attained at $\bar{\pi}$ and $\bar{\eta}$, respectively:

$$\bar{\pi} = \bar{\pi}(t,x,p) = \gamma^2(\Sigma\Sigma^*)^{-1}(a + Ax - r\mathbf{1}) + H_t^* p,\ \bar{\eta} = r\mathbf{1}.$$

Proof. Note that

$$\begin{aligned} &\langle H_t\eta, p\rangle + \langle Q(x,\pi),\eta\rangle - \frac{1}{2\gamma^2}|\Sigma^*(\Sigma\Sigma^*)^{-1}\eta|^2 \\ &= -\frac{\gamma^2}{2}\|\eta - \gamma^{-2}\Sigma\Sigma^*(H_t^* p + Q(x,\pi))\|^2_{(\Sigma\Sigma^*)^{-1}} \\ &\qquad + \frac{1}{2\gamma^2}\|\Sigma\Sigma^*(H_t^* p + Q(x,\pi))\|^2_{(\Sigma\Sigma^*)^{-1}}, \end{aligned}$$

where $\|x\|^2_{(\Sigma\Sigma^*)^{-1}} = x^*(\Sigma\Sigma^*)^{-1}x$ for $x \in \mathbb{R}^n$. Then we have

$$\begin{aligned} &\sup_{\eta\in\mathbb{R}^n} \{\langle F_t + G_t x + H_t\eta, p\rangle + L(x,\pi,\eta)\} \\ &= \langle F_t + G_t x, p\rangle - \frac{1}{2\gamma^2}Q(x,\pi)^*\Sigma\Sigma^* Q(x,\pi) + \Phi(x,\pi) \\ &\qquad + \frac{1}{2\gamma^2}(H_t^* p + Q(x,\pi))^*\Sigma\Sigma^*(H_t^* p + Q(x,\pi)) \\ &= \langle F_t + G_t x, p\rangle + \frac{1}{2\gamma^2}\pi^*\Sigma\Sigma^*\pi - \langle \pi, a + Ax - r\mathbf{1}\rangle - r \\ &\quad + \frac{1}{\gamma^2}(\gamma^2(a + Ax) - \Sigma\Sigma^*\pi)^* H_t^* p + \frac{1}{2\gamma^2}p^* H_t\Sigma\Sigma^* H_t^* p. \end{aligned}$$

By completing square on π, we have

$$\begin{aligned} \overline{\mathcal{H}}(t,x,p) &= \langle F_t + G_t x + rH_t\mathbf{1}, p\rangle \\ &\quad - \frac{\gamma^2}{2}(a + Ax - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(a + Ax - r\mathbf{1}) - r \end{aligned}$$

and inf in $\overline{\mathcal{H}}(t,x,p)$ is attained at $\pi = \gamma^2(\Sigma\Sigma^*)^{-1}(a + Ax - r\mathbf{1}) + H_t^* p$.

We look at

$$\begin{aligned} &\langle Q(x,\pi),\eta\rangle - \frac{1}{2\gamma^2}Q(x,\pi)^*\Sigma\Sigma^* Q(x,\pi) + \Phi(x,\pi) \\ &= \langle \pi, r\mathbf{1} - \eta\rangle + \gamma^2\langle(\Sigma\Sigma^*)^{-1}(a + Ax),\eta\rangle \\ &\qquad - \frac{\gamma^2}{2}(a + Ax)^*(\Sigma\Sigma^*)^{-1}(a + Ax) - r. \end{aligned}$$

Then we can see that

$$\inf_{\pi\in\mathbb{R}^n} \{\langle F_t + G_t x + H_t\eta, p\rangle + L(x,\pi,\eta)\} = \begin{cases} \langle F_t + G_t x + rH_t\mathbf{1}, p\rangle \\ \quad - \frac{\gamma^2}{2}(a + Ax - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(a + Ax - r\mathbf{1}) - r \\ \qquad \text{if } \eta = r\mathbf{1}, \\ -\infty \text{ if } \eta \ne r\mathbf{1}. \end{cases}$$

Hence we can obtain

$$\underline{\mathcal{H}}(t,x,p) = \langle F_t + G_t x + rH_t\mathbf{1}, p\rangle - \frac{\gamma^2}{2}(a + Ax - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(a + Ax - r\mathbf{1}) - r$$

and sup in $\underline{\mathcal{H}}(t,x,p)$ is attained at $\eta = r\mathbf{1}$. □

B. Classical solution of Isaacs PDEs and verification

By Proposition 5.1, the lower Isaacs PDE for $V(t,x)$ ($(t,x) \in [0,T] \times \mathbb{R}^m$) is reduced to a linear PDE of first order:

$$\frac{\partial V}{\partial t} + \langle F_t + G_t x + rH_t\mathbf{1}, V_x(t,x)\rangle - \frac{\gamma^2}{2}(a + Ax - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(a + Ax - r\mathbf{1}) - r = 0, \quad (46)$$

$$V(T,x) = 0, \ x \in \mathbb{R}^m. \quad (47)$$

As seen in the following proposition, we can find a classical solution of (46) and (47).

Proposition 5.2: Let $U : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ be

$$U(t,x) = \frac{1}{2}x^*\Xi_t x + \langle \eta_t, x\rangle + \rho_t, \ (t,x) \in [0,T] \times \mathbb{R}^m \quad (48)$$

where Ξ_t ($m \times m$ symmetric matrix), $\eta_t \in \mathbb{R}^m$, $\rho_t \in \mathbb{R}$ satisfy the system of the backward ODEs:

$$\dot{\Xi}_t + G_t^*\Xi_t + \Xi_t G_t - \gamma^2 A^*(\Sigma\Sigma^*)^{-1}A = O, \Xi_T = O, \quad (49)$$

$$\dot{\eta}_t + G_t^*\eta_t + \Xi_t(F_t + rH_t\mathbf{1}) - \gamma^2 A^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1}) = 0, \quad \eta_T = 0, \quad (50)$$

$$\dot{\rho}_t + \langle F_t + rH_t\mathbf{1}, \eta_t\rangle - \frac{\gamma^2}{2}(a - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1}) = 0, \quad \rho_T = 0. \quad (51)$$

Then $U(t,x)$ is a classical solution of (46) and (47).

Proof. Suppose that $U(t,x)$ of the form (48) satisfies (46) and (47). If we plug $U(t,x)$ of the form (48) into (46), we have

$$\frac{1}{2}x^*\dot{\Xi}_t x + \langle \dot{\eta}_t, x\rangle + \dot{\rho}_t + \langle F_t + G_t x + H_t r\mathbf{1}, \Xi_t x + \eta_t\rangle - \frac{\gamma^2}{2}(a + Ax - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(a + Ax - r\mathbf{1}) - r = 0 \quad (52)$$

By comparing the coefficients of x for each order, we have (49), (50), (51). Conversely, if we define (48) with Ξ_t, η_t, ρ_t satisfying (49), (50), (51), $U(t,x)$ satisfies (46) and (47). □

Due to a special structure arising from our problem, we can obtain a verification theorem for the differential game.

Theorem 5.3: Let $U(t,x)$ be the solution of (46) and (47) obtained in Proposition 5.2. Then (i) and (ii) hold:

(i) For any $\theta \in \Gamma[t,T]$,

$$U(t,x) \leq J(t,x;\theta), \ (t,x) \in [0,T] \times \mathbb{R}^m.$$

(ii) Let $\bar{\theta} \in \Gamma[t,T]$ be defined by

$$\bar{\theta}[\dot{y}]_s = \gamma^2(\Sigma\Sigma^*)^{-1}(a + Am_s - r\mathbf{1}) + H_s^* U_x(s, m_s), \ t \leq s \leq T,$$

where m_s is the solution of

$$\dot{m}_s = F_s + G_s m_s + H_s\dot{y}_s, \ t \leq s \leq T, \ m_t = x. \quad (53)$$

Then the following holds:

$$J(t,x;\bar{\theta}) \leq U(t,x), \ (t,x) \in [0,T] \times \mathbb{R}^m.$$

Proof. (i) For $\pi \in \Pi[t,T]$ and $\dot{y} \in \dot{\mathcal{Y}}[t,T]$, letting m_s be the solution of (37),

$$\begin{aligned} &U(T, m_T) \\ &= U(t,x) + \int_t^T \frac{d}{ds}U(s,m_s)ds \\ &= \int_t^T \{\frac{\partial U}{\partial s}(s,m_s) + \mathcal{K}(s,m_s,U_x(s,m_s),\pi_s,\dot{y}_s)\}ds \\ &\quad - \int_t^T L(m_s,\pi_s,\dot{y}_s)ds, \end{aligned}$$

where

$$\mathcal{K}(s,x,p,\pi,\eta) = \langle F_s + G_s x + H_s\eta, p\rangle + L(x,\pi,\eta).$$

Since $U(T,x) = 0$, we have

$$\int_t^T L(m_s,\pi_s,\dot{y}_s)ds = U(t,x) + \int_t^T \{\frac{\partial U}{\partial s}(s,m_s) + \mathcal{K}(s,m_s,U_x(s,m_s),\pi_s,\dot{y}_s)\}ds. \quad (54)$$

Let $\theta \in \Gamma[t,T]$ be arbitrarily taken. Taking $\dot{y}_s = r\mathbf{1}$, $\pi_s = \theta[\dot{y}]_s = \theta[r\mathbf{1}]_s$ in (54),

$$\begin{aligned} &\int_t^T L(m_s,\theta[r\mathbf{1}]_s,r\mathbf{1})ds \\ &= U(t,x) \\ &+ \int_t^T \{\frac{\partial U}{\partial s}(s,m_s) + \mathcal{K}(s,m_s,U_x(s,m_s),\theta[r\mathbf{1}]_s,r\mathbf{1})ds \\ &\geq U(t,x) \\ &+ \int_t^T \{\frac{\partial U}{\partial s}(s,m_s) + \inf_{\pi\in\mathbb{R}^n}\{\mathcal{K}(s,m_s,U_x(s,m_s),\pi,r\mathbf{1})\}ds, \end{aligned} \quad (55)$$

where m_s is the solution of

$$\dot{m}_s = F_s + G_s m_s + rH_s\mathbf{1}, \ t \leq s \leq T, m_t = x.$$

Since the sup in $\underline{\mathcal{H}}(t,x,p)$ is attained at $r\mathbf{1}$ which is independent of (t,x,p),

$$\text{RHS of (55)} = U(t,x) + \int_t^T \{\frac{\partial U}{\partial s}(s,m_s) + \underline{\mathcal{H}}(s,m_s,U_x(s,m_s))\}ds.$$

By Propositions 5.1 and 5.2, we have

$$U(t,x) \leq \int_t^T L(m_s,\theta[r\mathbf{1}]_s,r\mathbf{1})ds.$$

Therefore we can see that

$$U(t,x) \leq \sup_{\dot{y}\in\dot{\mathcal{Y}}[t,T]} \int_t^T L(m_s,\theta[\dot{y}]_s,\dot{y}_s)ds = J(t,x;\theta).$$

(ii) Let $\dot{y} \in \dot{\mathcal{Y}}[t,T]$ be arbitrarily given. If we take $\pi_s = \bar{\theta}[\dot{y}]_s$ in (54),

$$\begin{aligned}&\int_t^T L(m_s, \bar{\theta}[\dot{y}]_s, \dot{y}_s)ds \\ &= U(t,x) \\ &+\int_t^T \{\frac{\partial U}{\partial s}(s,m_s) + \mathcal{K}(s,m_s,U_x(s,m_s),\bar{\theta}[\dot{y}]_s,\dot{y}_s)\}ds \\ &\le U(t,x) \\ &+\int_t^T \{\frac{\partial U}{\partial s}(s,m_s) + \sup_{\eta\in\mathbb{R}^n} \mathcal{K}(s,m_s,U_x(s,m_s),\bar{\theta}[\dot{y}]_s,\eta)\}ds\end{aligned} \tag{56}$$

Note that the inf in $\overline{\mathcal{H}}(s,m_s,U_x(s,m_s))$ is attained at

$$\bar{\pi} = \bar{\pi}(s,m_s,U_x(s,m_s)) = \bar{\theta}[\dot{y}]_s.$$

Therefore we have from Propositions 5.1 and 5.2,

$$\begin{aligned}&\text{RHS of (56)} \\ &= U(t,x) + \int_t^T \{\frac{\partial U}{\partial s}(s,m_s) + \overline{\mathcal{H}}(s,m_s,U_x(s,m_s))\}ds \\ &= U(t,x).\end{aligned}$$

Hence we obtain

$$\int_t^T L(m_s, \bar{\theta}[\dot{y}]_s, \dot{y}_s)ds \le U(t,x).$$

Since $\dot{y} \in \dot{\mathcal{Y}}[t,T]$ is arbitrarily taken,

$$J(t,x;\bar{\theta}) = \sup_{\dot{y}\in\dot{\mathcal{Y}}[t,T]} \int_t^T L(m_s,\bar{\theta}[\dot{y}]_s,\dot{y}_s)ds \le U(t,x). \ \square$$

Remark 5.4: In Theorem 5.5 (i), we can compare $U(t,x)$ with $J(t,x;\theta)$ for any $\theta \in \Gamma[t,T]$ by an elementary direct calculation. This can be done because the maximizer $r\mathbf{1}$ of $\underline{\mathcal{H}}(t,x,p)$ does not depend on (t,x,p)

As a corollary, we can finally obtain a complete answer to investment problem $\widehat{\text{(IP)}}$

Theorem 5.5: Let $\Xi_t,\ \eta_t,\ \rho_t$ be the solutions of (49), (50),(51). Then the optimal value of $\widehat{\text{(IP)}}$ over the set of Elliott-Kalton strategies is given by

$$\begin{aligned}&\inf_{\theta\in\Gamma[0,T]}\left\{-\log v + \sup_{\dot{y}\in\dot{\mathcal{Y}}[0,T]} \hat{I}(\theta[\dot{y}],\dot{y})\right\} \\ &\qquad = -\log v + \frac{1}{2}\mu^*\Xi_0\mu + \langle\eta_0,\mu\rangle + \rho_0\end{aligned} \tag{57}$$

The inf of (57) is attained at $\bar{\theta} \in \Gamma[t,T]$ defined by

$$\bar{\theta}[\dot{y}]_s = \gamma^2(\Sigma\Sigma^*)^{-1}(a+Am_s-r\mathbf{1})+H_s^*(\Xi_s m_s+\eta_s), \tag{58}$$

where m_s is the solution of the ODE driven by $\dot{y}_s$

$$\dot{m}_s = F_s + G_s m_s + H_s\dot{y}_s,\ 0 \le s \le T,\ m_0 = \mu$$

Remark 5.6: In Theorem 5.5, disturbance attenuation level $\gamma > 0$ can be chosen arbitrarily. This is a different phenomenon from typical H^∞-control problems where disturbance attenuation levels usually have lower bounds. The investor can choose any $\gamma > 0$ based on his/her risk-averse preference.

Remark 5.7: Our investment problem with logarithmic utility function is obtained from risk-averse/small noise limit in a stochastic model. Consider SDEs for risky assets and an economic factor with small diffusion:

$$\begin{aligned}dS_t^i/S_t^i &= (a+AX_t)_i dt + \sqrt{\epsilon}\sum_{j=1}^{m+n}\sigma_{ij}dW_t^j, \\ dX_t &= (b+BX_t)dt + \sqrt{\epsilon}\Lambda dW_t,\end{aligned}$$

where $\epsilon > 0$ is a small noise parameter. $\{W_t\}$ is $m+n$-dimensional Brownian motion. Let V_T be the wealth corresponding to a portfolio π. For $\theta < 0$, we consider a maximization problem of expected power utility function:

$$\sup_\pi (1/\theta)E[V_T^\theta].$$

[13] studies this investment problem under unobserved factor. If we take $\theta = \theta_\epsilon = -\gamma^2/\epsilon$, it can be seen that $(1/\theta_\epsilon)\log\sup_\pi(1/\theta_\epsilon)E[V_T^{\theta_\epsilon}]$ converges to (57) as $\epsilon \to 0$.

Remark 5.8: We focused on logarithmic utility because we can take advantage of the affine/quadratic structure. It will be important to find some class of utilities where the problems are practically solvable.

It is known that there is literature on portfolio choices under partial information (cf. [8]). It may be interesting to compare the methods there with ours and give interpretations from viewpoints of portfolio selections.

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