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Algorithms for distributed solution of the Optimal Power Flow problem

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***Abstract*— Modern power networks often have partitioned structures, with disjoint parts of the network being operated by competing agents. It is therefore of significant interest to be able to solve the Optimal Power Flow (OPF) problem for the entire network in a distributed manner without requiring significant exchange of sensitive operational information between rival operators. For networks whose regions are interconnected over a tree structure, we consider a comparison of two potential distributed schemes for solving the OPF problem. For the augmented algorithm, with incorporated higher-order dynamics, we prove a result of guaranteed convergence to the set of solutions of the global problem. By contrast, for classical unregularized dual decompositions, we demonstrate that there exist network scenarios in which convergence breaks down and the algorithm fails to give a solution of the OPF problem.**

***Index Terms*— distributed algorithms, optimization, power systems, stability analysis**

I. INTRODUCTION

Power flow studies investigate the flow of electrical power through networks with topologies that represent real-world power grids, incorporating known power loads and losses together with unknown amounts of power generation. The classical Optimal Power Flow (OPF) problem involves minimizing the total cost of active power generation over all of the network buses capable of generating power, subject to satisfying all of the given loads and a collection of inequality constraints corresponding to physical limits on the system's operation [1], [2], [3]. Power networks today are generally partitioned in structure, with different operators running disjoint components of the overall network. Given this structure, it is desirable to investigate approaches to solving the OPF problem that are distributed in nature, whereby each operator may solve its own local optimization problem independently. In particular, each operator would prefer to be able not to disclose the exact operational details of its own subsystem to the other operators, who may be its rivals and competitors. A number of distributed approaches to solving the OPF problem have been studied, for example in [4] and [5].

In the previous work [6], we proposed an algorithmic approach incorporating the method of dual decomposition with additional quadratic regularization in terms of locally-calculated auxiliary variables, leading to an update scheme with higher-order dynamics. In this paper, we will justify the importance of these regularization terms by first furnishing a convergence result for the continuous-time approximation to this algorithm, and then demonstrating that there exist physically-relevant systems for which a simple unregularized dual decomposition fails to converge to a globally optimal solution. The paper is structured as follows. We begin in section II by recalling the problem formulation from [6] and stating formally the two algorithms to be considered. A convergence result for the augmented algorithm is provided in section III, and examples are studied in section IV that demonstrate the failure of the unregularized algorithm to converge. Finally, conclusions are drawn in section V.

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II. DISTRIBUTED ALGORITHMS

In what follows, we will be interested in solving, in a distributed manner, the OPF problem for an arbitrary network composed of distinct regions, indexed by $\mathcal{I} \in \mathcal{E} := \{1, 2, \ldots, R\}$, interconnected over a tree structure by single inter-regional tie-lines. Each region consists of a set of buses $\hat{\mathcal{N}}^{\mathcal{I}}$, a set of generator buses $\hat{\mathcal{G}}^{\mathcal{I}}$, and a collection of flow lines connecting them. We then model a tie-line between regions $\mathcal{I}$ and $\mathcal{J}$ by adjoining to this subsystem $\mathcal{I}$ a single dummy generator bus by means of a flow line of twice the tie-line admittance connecting to the bus in $\hat{\mathcal{N}}^{\mathcal{I}}$ from which the tie-line in question emanates [6], and likewise for $\mathcal{J}$. Doing this for all regions forms the extended bus and generator sets $\mathcal{N}^{\mathcal{I}}$ and $\mathcal{G}^{\mathcal{I}}$, which we now label in such a way that the index $\ell_{\mathcal{I}\mathcal{J}}$ denotes the dummy bus corresponding to the $\mathcal{I} \to \mathcal{J}$ tie-line. Let us additionally introduce the notation:

- The functions $f_k^{\mathcal{I}}$ represent the local cost functions of active power generation at each generator bus $k \in \hat{\mathcal{G}}^{\mathcal{I}}$.
- The known quantities $P_{D_k}^{\mathcal{I}} + iQ_{D_k}^{\mathcal{I}}$ denote the drawn power at each bus $k \in \hat{\mathcal{N}}^{\mathcal{I}}$.
- The set $\mathcal{C}(\mathcal{I})$ is defined as the set of regions $\mathcal{J}$ that are connected to region $\mathcal{I}$ by means of a tie-line.
- The set $\mathcal{D}(\mathcal{I})$ is defined as the set of regions $\mathcal{J} \in \mathcal{C}(\mathcal{I})$ having index $\mathcal{J} < \mathcal{I}$.
- The quantity $\delta_{\mathcal{I}\mathcal{J}}$ equals 1 if $\mathcal{I} > \mathcal{J}$ and -1 if $\mathcal{I} < \mathcal{J}$.
- The matrices $\mathbf{Y}_k^{\mathcal{I}}$, $\bar{\mathbf{Y}}_k^{\mathcal{I}}$, $\mathbf{M}_k^{\mathcal{I}}$ at each bus $k \in \mathcal{N}^{\mathcal{I}}$ and the matrices $W^{\mathcal{I}}$ associated with the voltages are as defined in [7], [8] for each of the extended regions.

In this formulation, the classical active power generation, reactive power generation, and nodal squared voltage magnitude optimization variables may be written as $P_{D_k}^{\mathcal{I}} + \mathrm{Tr}\left(\mathbf{Y}_k^{\mathcal{I}} W^{\mathcal{I}}\right)$, $Q_{D_k}^{\mathcal{I}} + \mathrm{Tr}\left(\bar{\mathbf{Y}}_k^{\mathcal{I}} W^{\mathcal{I}}\right)$, and $\mathrm{Tr}\left(\mathbf{M}_k^{\mathcal{I}} W^{\mathcal{I}}\right)$ respectively. Relaxing the OPF problem as in [7], [8] and denoting by $\mathcal{R}^{\mathcal{I}}$ the convex feasible regions of all positive semidefinite matrices $W^{\mathcal{I}}$ that satisfy the network constraints[1] gives the

[1]These include the load satisfaction equality and physical inequality constraints on the nodal powers and nodal squared voltage magnitudes.

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convex matrix optimization problem

$$\begin{array}{ll} \underset{(W^{\mathcal{I}})_{\mathcal{I}\in\mathcal{E}}}{\text{minimize}} & \displaystyle\sum_{\mathcal{I}\in\mathcal{E}}\sum_{k\in\hat{\mathcal{G}}^{\mathcal{I}}} f_k^{\mathcal{I}}\left(P_{D_k}^{\mathcal{I}} + \operatorname{Tr}\left(\mathbf{Y}_k^{\mathcal{I}} W^{\mathcal{I}}\right)\right) \\ \underset{\forall \mathcal{I}\in\mathcal{E},\mathcal{J}\in\mathcal{D}(\mathcal{I})}{\text{subject to}} & \operatorname{Tr}\left(\mathbf{Y}_{\ell_{\mathcal{I}\mathcal{J}}}^{\mathcal{I}} W^{\mathcal{I}}\right) + \operatorname{Tr}\left(\mathbf{Y}_{\ell_{\mathcal{J}\mathcal{I}}}^{\mathcal{J}} W^{\mathcal{J}}\right) = 0, \\ & \operatorname{Tr}\left(\bar{\mathbf{Y}}_{\ell_{\mathcal{I}\mathcal{J}}}^{\mathcal{I}} W^{\mathcal{I}}\right) + \operatorname{Tr}\left(\bar{\mathbf{Y}}_{\ell_{\mathcal{J}\mathcal{I}}}^{\mathcal{J}} W^{\mathcal{J}}\right) = 0, \\ & \operatorname{Tr}\left(\mathbf{M}_{\ell_{\mathcal{I}\mathcal{J}}}^{\mathcal{I}} W^{\mathcal{I}}\right) - \operatorname{Tr}\left(\mathbf{M}_{\ell_{\mathcal{J}\mathcal{I}}}^{\mathcal{J}} W^{\mathcal{J}}\right) = 0, \\ & W^{\mathcal{I}} \in \mathcal{R}^{\mathcal{I}}. \end{array} \tag{1}$$

As discussed in [6], the method of dual decomposition may be applied to (1), introducing dual variables for each of the tie-line equality constraints and then adding them on to the objective function. The resulting primal optimization problem then separates into independent optimization problems local to each region $\mathcal{I}$, for any given values of the dual variables. The dual variables are then updated by a simple gradient-based update scheme, yielding the iterative solution method detailed in Algorithm 1. In [6], we also propose the modification of this scheme given in Algorithm 2, which incorporates higher-order dynamics and regularization in terms of local auxiliary variables in order to improve convergence properties without requiring any additional information exchange between regions as compared with Algorithm 1. These algorithms are both presented in terms of $\boldsymbol{\lambda}_{\mathcal{I}\mathcal{J}} = (\lambda_{\mathcal{I}\mathcal{J}}^P, \lambda_{\mathcal{I}\mathcal{J}}^Q, \lambda_{\mathcal{I}\mathcal{J}}^V)^T$ and $\mathbf{z}_{\mathcal{I}\mathcal{J}} = (z_{\mathcal{I}\mathcal{J}}^P, z_{\mathcal{I}\mathcal{J}}^Q, z_{\mathcal{I}\mathcal{J}}^V)^T$, which are stacked to give the global vectors $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\mathcal{I}\mathcal{J}}^T)_{\mathcal{I}\in\mathcal{E},\mathcal{J}\in\mathcal{D}(\mathcal{I})}^T$ and $\mathbf{z} = (\mathbf{z}_{\mathcal{I}\mathcal{J}}^T)_{\mathcal{I}\in\mathcal{E},\mathcal{J}\in\mathcal{C}(\mathcal{I})}^T$. The notation $\mathbf{T}_{\mathcal{I}\mathcal{J}}(W^{\mathcal{I}})$ denotes the vector of traces of a matrix $W^{\mathcal{I}}$, $\mathbf{T}_{\mathcal{I}\mathcal{J}}(W^{\mathcal{I}}) = (\operatorname{Tr}(\mathbf{Y}_{\ell_{\mathcal{I}\mathcal{J}}}^{\mathcal{I}} W^{\mathcal{I}}), \operatorname{Tr}(\bar{\mathbf{Y}}_{\ell_{\mathcal{I}\mathcal{J}}}^{\mathcal{I}} W^{\mathcal{I}}),\ \delta_{\mathcal{I}\mathcal{J}} \operatorname{Tr}(\mathbf{M}_{\ell_{\mathcal{I}\mathcal{J}}}^{\mathcal{I}} W^{\mathcal{I}}))^T$. Then $\mathbf{T}((W^{\mathcal{I}})_{\mathcal{I}\in\mathcal{E}}) = (\mathbf{T}_{\mathcal{I}\mathcal{J}}(W^{\mathcal{I}})^T)_{\mathcal{I}\in\mathcal{E},\mathcal{J}\in\mathcal{C}(\mathcal{I})}^T$, stacked in corresponding order to $\mathbf{z}$, and $\mathbf{U}((W^{\mathcal{I}})_{\mathcal{I}\in\mathcal{E}}) = (\mathbf{T}_{\mathcal{I}\mathcal{J}}(W^{\mathcal{I}})^T + \mathbf{T}_{\mathcal{J}\mathcal{I}}(W^{\mathcal{J}})^T)_{\mathcal{I}\in\mathcal{E},\mathcal{J}\in\mathcal{D}(\mathcal{I})}^T$, stacked in corresponding order to $\boldsymbol{\lambda}$. Finally, we define the objective functions

$$\operatorname{obj}_{\boldsymbol{\lambda}}^{\mathcal{I}}(W^{\mathcal{I}}) = \sum_{k\in\hat{\mathcal{G}}^{\mathcal{I}}} f_k^{\mathcal{I}}\left(P_{D_k}^{\mathcal{I}} + \operatorname{Tr}\left(\mathbf{Y}_k^{\mathcal{I}} W^{\mathcal{I}}\right)\right) + \sum_{\mathcal{J}\in\mathcal{C}(\mathcal{I})} \boldsymbol{\lambda}_{\mathcal{I}\mathcal{J}} \cdot \mathbf{T}_{\mathcal{I}\mathcal{J}}$$

and $\overline{\operatorname{obj}}_{\boldsymbol{\lambda},\mathbf{z}}^{\mathcal{I},\rho}(W^{\mathcal{I}}) = \operatorname{obj}_{\boldsymbol{\lambda}}^{\mathcal{I}}(W^{\mathcal{I}}) + \frac{\rho}{2}\sum_{\mathcal{J}\in\mathcal{C}(\mathcal{I})} \|T_{\mathcal{I}\mathcal{J}}(W^{\mathcal{I}}) - z_{\mathcal{I}\mathcal{J}}\|_2^2$. In this way, we obtain two potential distributed schemes for solving the relaxed OPF problem (1). The convergence properties of these schemes are studied below.

III. GUARANTEED CONVERGENCE RESULT

We shall begin by investigating the convergence of the modified scheme given in Algorithm 2. To gain insight, let us consider the differential equation approximation to the update rules (7) and (8) that is obtained in the continuous-time limit as the step sizes become small. This gives us the ordinary differential equation system

$$\frac{d\boldsymbol{\lambda}^{\rho}}{dt} = \kappa \mathbf{U}\left((W^{\mathcal{I},\rho}(\boldsymbol{\lambda}^{\rho}, \mathbf{z}^{\rho})_{\mathcal{I}\in\mathcal{E}})\right), \tag{2}$$

$$\frac{d\mathbf{z}^{\rho}}{dt} = \rho\kappa\left(\mathbf{T}\left((W^{\mathcal{I},\rho}(\boldsymbol{\lambda}^{\rho}, \mathbf{z}^{\rho}))_{\mathcal{I}\in\mathcal{E}}\right) - \mathbf{z}_n^{\rho}\right), \tag{3}$$

where $W^{\mathcal{I},\rho}(\boldsymbol{\lambda}^{\rho}, \mathbf{z}^{\rho}) = \operatorname{argmin}_{W^{\mathcal{I}}\in\mathcal{R}^{\mathcal{I}}} \overline{\operatorname{obj}}_{\boldsymbol{\lambda}^{\rho},\mathbf{z}^{\rho}}^{\mathcal{I},\rho}(W^{\mathcal{I}})$ subject to the consistency constraint imposed in Algorithm 2 and κ is a positive feedback gain constant. Since the set of fixed points of Algorithm 2 is nonempty [6] and any fixed point $((W_*^{\mathcal{I},\rho})_{\mathcal{I}\in\mathcal{E}}, \boldsymbol{\lambda}_*^{\rho}, \mathbf{z}_*^{\rho})$ corresponds to an equilibrium solution $\boldsymbol{\lambda}^{\rho} = \boldsymbol{\lambda}_*^{\rho}$, $\mathbf{z}^{\rho} = \mathbf{z}_*^{\rho}$ of the system (2)–(3), we will investigate the asymptotic stability of the nonempty set

$$E^{\rho} = \left\{ \begin{pmatrix} \boldsymbol{\lambda}^{\rho} \\ \mathbf{z}^{\rho} \end{pmatrix} : \mathbf{U}\big((W^{\mathcal{I},\rho}(\boldsymbol{\lambda}^{\rho}, \mathbf{z}^{\rho}))_{\mathcal{I}\in\mathcal{E}}\big) = 0, \right.$$
$$\left. \text{and } \mathbf{T}\big((W^{\mathcal{I},\rho}(\boldsymbol{\lambda}^{\rho}, \mathbf{z}^{\rho}))_{\mathcal{I}\in\mathcal{E}}\big) = \mathbf{z}^{\rho} \right\}.$$

Algorithm 1 Distributed dual decomposition algorithm for the relaxed OPF problem.

while $\Delta_n >$ tolerance **do**

1. Optimize regional problems to obtain $W_{n+1}^{\mathcal{I}}$ as

$$W_{n+1}^{\mathcal{I}} = \underset{W^{\mathcal{I}}\in\mathcal{R}^{\mathcal{I}}}{\operatorname{argmin}}\ \operatorname{obj}_{\boldsymbol{\lambda}_n}^{\mathcal{I}}\left(W^{\mathcal{I}}\right), \tag{4}$$

according to the consistency criterion that if given the same values of $\lambda_{\mathcal{I}\mathcal{J},n}^P$, $\lambda_{\mathcal{I}\mathcal{J},n}^Q$, $\lambda_{\mathcal{I}\mathcal{J},n}^V$ on multiple occasions the optimization method employed should on each occasion yield the same answer.

2. Update the dual variables by

$$\boldsymbol{\lambda}_{n+1} = \boldsymbol{\lambda}_n + \alpha_n \mathbf{U}\left((W_{n+1}^{\mathcal{I}})_{\mathcal{I}\in\mathcal{E}}\right). \tag{5}$$

Update the remaining dual variables by the symmetry rule $\lambda_{\mathcal{J}\mathcal{I}}^P = \lambda_{\mathcal{I}\mathcal{J}}^P$, $\lambda_{\mathcal{J}\mathcal{I}}^Q = \lambda_{\mathcal{I}\mathcal{J}}^Q$, and $\lambda_{\mathcal{J}\mathcal{I}}^V = \lambda_{\mathcal{I}\mathcal{J}}^V$.

3. Update the residual error by setting Δ_{n+1} equal to

$$\Delta\left((W_{n+1}^{\mathcal{I}})_{\mathcal{I}\in\mathcal{E}}\right) = \left\|\mathbf{U}\left((W_{n+1}^{\mathcal{I}})_{\mathcal{I}\in\mathcal{E}}\right)\right\|_{\infty}.$$

end while

Algorithm 2 Distributed dual decomposition algorithm for the relaxed OPF problem with quadratic regularization terms.

while $\Delta_n^{\rho} >$ tolerance **do**

1. Optimize regional problems to obtain $W_{n+1}^{\mathcal{I},\rho}$ as

$$W_{n+1}^{\mathcal{I},\rho} = \underset{W^{\mathcal{I}}\in\mathcal{R}^{\mathcal{I}}}{\operatorname{argmin}}\ \overline{\operatorname{obj}}_{\boldsymbol{\lambda}_n^{\rho},\mathbf{z}_n^{\rho}}^{\mathcal{I},\rho}\left(W^{\mathcal{I}}\right), \tag{6}$$

according to the consistency criterion that if given the same values of $\lambda_{\mathcal{I}\mathcal{J},n}^{P,\rho}$, $\lambda_{\mathcal{I}\mathcal{J},n}^{Q,\rho}$, $\lambda_{\mathcal{I}\mathcal{J},n}^{V,\rho}$, $z_{\mathcal{I}\mathcal{J},n}^{P,\rho}$, $z_{\mathcal{I}\mathcal{J},n}^{Q,\rho}$, and $z_{\mathcal{I}\mathcal{J},n}^{V,\rho}$ on multiple occasions the optimization method employed should on each occasion yield the same answer.

2. Update the dual variables by

$$\boldsymbol{\lambda}_{n+1}^{\rho} = \boldsymbol{\lambda}_n^{\rho} + \alpha_n \mathbf{U}\left((W_{n+1}^{\mathcal{I},\rho})_{\mathcal{I}\in\mathcal{E}}\right). \tag{7}$$

Update the remaining dual variables by the symmetry rule $\lambda_{\mathcal{J}\mathcal{I}}^{P,\rho} = \lambda_{\mathcal{I}\mathcal{J}}^{P,\rho}$, $\lambda_{\mathcal{J}\mathcal{I}}^{Q,\rho} = \lambda_{\mathcal{I}\mathcal{J}}^{Q,\rho}$, and $\lambda_{\mathcal{J}\mathcal{I}}^{V,\rho} = \lambda_{\mathcal{I}\mathcal{J}}^{V,\rho}$.

3. Update the local auxiliary variables by

$$\mathbf{z}_{n+1}^{\rho} = \mathbf{z}_n^{\rho} + \rho\alpha_n\left(\mathbf{T}\left((W_{n+1}^{\mathcal{I},\rho})_{\mathcal{I}\in\mathcal{E}}\right) - \mathbf{z}_n^{\rho}\right). \tag{8}$$

4. Update the residual error by setting Δ_{n+1}^{ρ} equal to

$$\Delta^{\rho}\left((W_{n+1}^{\mathcal{I},\rho})_{\mathcal{I}\in\mathcal{E}}\right) = \left\|\mathbf{U}\left((W_{n+1}^{\mathcal{I},\rho})_{\mathcal{I}\in\mathcal{E}}\right)\right\|_{\infty}.$$

end while

We now state our main convergence result[2] for Algorithm 2. This extends the stability proof in [6] by additionally showing that the primal residual value converges to zero. As will be seen in section IV, this is important since, by contrast, for the unregularized scheme in Algorithm 1 there exist cases for which this primal residual convergence can fail to hold.

Theorem 1: Suppose that $\rho \neq 0$. Then, given any initial conditions $\boldsymbol{\lambda}^\rho(0) = \boldsymbol{\lambda}_0^\rho$, $\mathbf{z}^\rho(0) = \mathbf{z}_0^\rho$ in the differential equation system (2)–(3), the following hold:

1) *Dual/auxiliary convergence:* The dual and auxiliary variables satisfy $(\boldsymbol{\lambda}^\rho(t)^T, \mathbf{z}^\rho(t)^T)^T \to E^\rho$ as $t \to \infty$.
2) *Primal residual convergence:* The primal residual error satisfies $\Delta^\rho((W^{\mathcal{I},\rho}(\boldsymbol{\lambda}^\rho(t), \mathbf{z}^\rho(t)))_{\mathcal{I}\in\mathcal{E}}) \to 0$ as $t \to \infty$.

Proof: We sketch the method of the proof here.

1) Using convexity and optimality arguments, the function $V(\boldsymbol{\lambda}^\rho, \mathbf{z}^\rho) = \|\boldsymbol{\lambda}^\rho - \boldsymbol{\lambda}_*^\rho\|_2^2 + \|\mathbf{z}^\rho - \mathbf{z}_*^\rho\|_2^2$ can be shown to be a radially unbounded Lyapunov function for the system (2)–(3). Moreover, it is seen that any trajectory along which $\frac{d}{dt}V(\boldsymbol{\lambda}^\rho, \mathbf{z}^\rho) \equiv 0$ must be an equilibrium point, whence Lasalle's Theorem applies and guarantees that, given any initial conditions $\boldsymbol{\lambda}^\rho(0) = \boldsymbol{\lambda}_0^\rho$, $\mathbf{z}^\rho(0) = \mathbf{z}_0^\rho$, the solutions of (2)–(3) must satisfy $(\boldsymbol{\lambda}^\rho(t)^T, \mathbf{z}^\rho(t)^T)^T \to E^\rho$ as $n \to \infty$.

2) We prove, by an argument similar to that in [9], a continuity result for the optimal solutions of the primal minimization problems (6) near any equilibrium $((\boldsymbol{\lambda}_*^\rho)^T, (\mathbf{z}_*^\rho)^T)^T \in E^\rho$ in terms of the pseudometric $d^{\mathcal{I}}(W^{\mathcal{I}}, \tilde{W}^{\mathcal{I}}) = \sum_{\mathcal{J}\in\mathcal{C}(\mathcal{I})} \|T_{\mathcal{I}\mathcal{J}}(W^{\mathcal{I}}) - T_{\mathcal{I}\mathcal{J}}(\tilde{W}^{\mathcal{I}})\|_2$. Then we invoke the result of part 1) to conclude that therefore $\Delta^\rho((W^{\mathcal{I},\rho}(\boldsymbol{\lambda}^\rho(t), \mathbf{z}^\rho(t)))_{\mathcal{I}\in\mathcal{E}}) \to 0$ as $t \to \infty$. ∎

Theorem 1 provides a guarantee that the equilibrium set E^ρ of the system (2)–(3) is globally asymptotically stable, and furthermore that the primal residual decays to zero along all system trajectories. From these results we may infer that, provided that the step sizes are small enough, Algorithm 2 will terminate in finite time and the collection of primal iterates will converge to a solution of the global problem (1).

IV. EXAMPLES

In this section, we will show, through both analytical study and numerical simulations, that there exist cases in which the convergence of Algorithm 1 to a solution of the global problem (1) fails, thus emphasizing the importance of incorporating modifications as in Algorithm 2. Numerical simulations of an application of Algorithm 2 to the example studied are also included, illustrating Theorem 1.

A. Formulation of the example

We begin by describing in detail the example system that is to be considered.

Note firstly that, in order to avoid unboundedness[3] in the primal problems (4) within Algorithm 1, we must impose constraints on the voltage magnitudes at the dummy buses. This can be done by incorporating within our original model constraints along the flow lines of the form discussed in [8]. In their full form, such constraints consist of coupling linear inequality constraints, which may be incorporated within our framework by Lagrangian relaxation in terms of an additional set of dual variables. However, in order to clarify the analysis in what follows, we shall instead impose within the definitions of the feasible sets $\mathcal{R}^{\mathcal{I}}$ the consequent simpler independent limits $(\underline{V}^{\mathcal{I}}_{\ell_{\mathcal{I}\mathcal{J}}})^2 \leq \mathrm{Tr}(\mathbf{M}^{\mathcal{I}}_{\ell_{\mathcal{I}\mathcal{J}}} W^{\mathcal{I}}) \leq (\overline{V}^{\mathcal{I}}_{\ell_{\mathcal{I}\mathcal{J}}})^2$. Note that in general the allowed range $[(\underline{V}^{\mathcal{I}}_{\ell_{\mathcal{I}\mathcal{J}}})^2, (\overline{V}^{\mathcal{I}}_{\ell_{\mathcal{I}\mathcal{J}}})^2]$ can be significantly wider than the permitted ranges of the corresponding nodal voltage restrictions at buses in $\tilde{\mathcal{N}}^{\mathcal{I}}$ and that symmetry implies the relations $\underline{V}^{\mathcal{J}}_{\ell_{\mathcal{J}\mathcal{I}}} = \underline{V}^{\mathcal{I}}_{\ell_{\mathcal{I}\mathcal{J}}}$ and $\overline{V}^{\mathcal{J}}_{\ell_{\mathcal{J}\mathcal{I}}} = \overline{V}^{\mathcal{I}}_{\ell_{\mathcal{I}\mathcal{J}}}$ for all $\mathcal{J} \in \mathcal{C}(\mathcal{I})$.

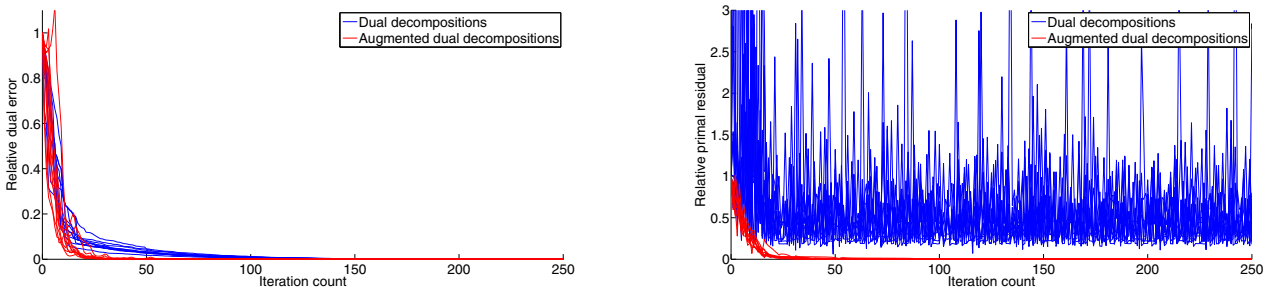
The simple network of interest here consists of three regions, each having a single bus, connected in a linear arrangement. The central bus, constituting region 1, is connected to a given active and reactive load, while the two end buses, respectively constituting regions 2 and 3, are generators with increasing local cost functions. Such a system can be thought of as a large-scale model of three interconnected complex systems in which the central system is known to be a net consumer, while the end systems are known to both be net generators. Let us now assume that the generators 2 and 3 are both online and have nonzero minimal active generation limits $\underline{P}^2$ and $\underline{P}^3$ and that generator 3 is arbitrarily declared the slack bus for the system. Then, consider a situation in which the active load at 1 has dropped to a level significantly below the total minimal active generation, however, operational reasons require that both generators remain connected to the network. Such a situation is realistic in cases where the generators' specifications include prohibitive start-up costs or slow ramp-rates.

B. Analytical study

Analytical solution of the centralized problem (1) shows that the optimal Lagrange multipliers for this problem are given by $\boldsymbol{\lambda}_* = \mathbf{0}$. This corresponds to the fact that, due to the low-load condition, the global system may be operated optimally at a range of voltage values. In particular, it can be shown that, when $\boldsymbol{\lambda} = \boldsymbol{\lambda}_*$, the local problems corresponding to (4) admit optimal solutions $\hat{W}^1$, $\hat{W}^2$, $\hat{W}^3$ satisfying the voltage levels $\mathrm{Tr}(\mathbf{M}^1_{\ell_{12}}\hat{W}^1) = (\overline{V}^1_{\ell_{12}})^2$, $\mathrm{Tr}(\mathbf{M}^1_{\ell_{13}}\hat{W}^1) = (\overline{V}^1_{\ell_{13}})^2$, $\mathrm{Tr}(\mathbf{M}^2_{\ell_{21}}\hat{W}^2) = (\underline{V}^2_{\ell_{21}})^2$, $\mathrm{Tr}(\mathbf{M}^3_{\ell_{31}}\hat{W}^3) = (\underline{V}^3_{\ell_{31}})^2$, and the power equalities $\mathrm{Tr}\left(\mathbf{Y}^2_{\ell_{21}}\hat{W}^2\right) + \mathrm{Tr}\left(\mathbf{Y}^1_{\ell_{12}}\hat{W}^1\right) = 0$, $\mathrm{Tr}\left(\bar{\mathbf{Y}}^2_{\ell_{21}}\hat{W}^2\right) + \mathrm{Tr}\left(\bar{\mathbf{Y}}^1_{\ell_{12}}\hat{W}^1\right) = 0$, $\mathrm{Tr}\left(\mathbf{Y}^3_{\ell_{31}}\hat{W}^3\right) + \mathrm{Tr}\left(\mathbf{Y}^1_{\ell_{13}}\hat{W}^1\right) = 0$, $\mathrm{Tr}\left(\bar{\mathbf{Y}}^3_{\ell_{31}}\hat{W}^3\right) + \mathrm{Tr}\left(\bar{\mathbf{Y}}^1_{\ell_{13}}\hat{W}^1\right) = 0$. Consequently, it is clear that if $\boldsymbol{\lambda}_n = (0, 0, \mu, 0, 0, \nu)^T$ with μ, $\nu > 0$, then the matrices $\hat{W}^1$, $\hat{W}^2$, $\hat{W}^3$ are in fact optimal for the local primal problems (4) at the $(n+1)^{\text{st}}$ iteration. Thus, suppose that $W^1_{n+1} = \hat{W}^1$, $W^2_{n+1} = \hat{W}^2$, $W^3_{n+1} = \hat{W}^3$. Then $\mathbf{U}((W^{\mathcal{I}}_{n+1})_{\mathcal{I}\in\{1,2,3\}}) = (0, 0, (\underline{V}^2_{\ell_{21}})^2 - (\overline{V}^1_{\ell_{12}})^2, 0, 0, (\underline{V}^3_{\ell_{31}})^2 - (\overline{V}^1_{\ell_{13}})^2)^T$, whence, for sufficiently small step sizes, the update rules (5) imply that $\boldsymbol{\lambda}_{n+1}$ has

[2]It should be noted that, unlike alternative schemes such as ADMM, the regularization method invoked here requires no additional information exchange between regions compared to the original dual decomposition scheme, maintaining the fully distributed structure.

[3]Such unboundedness is avoided in Algorithm 2 by the strict convexity of (6) in the complicating variables. Thus, Algorithm 2 is immediately applicable to a wide class of problems for which Algorithm 1 will fail.



(a) (b)

Fig. 1. The evolution of (a) the relative error in the dual variables and (b) the relative value of the primal residual for Algorithms 1 and 2 (with the parameter ρ set to 1) applied to the three-region example system with ten randomly chosen initial conditions.

the same form as $\boldsymbol{\lambda}_n$ and this analysis applies also at iteration $n+2$. Inductively, this implies that, whenever $\boldsymbol{\lambda}_0$ has the structural form $(0,0,+,0,0,+)^T$, it is possible to choose step sizes such that $\boldsymbol{\lambda}_n \to \boldsymbol{\lambda}_*$ as $n \to \infty$ and yet $\Delta_{n+1} = \max\{(\overline{V}^1_{\ell_{12}})^2 - (\underline{V}^2_{\ell_{21}})^2, (\overline{V}^1_{\ell_{13}})^2 - (\underline{V}^3_{\ell_{31}})^2\} > 0$ for all $n \in \mathbb{N}$. Therefore, the primal iterates fail to converge toward feasibility for (1), and Algorithm 1 fails.

The significant point in the foregoing analysis is that the objective function $\mathrm{obj}^{\mathcal{I}}_{\boldsymbol{\lambda}}(W^{\mathcal{I}})$ is not strictly convex in the complicating variables $T_{\mathcal{I}\mathcal{J}}(W^{\mathcal{I}})$. This fact permits there to be multiple possible solutions for $W^{\mathcal{I}}_{n+1}$ in (4) with differing values of these complicating variables, admitting the possibility of primal divergent behavior even when the dual variables converge to optimality. By contrast, the quadratic regularization terms present in $\overline{\mathrm{obj}}^{\mathcal{I},\rho}_{\boldsymbol{\lambda},\mathbf{z}}(W^{\mathcal{I}})$ make this augmented objective function strictly convex in each of these complicating variables. Consequently, these terms must have unique values at any possible optimal solution $W^{\mathcal{I},\rho}_{n+1}$ of (6), avoiding such problematic possibilities.

C. Numerical simulations

In order to demonstrate the performance of the two algorithms applied with general initial values to this particular example, we plot in Fig. 1 both the evolution of the error in the dual variables, in (a), and the evolution of the primal residual, in (b). We see clearly that, while both methods yield convergence of the dual variables to their optimal values, only Algorithm 2 shows convergence of the primal residual to zero. By contrast, the primal residual from Algorithm 1 is seen to oscillate among values significantly away from zero, with no decay being observed over the 250 iterations simulated. This means that the value of Δ_{n+1} does not approach zero as $n \to \infty$, which implies that the primal iterates obtained from Algorithm 1 do not tend toward satisfaction of the tie-line equality constraints featuring in (1). Therefore, we conclude that there exist general, physically-relevant, network situations for which Algorithm 1 fails to solve (1) and indeed modified schemes such as that suggested in Algorithm 2 are required to obtain a global solution. Since the precise network conditions are not, in general, known a priori, such convergence breakdowns represent a significant problem with Algorithm 1 and justify the consideration of the more involved dynamics present in Algorithm 2.

V. CONCLUSIONS

We have considered a comparison of two potential distributed algorithmic approaches to the solution of the OPF problem in a general network composed of distinct regions interconnected over a tree structure. For the algorithm that involves a regularization by means of local auxiliary variables, we used Lyapunov theory and quantitative stability analysis to prove a convergence result, for the equivalent continuous-time formulation, for both the dual and auxiliary variables and the primal residual. This implies that, for sufficiently small step size, this augmented scheme will terminate in finite time and yield an optimal solution for the global network OPF problem. By contrast, we then investigated an example of a physically-relevant network scenario in which the straightforward unregularized dual decomposition algorithm fails to converge to a globally optimal solution, showing that there exist situations in which this simpler approach is insufficient for solving the global OPF problem. Numerical simulations have been provided to illustrate these results. Preliminary simulations also illustrate that the augmented algorithm can offer an improved convergence rate, and quantifying this is part of ongoing work.

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