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# Product convolutional codes from linear system point of view*

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***Abstract*— In this work, from a controllable and observable input-state-output representation of two convolutional codes, we present an input-state-output representation of a convolutional product code. We also establish a lower bound on the free distance of the product code in terms of the free distance of the constituent codes.**

## I. Introduction

Combining known codes is a powerful method to obtain a new code with better error correction capability avoiding the exponential increase of decoding complexity. For convolutional codes, we can find in the literature some powerful combining methods as turbo codes [1] and woven convolutional codes [2], [3]. More recently, as a natural extension of the direct product codes introduced by Elias [4], Bossert, Medina and Sidorenko [5] introduce the product of convolutional codes and they show that every convolutional product code can be represented as a woven code.

On the other hand, Climent, Herranz and Perea [6] and Herrnaz [7], using the input-state-output representatation of convolutional codes introduced by Rosenthal and York [8], introduce the input-state-output representation of different serial and parallel concatenation convolutional codes. With this concatenated convolutional codes they also present a construction of new codes with prescribed distance.

The paper is structured as follows. In Section II we review the way in which convolutional codes have been described in the context of linear systems theory. We also describe how product convolutional codes are constructed as convolutional counterpart of the direct product of block codes. In Section III we present an input-state-output representation of a product code from the input-state-output representations of the constituent codes. We also give a lower bound of the free distance of the product code in terms of the free distances of the constituent codes. Finally, conclusions are presented in Section IV.

## II. Preliminaries

### A. Input-State-Output Representation of a Convolutional Code

In this paper, we denote by $\mathbb{F}$ a finite field and by $\mathbb{F}[z]$ the polynomial ring on the variable $z$ with coefficients in $\mathbb{F}$.

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A *convolutional code* of rate $k/n$ is a submodule $\mathcal{C}$ of $\mathbb{F}^n[z]$ with rank $k$. So, there exists an $n \times k$ polynomial encoder matrix $G(z) \in \mathbb{F}[z]^{n\times k}$, which is basic and minimal, such that

$$\mathcal{C} = \{\boldsymbol{v}(z) \in \mathbb{F}[z]^n \mid \boldsymbol{v}(z) = G(z)\boldsymbol{u}(z) \text{ with } \boldsymbol{u}(z) \in \mathbb{F}[z]^k\}.$$

A convolutional code is *observable* if one, and therefore any, generator matrix $G(z)$ is right prime (see [9]). Furthermore, if $G(z)$ is a generator matrix of an observable convolutional code, then $G(z)$ is a noncatastrophic generator matrix (see [9]). The *degree* or *complexity* of $\mathcal{C}$ is the sum of the column degrees of one, and hence any, minimal basic generator matrix. A convolutional code $\mathcal{C}$ of rate $k/n$ and degree $\delta$, called an $(n, k, \delta)$-code, can be given by the *input-state-output (ISO) representation* (see [10], [11], [8], [12])

$$\begin{aligned} \boldsymbol{x}_{t+1} &= A\boldsymbol{x}_t + B\boldsymbol{u}_t, \\ \boldsymbol{y}_t &= C\boldsymbol{x}_t + D\boldsymbol{u}_t, \\ \boldsymbol{v}_t &= \begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{u}_t \end{pmatrix}, \quad \boldsymbol{x}_0 = \boldsymbol{0}, \end{aligned} \tag{1}$$

where for each instant $t$, $\boldsymbol{x}_t \in \mathbb{F}^\delta$ is the *state vector*, $\boldsymbol{u}_t \in \mathbb{F}^k$ is the *information vector*, $\boldsymbol{y}_t \in \mathbb{F}^{n-k}$ is the *parity vector*, and $\boldsymbol{v}_t \in \mathbb{F}^n$ is a codeword of $\mathcal{C}$. In this case, $\mathcal{C}$ is said to be generated by $(A, B, C, D)$ and denoted by $\mathcal{C}(A, B, C, D)$. Here, $A$, $B$, $C$, and $D$ are matrices of sizes $\delta \times \delta$, $\delta \times k$, $(n - k) \times \delta$ and $(n - k) \times k$, respectively; that is, $(A, B, C, D)$ is a *minimal representation* and it is characterized through the condition that the pair $(A, B)$ is *controllable*. Note that the concept of minimality of an ISO representation is different from the concept of minimality of a representation in classical linear systems theory. In fact, if $(A, B)$ is controllable, then the observability of $(A, C)$ ensures that the linear system (1) describes a noncatastrophic convolutional code (see [8, Lemma 2.11]).

The *free distance* of a convolutional code $\mathcal{C}$ can be characterized (see [10]) as

$$d_{free}(\mathcal{C}) = \min\left(\sum_{t=0}^{\infty} \mathrm{wt}(\boldsymbol{u}_t) + \sum_{t=0}^{\infty} \mathrm{wt}(\boldsymbol{y}_t)\right)$$

where the minimum has to be taken over all possible nonzero codewords and where wt denotes the Hamming weight. The free distance of an $(n, k, \delta)$-code $\mathcal{C}$ is always upper-bounded (see [13]) by the *generalized Singleton bound*

$$d_{free}(\mathcal{C}) \leq (n - k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + 1.$$

In addition, the convolutional code $\mathcal{C}$ is called *maximum-distance separable* (MDS) if its free distance is equal to the generalized Singleton bound.

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### *B. Product Convolutional Codes*

Let $\mathcal{C}^{\shortmid}$ and $\mathcal{C}^{-}$ be "vertical" $(n^{\shortmid}, k^{\shortmid}, \delta^{\shortmid})$ and "horizontal" $(n^{-}, k^{-}, \delta^{-})$-codes respectively. Then, the *product convolutional code* $\mathcal{C}^{\shortmid} \otimes \mathcal{C}^{-}$ is defined to be the convolutional code whose codewords consist of all $n^{\shortmid} \times n^{-}$ matrices with all columns in $\mathcal{C}^{\shortmid}$ and all rows in $\mathcal{C}^{-}$ (see [5], [14]). It is an $(n^{\shortmid}n^{-}, k^{\shortmid}k^{-}, \delta^{\shortmid}k^{-} + \delta^{-}k^{\shortmid})$-code.

Encoding of the product convolutional code $\mathcal{C}^{\shortmid} \otimes \mathcal{C}^{-}$ can be done as follows (see [5], [14]). Let $G^{\shortmid}(z)$ and $G^{-}(z)$ be generator matrices of the component convolutional codes $\mathcal{C}^{\shortmid}$ and $\mathcal{C}^{-}$, respectively. Denote by $U(z)$ a $k^{\shortmid} \times k^{-}$ information matrix over $\mathbb{F}[z]$. Now, every column of $U(z)$ is encoded using $G^{\shortmid}(z)$, and then every row of the resulting matrix is encoding using $G^{-}(z)$. Thus, the encoded codeword matrix $V(z)$ is given by

$$V(z) = G^{\shortmid}(z)U(z)G^{-}(z)^{T}.$$

Note that $V(z)$ can be described in terms of the Kronecker product of the generator matrices $G^{\shortmid}(z)$ and $G^{-}(z)$ as follows:

$$\operatorname{vect}(V(z)) = \left(G^{-}(z) \otimes G^{\shortmid}(z)\right) \operatorname{vect}(U(z))$$

where $\operatorname{vect}(\cdot)$ denotes the vectorization of a matrix formed by stacking its columns into a single column vector.

## III. Input-State-Output Representation of a Product Convolutional Code

In this section we obtain the matrices that describes an ISO representation of the product code from ISO representations of the vertical and horizontal codes $\mathcal{C}^{\shortmid}$ and $\mathcal{C}^{-}$.

Assume that $(A^{\shortmid}, B^{\shortmid}, C^{\shortmid}, D^{\shortmid})$ is an ISO representation of the vertical $(n^{\shortmid}, k^{\shortmid}, \delta^{\shortmid})$ code $\mathcal{C}^{\shortmid}$. Analogously, let $(A^{-}, B^{-}, C^{-}, D^{-})$ be an ISO representation of the horizontal $(n^{-}, k^{-}, \delta^{-})$ code $\mathcal{C}^{-}$. Assume also that the $k^{\shortmid} \times k^{-}$ matrix $U_t$ is the information matrix of the product code $\mathcal{C}^{\shortmid} \otimes \mathcal{C}^{-}$.

By using the vertical code $\mathcal{C}^{\shortmid}$, we can encode the columns of $U_t$ as

$$\begin{aligned} \boldsymbol{x}_{t+1}^{(1)} &= A_1 \boldsymbol{x}_t^{(1)} + B_1 \boldsymbol{u}_t^{(1)}, \\ \boldsymbol{y}_t^{(1)} &= C_1 \boldsymbol{x}_t^{(1)} + D_1 \boldsymbol{u}_t^{(1)}, \\ \boldsymbol{v}_t^{(1)} &= \begin{pmatrix} \boldsymbol{y}_t^{(1)} \\ \boldsymbol{u}_t^{(1)} \end{pmatrix}, \quad \boldsymbol{x}_0^{(1)} = \mathbf{0}, \end{aligned} \tag{2}$$

where $\boldsymbol{u}_t^{(1)} = \operatorname{vect}(U_t)$ and

$$\begin{aligned} A_1 &= I_{k^-} \otimes A^{\shortmid}, \quad B_1 = I_{k^-} \otimes B^{\shortmid}, \\ C_1 &= I_{k^-} \otimes C^{\shortmid}, \quad D_1 = I_{k^-} \otimes D^{\shortmid}. \end{aligned}$$

Note that $A_1$, $B_1$, $C_1$, and $D_1$ are matrices of sizes $\delta^{\shortmid}k^{-} \times \delta^{\shortmid}k^{-}$, $\delta^{\shortmid}k^{-} \times k^{\shortmid}k^{-}$, $(n^{\shortmid}-k^{\shortmid})k^{-} \times \delta^{\shortmid}k^{-}$, and $(n^{\shortmid}-k^{\shortmid})k^{-} \times k^{\shortmid}k^{-}$, respectively.

Now, by using the horizontal code $\mathcal{C}^{-}$, we can encode the rows of $U_t$. For this, we need transform each row of matrix $U_t$ in an appropriate column vector; we can carry out this process as $\boldsymbol{u}_t^{(2)} = Q\boldsymbol{u}_t^{(1)}$ where $Q$ is the permutation matrix given by

$$Q = \begin{pmatrix} I_{k^-} \otimes \boldsymbol{e}_1^T \\ I_{k^-} \otimes \boldsymbol{e}_2^T \\ \vdots \\ I_{k^-} \otimes \boldsymbol{e}_{k^{\shortmid}}^T \end{pmatrix}.$$

Here $\boldsymbol{e}_i$ is the $k^{\shortmid}$-column vector with a 1 in the $i$th entry and 0 otherwise. So, the horizontal code $\mathcal{C}^{-}$ encodes the vector $\boldsymbol{u}_t^{(2)}$ as

$$\begin{aligned} \boldsymbol{x}_{t+1}^{(2)} &= A_2 \boldsymbol{x}_t^{(2)} + B_2 \boldsymbol{u}_t^{(2)}, \\ \boldsymbol{y}_t^{(2)} &= C_2 \boldsymbol{x}_t^{(2)} + D_2 \boldsymbol{u}_t^{(2)}, \\ \boldsymbol{v}_t^{(2)} &= \begin{pmatrix} \boldsymbol{y}_t^{(2)} \\ \boldsymbol{u}_t^{(2)} \end{pmatrix}, \quad \boldsymbol{x}_0^{(2)} = \mathbf{0}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} A_2 &= I_{k^{\shortmid}} \otimes A^{-}, \quad B_2 = I_{k^{\shortmid}} \otimes B^{-}, \\ C_2 &= I_{k^{\shortmid}} \otimes C^{-}, \quad D_2 = I_{k^{\shortmid}} \otimes D^{-}. \end{aligned}$$

Note that $A_2$, $B_2$, $C_2$, and $D_2$ are matrices of sizes $\delta^{-}k^{\shortmid} \times \delta^{-}k^{\shortmid}$, $\delta^{-}k^{\shortmid} \times k^{\shortmid}k^{-}$, $(n^{-}-k^{-})k^{\shortmid} \times \delta^{-}k^{\shortmid}$, and $(n^{-}-k^{-})k^{\shortmid} \times k^{-}k^{\shortmid}$, respectively.

Following a reciprocal process to the one used to obtain the column vector $\boldsymbol{u}_t^{(1)}$ from the matrix $U_t$; that is, $\boldsymbol{u}_t^{(1)} = \operatorname{vect}(U_t)$, we obtain the matrix $Y_t$ from the parity vector $\boldsymbol{y}_t^{(1)}$; that is, $\boldsymbol{y}_t^{(1)} = \operatorname{vect}(Y_t)$. Then, by using again the horizontal code $\mathcal{C}^{-}$ we can encode the rows of $Y_t$. Similarly to the previous case, we consider the column vector $\boldsymbol{u}_t^{(3)} = P\boldsymbol{y}_t^{(1)}$ where $P$ is the permutation matrix given by

$$P = \begin{pmatrix} I_{k^-} \otimes \boldsymbol{d}_1^T \\ I_{k^-} \otimes \boldsymbol{d}_2^T \\ \vdots \\ I_{k^-} \otimes \boldsymbol{d}_{n^{\shortmid}-k^{\shortmid}}^T \end{pmatrix}$$

where $\boldsymbol{d}_i$ is the $(n^{\shortmid}-k^{\shortmid})$-column vector with a 1 in the $i$th entry and 0 otherwise. So, the horizontal code $\mathcal{C}^{-}$ encodes the vector $\boldsymbol{u}_t^{(3)}$ as

$$\begin{aligned} \boldsymbol{x}_{t+1}^{(3)} &= A_3 \boldsymbol{x}_t^{(3)} + B_3 P\boldsymbol{y}_t^{(1)}, \\ \boldsymbol{y}_t^{(3)} &= C_3 \boldsymbol{x}_t^{(3)} + D_3 P\boldsymbol{y}_t^{(1)}, \\ \boldsymbol{v}_t^{(3)} &= \begin{pmatrix} \boldsymbol{y}_t^{(3)} \\ \boldsymbol{u}_t^{(3)} \end{pmatrix}, \quad \boldsymbol{x}_0^{(3)} = \mathbf{0}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} A_3 &= I_{n^{\shortmid}-k^{\shortmid}} \otimes A^{-}, \quad B_3 = I_{n^{\shortmid}-k^{\shortmid}} \otimes B^{-}, \\ C_3 &= I_{n^{\shortmid}-k^{\shortmid}} \otimes C^{-}, \quad D_3 = I_{n^{\shortmid}-k^{\shortmid}} \otimes D^{-}. \end{aligned}$$

Note that $A_3$, $B_3$, $C_3$, and $D_3$ are matrices of sizes $(n^{\shortmid}-k^{\shortmid})\delta^{-} \times (n^{\shortmid}-k^{\shortmid})\delta^{-}$, $(n^{\shortmid}-k^{\shortmid})\delta^{-} \times (n^{\shortmid}-k^{\shortmid})k^{-}$, $(n^{\shortmid}-k^{\shortmid})(n^{-}-k^{-}) \times (n^{\shortmid}-k^{\shortmid})\delta^{-}$, and $(n^{\shortmid}-k^{\shortmid})(n^{-}-k^{-}) \times (n^{\shortmid}-k^{\shortmid})k^{-}$, respectively.

Finally, taking into account that the state vector of the product code is $\boldsymbol{x}_t = (\boldsymbol{x}_t^{(3)}, \boldsymbol{x}_t^{(2)}, \boldsymbol{x}_t^{(1)})^T$ and that the parity vector is $\boldsymbol{y}_t = (\boldsymbol{y}_t^{(3)}, \boldsymbol{y}_t^{(2)}, \boldsymbol{y}_t^{(1)})^T$, from expressions (2), (3), and (4) we obtain an ISO representation of the product code.

**Theorem 1:** *Assume that $(A^{\shortmid}, B^{\shortmid}, C^{\shortmid}, D^{\shortmid})$ is an ISO representation of the vertical $(n^{\shortmid}, k^{\shortmid}, \delta^{\shortmid})$ code $\mathcal{C}^{\shortmid}$. Analogously, let $(A^{-}, B^{-}, C^{-}, D^{-})$ be an ISO representation of the horizontal $(n^{-}, k^{-}, \delta^{-})$ code $\mathcal{C}^{-}$. Then an ISO representation for the product code $\mathcal{C}^{\shortmid} \otimes \mathcal{C}^{-}$ is given by (1), with*

$$A = \begin{pmatrix} I_{n^{\shortmid}-k^{\shortmid}} \otimes A^{-} & O & (I_{n^{\shortmid}-k^{\shortmid}} \otimes B^{-})P(I_{k^{-}} \otimes C^{\shortmid}) \\ O & I_{k^{\shortmid}} \otimes A^{-} & O \\ O & O & I_{k^{-}} \otimes A^{\shortmid} \end{pmatrix},$$

$$B = \begin{pmatrix} (I_{n^{\shortmid}-k^{\shortmid}} \otimes B^{-})P(I_{k^{-}} \otimes D^{\shortmid}) \\ (I_{k^{\shortmid}} \otimes B^{-})Q \\ I_{k^{-}} \otimes B^{\shortmid} \end{pmatrix},$$

$$C = \begin{pmatrix} I_{n^{\shortmid}-k^{\shortmid}} \otimes C^{-} & O & (I_{n^{\shortmid}-k^{\shortmid}} \otimes D^{-})P(I_{k^{-}} \otimes C^{\shortmid}) \\ O & I_{k^{\shortmid}} \otimes C^{-} & O \\ O & O & I_{k^{-}} \otimes C^{\shortmid} \end{pmatrix},$$

$$D = \begin{pmatrix} (I_{n^{\shortmid}-k^{\shortmid}} \otimes D^{-})P(I_{k^{-}} \otimes D^{\shortmid}) \\ (I_{k^{\shortmid}} \otimes D^{-})Q \\ I_{k^{-}} \otimes D^{\shortmid} \end{pmatrix},$$

*where $O$ denotes the null matrix of the appropriate size.*

Once we have obtained an ISO representation of the product code $\mathcal{C}^{\shortmid} \otimes \mathcal{C}^{-}$, we are interested in the properties to be satisfied by matrices $A^{\shortmid}$, $B^{\shortmid}$, $C^{\shortmid}$, $D^{\shortmid}$, $A^{-}$, $B^{-}$, $C^{-}$ and $D^{-}$. The next example shows that is not enough for the pairs $(A^{\shortmid}, B^{\shortmid})$ and $(A^{-}, B^{-})$ to be controllable in order to get a controllable pair $(A, B)$ for the product code.

**Example 1:** Let $\alpha$ be a primitive element of the Galois field $\mathbb{F} = GF(8)$ with $\alpha^3 + \alpha + 1 = 0$, and consider the $(6, 4, 2)$-vertical code $\mathcal{C}^{\shortmid}(A^{\shortmid}, B^{\shortmid}, C^{\shortmid}, D^{\shortmid})$, where

$$A^{\shortmid} = \begin{pmatrix} \alpha^3 & \alpha \\ \alpha & \alpha^4 \end{pmatrix}, \quad B^{\shortmid} = \begin{pmatrix} 1 & \alpha & \alpha^3 & \alpha^4 \\ \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 \end{pmatrix},$$
$$C^{\shortmid} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}, \quad D^{\shortmid} = \begin{pmatrix} \alpha^3 & \alpha^4 & \alpha^3 & \alpha^4 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

and a $(5, 2, 1)$-horizontal code $\mathcal{C}^{-}(A^{-}, B^{-}, C^{-}, D^{-})$, where

$$A^{-} = \begin{pmatrix} \alpha \end{pmatrix}, \quad B^{-} = \begin{pmatrix} \alpha^3 & 0 \end{pmatrix},$$
$$C^{-} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad D^{-} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & 0 \end{pmatrix}.$$

For all $z \in \overline{\mathbb{F}}$ we have that

$$\operatorname{rank} \begin{pmatrix} zI_{\delta^{\shortmid}} - A^{\shortmid} & B^{\shortmid} \end{pmatrix} = 2$$
$$\operatorname{rank} \begin{pmatrix} zI_{\delta^{-}} - A^{-} & B^{-} \end{pmatrix} = 1$$

and therefore, the pairs $(A^{\shortmid}, B^{\shortmid})$ and $(A^{-}, B^{-})$ are controllable.

Now, if we compute the matrices $A$ and $B$ of the product code, we observe that

$$\operatorname{rank} \begin{pmatrix} \alpha I_{10} - A & B \end{pmatrix} = 8,$$

so the pair $(A, B)$ is not controllable.

We provide, in the next theorem, a lower bound for the free distance of $\mathcal{C}^{\shortmid} \otimes \mathcal{C}^{-}$ in terms of the free distances of $\mathcal{C}^{\shortmid}$ and $\mathcal{C}^{-}$.

**Theorem 2:** *Let $\mathcal{C}^{\shortmid}$ and $\mathcal{C}^{-}$ be $(n^{\shortmid}, k^{\shortmid}, \delta^{\shortmid})$ and $(n^{-}, k^{-}, \delta^{-})$ codes. Then, the free distance of the product code $\mathcal{C}^{\shortmid} \otimes \mathcal{C}^{-}$ verifies*

$$d_{free}(\mathcal{C}^{\shortmid} \otimes \mathcal{C}^{-}) \geq k^{-} d_{free}(\mathcal{C}^{\shortmid}) + k^{\shortmid} d_{free}(\mathcal{C}^{-}) - k^{-} k^{\shortmid}.$$

## IV. Conclusions

The results introduced in this paper show a input-state-output representation of a product code and a lower bound on its free distance. This representation will allow us to introduce, from a given vertical and horizontal codes, a new product convolutional code with fixed distance.

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