

An iterative procedure for computing the stabilizing solution of discrete-time periodic Riccati equations with an indefinite sign

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Abstract—In this note, the problem of computation of the stabilizing solution of a class of periodic discrete-time Riccati equation is addressed. Such an equation is closely related to the so called full information H_∞ control problem of discrete-time periodic systems. A globally convergent iterative algorithm is proposed for this purpose. The performances of the proposed algorithm are illustrated on some numerical examples.

I. INTRODUCTION

Periodic systems are an important sub-class of time-varying systems that received much attention during the past several decades. The theory of stability, optimal and robust control, as well as important applications of such systems, can be found in several references in the current literature. One can refer to the recent monographs [3], [9] and the references therein.

Several aspects regarding the Riccati equations with periodic coefficients and their applications in various control problems may be found in [1], [14] for the continuous time case and [2], [3], [4] for the discrete-time case.

This note is devoted to the problem of numerical computation of the stabilizing solution of a class of periodic discrete-time Riccati equation with an indefinite sign of it's quadratic term. The motivation behind this problem is that such an equation is closely related to the so called full information H_∞ control of discrete-time periodic systems. Hence, the class of Riccati equations we deal with throughout the paper, will be referred to as H_∞ -periodic Riccati equation (H_∞ -PRE).

Very recently, we have proposed in [7] a globally convergent iterative algorithm for the computation of the stabilizing solution of H_∞ -PRE that can be viewed as an extension to the discrete-time time-varying case, of the results in [13] for the deterministic continuous-time time-invariant case and in [5], [6] for the stochastic continuous-time time-invariant case. The main idea behind this algorithm is to equivalently transform the original H_∞ -type problem into the problem of solving a sequence of H_2 -type PREs, that is, discrete-time Riccati equations with definite sign of the quadratic parts. The stabilizing solution of the original H_∞ -PRE is then

approximated by the sum of solutions of the H_2 -type PREs. The computation of the sequence of approximations of the stabilizing solution relies on the computation of a vanishing matrix sequence $\{Z^{(k)}(t)\}_{k \geq 0}$. We believe that the vanishing nature of the matrix sequence $\{Z^{(k)}(t)\}_{k \geq 0}$ could induce ill conditioning in its computation. This observation represents the main motivation for the work reported in this article. Indeed, our aim is to propose an alternative method for the computation of the sequence of approximations of the stabilizing solution. More specifically, we will propose an alternative globally convergent iterative algorithm for the computation of the stabilizing solution of the H_∞ -PRE that shares similar desirable characteristics than the algorithm given in [7] (namely simple initialization, global convergence and local quadratic rate of convergence). The main advantage of this new formulation is that the sequence of approximations of the stabilizing solution are computed **directly** without involving the computation of some intermediate (vanishing) matrix sequence. As a minor contribution, we propose also in this note an iterative procedure of Kleinman type for the solution of H_2 -type PREs.

This paper is organized as follows: In section 2, the problem setting is described. Section 3 introduces the proposed algorithm and states the main result of the paper and its proof. Some numerical examples are given in Section 4.

Notations. A^T stands for the transpose of the matrix A . The notation $X \geq Y$ ($X > Y$, respectively), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (positive definite, respectively). In block matrices, \star indicates symmetric terms: $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & \star \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & B \\ \star & C \end{bmatrix}$. The expression $MN\star$ is equivalent to MNM^T while $M\star$ is equivalent to MM^T .

II. THE PROBLEM SETTING

In the first part of this section we will state the problem we deal with in this paper. In a second part, we will motivate our study by showing the strong connection between the considered problem and the so-called full information H_∞ control.

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A. Problem formulation

Consider the discrete-time Riccati equation (DTRE):

$$\begin{aligned} X(t) &= A^T(t)X(t+1)A(t) \\ &\quad - [A^T(t)X(t+1)B(t) + C^T(t)D(t)] \\ &\quad \times [R_\gamma(t) + B^T(t)X(t+1)B(t)]^{-1} \star \\ &\quad + C^T(t)C(t), \quad t \in \mathbb{Z} \end{aligned} \quad (1)$$

where $A(t) \in \mathbb{R}^{n \times n}$, $B(t) = (B_1(t) \quad B_2(t))$, $B_k(t) \in \mathbb{R}^{n \times m_k}$, $k = 1, 2$, $C(t) \in \mathbb{R}^{p \times n}$, $D(t) = (D_1(t) \quad D_2(t))$, $D_k(t) \in \mathbb{R}^{p \times m_k}$, $k = 1, 2$, $R_\gamma(t) = D^T(t)D(t) + \begin{pmatrix} -\gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{m \times m}$, $m = m_1 + m_2$, $\gamma > 0$ is a given scalar. Throughout the paper we assume:

H1: There exists an integer $\theta \geq 1$ such that $A(t + \theta) = A(t)$, $B(t + \theta) = B(t)$, $C(t + \theta) = C(t)$, $D(t + \theta) = D(t)$ for $t \in \mathbb{Z}$.

In our approach, the class of admissible solutions consists of all bounded sequences $\{X(t)\}_{t \in \mathbb{Z}} \subset \mathcal{S}_n$ satisfying (1) and the following two sign conditions:

$$\begin{aligned} &D_1^T(t)D_1(t) + B_1^T(t)X(t+1)B_1(t) \\ &\quad - [B_1^T(t)X(t+1)B_2(t) + D_1^T(t)D_2(t)] \\ &\quad \times [B_2^T(t)X(t+1)B_2(t) + D_2^T(t)D_2(t)]^{-1} \star -\gamma^2 I_{m_1} < 0 \end{aligned} \quad (2)$$

$$D_2^T(t)D_2(t) + B_2^T(t)X(t+1)B_2(t) > 0 \quad (3)$$

$t \in \mathbb{Z}$. Therefore we are interested in the global, and bounded on \mathbb{Z} , solutions for which the matrices $R_\gamma(t) + B^T(t)X(t+1)B(t)$, $t \in \mathbb{Z}$ have not defined sign. Throughout the paper \mathcal{S}_d stands for the linear space of $d \times d$ real symmetric matrices.

Definition 2.1: An admissible solution $\{\tilde{X}(t)\}_{t \in \mathbb{Z}}$ is called **stabilizing solution**, if the zero state equilibrium of the discrete - time linear system on \mathbb{R}^n :

$$x(t+1) = [A(t) + B(t)\tilde{F}(t)]x(t) \quad (4)$$

is exponentially stable, where

$$\begin{aligned} \tilde{F}(t) &= - [R_\gamma(t) + B^T(t)\tilde{X}(t+1)B(t)]^{-1} \\ &\quad \times [B^T(t)\tilde{X}(t+1)A(t) + D^T(t)C(t)] \end{aligned} \quad (5)$$

The stabilizing solution of DTRE (1) is involved in the design of the solution of H_∞ control problem with level of attenuation γ associated to the discrete - time linear system:

$$x(t+1) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \quad (6)$$

$t \geq 0$, $x(0) = 0$, and the cost functional

$$\begin{aligned} J(u(\cdot), w(\cdot)) &= \sum_{t=0}^{\infty} (\|C(t)x(t) + D_1(t)w(t) \\ &\quad + D_2(t)u(t)\|^2 - \gamma^2 \|w(t)\|^2) \end{aligned} \quad (7)$$

where $u(\cdot) \in \mathbb{R}^{m_2}$ are the control parameters and $w(\cdot) \in$

$\ell_2(\mathbb{Z}, \mathbb{R}^{m_1})$ model the exogenous disturbances whose effect should be attenuated. This point will be detailed in the next sub-section. We recall that $\ell_2(\mathbb{Z}_+, \mathbb{R}^{m_1}) = \{\{w(t)\}_{t \in \mathbb{Z}_+} : \sum_{t=0}^{\infty} w^T(t)w(t) < \infty\}$.

The iterative procedures of Kleinman type (see e.g. [12]) cannot be directly applied for the numerical computation of the stabilizing solution of DTRE (1) because its quadratic term has not defined sign.

In this paper we propose an iterative procedure for the numerical computation of the stabilizing solution $\tilde{X}(\cdot)$ of (1).

B. Relation with the full information H_∞ control problem

Let us consider the so called **full information** control problem described by the controlled system (6) and the cost functional (7). This means that at each time instance t both the state vector $x(t)$ as well as the exogenous disturbance $w(t)$ are available for measurements. The class of admissible controls consists of the memoryless control laws $u_{KW}(t) = K(t)x(t) + W(t)w(t)$. Thus, for a given scalar $\gamma > 0$, \mathcal{A}_γ stands for the set of the pairs of the θ -periodic sequences $(\{K(t)\}_{t \in \mathbb{Z}}, \{W(t)\}_{t \in \mathbb{Z}}) \subset \mathbb{R}^{m_2 \times n} \times \mathbb{R}^{m_2 \times m_1}$ with the properties:

- i) The closed-loop system: $x(t+1) = (A(t) + B_2(t)K(t))x(t)$, is exponentially stable.
- ii) $J(u_{KW}(\cdot), w(\cdot)) < 0$ for all $0 \neq w(\cdot) \in \ell_2(\mathbb{Z}, \mathbb{R}^{m_1})$.

Often we shall write $(K(\cdot), W(\cdot)) \in \mathcal{A}_\gamma$ instead of $(\{K(t)\}_{t \in \mathbb{Z}}, \{W(t)\}_{t \in \mathbb{Z}}) \in \mathcal{A}_\gamma$. Under the assumptions:

- a) the assumption H1 is fulfilled;
- b) $D_2^T(t)D_2(t) > 0$, $t \in \mathbb{Z}$ and the pair $(\check{C}(\cdot), \check{A}(\cdot))$ is detectable, where $\check{C}(t) = [I_p - D_2(t)(D_2^T(t)D_2(t))^{-1}D_2^T(t)]C(t)$ and $\check{A}(t) = A(t) - B_2(t)(D_2^T(t)D_2(t))^{-1}D_2^T(t)C(t)$.

if $\{\tilde{X}(t)\}_{t \in \mathbb{Z}}$ is the stabilizing solution of DTRE (1) we define

$$\begin{aligned} \tilde{K}(t) &= - [D_2^T(t)D_2(t) + B_2^T(t)\tilde{X}(t+1)B_2(t)]^{-1} \\ &\quad \times [B_2^T(t)\tilde{X}(t+1)A(t) + D_2^T(t)C(t)] \end{aligned}$$

$$\begin{aligned} \tilde{W}(t) &= - [D_2^T(t)D_2(t) + B_2^T(t)\tilde{X}(t+1)B_2(t)]^{-1} \\ &\quad \times [D_2^T(t)D_1(t) + B_2^T(t)\tilde{X}(t+1)B_1(t)] \end{aligned}$$

$t \in \mathbb{Z}$.

One can show that the pair $(\tilde{K}(\cdot), \tilde{W}(\cdot))$ lies in \mathcal{A}_γ . For more details see for example [3], [8], [11]. It is worth mentioning that among the solutions of the full information H_∞ control problem, the control $u_{\tilde{K}, \tilde{W}}(t) = \tilde{K}(t)x(t) + \tilde{W}(t)w(t)$ has a privileged place, because it provides the solution of the mixed H_2/H_∞ control problem. That is why it is useful to have a reliable procedure for numerical computation of the stabilizing solution of DTRE of type (1).

III. MAIN RESULTS

Before giving the main result of this paper, we first recall here, for the reader convenience, an iterative procedure for

the numerical computation of the unique stabilizing and θ -periodic solution $\tilde{X}(\cdot)$ of (1) that has been recently proposed in [7]. The new proposed iterative method is largely based on the algorithm in [7] which may be described in the following steps:

Step 0. Take

$$X^{(0)}(t) = 0, \quad t \in \mathbb{Z}, \quad (8)$$

and compute $Z^{(0)}(\cdot)$ as the unique stabilizing and θ -periodic solution of the DTRE:

$$\begin{aligned} Z^{(0)}(t) &= A^T(t)Z^{(0)}(t+1)A(t) \\ &\quad - \left[A^T(t)Z^{(0)}(t+1)B_2(t) + C^T(t)D_2(t) \right] \\ &\quad \times \left[B_2^T(t)Z^{(0)}(t+1)B_2(t) + D_2^T(t)D_2(t) \right]^{-1} \star \\ &\quad + C^T(t)C(t) \end{aligned} \quad (9)$$

Step k, $k \geq 1$. Take $X^{(k)}(t) = Z^{(k-1)}(t) + X^{(k-1)}(t)$ and compute $Z^{(k)}(\cdot)$ as the unique stabilizing and θ -periodic solution of the DTRE with defined sign:

$$\begin{aligned} Z^{(k)}(t) &= \left(A(t) + B(t)F^{(k)}(t) \right)^T Z^{(k)}(t+1) \star \\ &\quad - \left[\left(A(t) + B(t)F^{(k)}(t) \right)^T Z^{(k)}(t+1)B_2(t) \right] \\ &\quad \times \left[R_2^{(k)}(t) + B_2(t)Z^{(k)}(t+1)B_2(t) \right]^{-1} \star \\ &\quad + M^{(k)}(t) \end{aligned} \quad (10)$$

where

$$\begin{aligned} F^{(k)}(t) &= - \left[R_\gamma(t) + B^T(t)X^{(k)}(t+1)B(t) \right]^{-1} \\ &\quad \times \left[B^T(t)X^{(k)}(t+1)A(t) + D^T(t)C(t) \right] \end{aligned} \quad (11)$$

$$R_2^{(k)}(t) = D_2^T(t)D_2(t) + B_2^T(t)X^{(k)}(t+1)B_2(t) \quad (12)$$

$$\begin{aligned} M^{(k)}(t) &= A^T(t)X^{(k)}(t+1)A(t) \\ &\quad - \left[A^T(t)X^{(k)}(t+1)B(t) + C^T(t)D(t) \right] \\ &\quad \times \left[R_\gamma(t) + B(t)X^{(k)}(t+1)B(t) \right]^{-1} \star \\ &\quad + C^T(t)C(t) - X^{(k)}(t). \end{aligned} \quad (13)$$

The algorithm stops if for some $k_0 \geq 0$ we have

$$0 \leq \lambda_{max}[M^{(k_0)}(t)] < \varepsilon$$

for all $0 \leq t \leq \theta - 1$, where $\varepsilon > 0$ is a prefixed level of accuracy. It is worth mentioning that (9) and (10) involved in the computation of $Z^{(k)}(\cdot)$, $k \geq 0$ are standard H_2 -periodic Riccati equation, or equivalently, discrete-time periodic Riccati equations arising in connection with linear quadratic optimization problem. Hence, (9) is related to the linear quadratic optimization problem described by the controlled system:

$$x(t+1) = A(t)x(t) + B_2(t)u(t), \quad x(0) = x_0$$

and the quadratic cost function:

$$J_0(x_0, u) = \sum_{t=0}^{\infty} |C(t)x(t) + D_2(t)u(t)|^2.$$

For $k \geq 1$ the Riccati equation (10) occurs in connection with the linear quadratic optimization problem described by the controlled system

$$x(t+1) = (A(t) + B(t)F^{(k)}(t))x(t) + B_2(t)u(t), \quad x(0) = x_0$$

and the quadratic cost function:

$$J_k(x_0, u) = \sum_{t=0}^{\infty} (x^T(t)M^{(k)}(t)x(t) + u^T(t)R_2^{(k)}(t)u(t)).$$

Theorem 3.1: Assume

- The assumption $H1$ is fulfilled;
- $D_2^T(t)D_2(t) > 0$, $t \in \mathbb{Z}$ and the pair $(\check{C}(\cdot), \check{A}(\cdot))$ is detectable, where $\check{C}(t) = [I_p - D_2(t)(D_2^T(t)D_2(t))^{-1}D_2^T(t)]C(t)$ and $\check{A}(t) = A(t) - B_2(t)(D_2^T(t)D_2(t))^{-1}D_2^T(t)C(t)$.
- the set \mathcal{A}_γ is not empty.

Under these conditions, the sequences $\{X^{(k)}(t)\}_{k \geq 0}$ and $\{Z^{(k)}(t)\}_{k \geq 0}$, $t \in \mathbb{Z}$ introduced via (9) - (13) are well defined for each $k \geq 0$. Furthermore, the sequences $\{X^{(k)}(t)\}_{k \geq 0}$ and $\{Z^{(k)}(t)\}_{k \geq 0}$, $t \in \mathbb{Z}$ are convergent and we have $\lim_{k \rightarrow \infty} X^{(k)}(t) = \tilde{X}(t)$, $\lim_{k \rightarrow \infty} Z^{(k)}(t) = 0$, $t \in \mathbb{Z}$, $\tilde{X}(\cdot)$ being the unique stabilizing and θ -periodic solution of (1).

Remark 3.1: From the proof of Theorem 3.1 (see [7]), we may display the following properties of the sequences of iterations $\{X^{(k)}(t)\}_{k \geq 0}$ and $\{Z^{(k)}(t)\}_{k \geq 0}$, $t \in \mathbb{Z}$ introduced via (3.1)-(3.6):

- $0 = X^{(0)}(t) \leq X^{(1)}(t) \leq \dots \leq X^{(k)}(t) \leq \dots \leq \tilde{X}(t)$, $\tilde{X}(\cdot)$ being the stabilizing and θ -periodic solution of (2.1).
- $X^{(k)}(t) = \sum_{j=0}^{k-1} Z^{(j)}(t)$, $k \geq 1$, $t \in \mathbb{Z}$;
- $\lim_{k \rightarrow \infty} X^{(k)}(t) = \tilde{X}(t)$ and $\lim_{k \rightarrow \infty} Z^{(k)}(t) = 0$, $t \in \mathbb{Z}$.

A. An alternative way to define the sequence of approximations of the stabilizing solution

It appears from the proposed algorithm in [7] (and recalled in the previous section) that the computation of the sequence of approximations of the stabilizing solution ($\{X^{(k)}(t)\}_{k \geq 0}$) relies on the computation of the matrix sequence $\{Z^{(k)}(t)\}_{k \geq 0}$. We believe that the vanishing nature of the matrix sequence $\{Z^{(k)}(t)\}_{k \geq 0}$ (see iii) in Remark 3.1) could induce ill conditioning in its computation. Hence, summation of the induced computation errors (see ii) in Remark 3.1) could affect the reliability of the computation of the stabilizing solution $\tilde{X}(t)$. This observation motivated us to look for an alternative method for the **direct** computation of the sequence $\{X^{(k)}(t)\}_{k \geq 0}$. This method will be presented in what follows. Before doing so, we first introduce several notations which are involved in the statement of the main results: $\mathcal{G}(\cdot, \cdot) : Dom \mathcal{G} \rightarrow \mathcal{S}_n$ is defined by the right hand

side of (1) and $\text{Dom } \mathcal{G}$ consists of all pairs $(t, X) \in \mathbf{Z} \times \mathcal{S}_n$ satisfying the sign conditions (2) and (3) with $X(t+1)$ replaced by X , where

$$\begin{cases} V_{11}^{(k+1)}(t) = [\gamma^2 I_{m_1} - D_1^T(t)D_1(t) \\ - B_1^T(t)X^{(k+1)}(t+1)B_1(t) + (D_1^T(t)D_2(t) \\ + B_1^T(t)X^{(k+1)}(t+1)B_2(t))(D_2^T(t)D_2(t) \\ + B_2^T(t)X^{(k+1)}(t+1)B_2(t))^{-1} \star]^{1/2} \\ V_{21}^{(k+1)}(t) = [D_2^T(t)D_2(t) + B_2^T(t)X^{(k+1)}(t+1)B_2(t)]^{-1/2} \\ \times [D_2^T(t)D_1(t) + B_2^T(t)X^{(k+1)}(t+1)B_1(t)] \\ V_{22}^{(k+1)}(t) = [D_2^T(t)D_2(t) + B_2^T(t)X^{(k+1)}(t+1)B_2(t)]^{1/2} \end{cases} \quad (14)$$

$k \geq 0$.

Theorem 3.2: Consider the same assumptions a), b), c) as in Theorem 3.1. Let $X^{(1)}(t)$ be the unique stabilizing and θ -periodic solution of the DTRE with defined sign (9) and let $X^{(k)}(t)$, $k \geq 2$, be the unique stabilizing and θ -periodic solution of the DTRE with defined sign:

$$\begin{aligned} X^{(k)}(t) &= (\tilde{\mathbf{A}}^{(k)}(t))^T X^{(k)}(t+1) \tilde{\mathbf{A}}^{(k)}(t) \\ &\quad - [(\tilde{\mathbf{A}}^{(k)}(t))^T X^{(k)}(t+1)B_2(t) + \mathbf{L}^{(k-1)}(t)] \\ &\quad \times (D_2^T(t)D_2(t) + B_2^T(t)X^{(k)}(t+1)B_2(t))^{-1} \star \\ &\quad + \mathbf{M}^{(k-1)}(t) \end{aligned} \quad (15)$$

where:

$$\begin{cases} \tilde{\mathbf{A}}^{(k)}(t) = A(t) + B_1(t)F_1^{k-1}(t) \\ F_1^{(k-1)}(t) = (I_{m_1} \quad 0) F^{(k)}(t) \\ \mathbf{L}^{(k-1)}(t) = (C(t) + D_1(t)F_1^{(k-1)}(t))^T D_2(t) \\ \mathbf{M}^{(k-1)}(t) = (C(t) + D_1(t)F_1^{(k-1)}(t))^T \star \\ -\gamma^2 (F_1^{(k-1)}(t))^T F_1^{(k-1)}(t) \end{cases} \quad (16)$$

The sequence $\{X^{(k)}(t)\}_{k \geq 1}$ is well defined for each $k \geq 0$. Furthermore, the sequence $\{X^{(k)}(t)\}_{k \geq 1}$ is convergent and we have $\lim_{k \rightarrow \infty} X^{(k)}(t) = \tilde{X}(t)$, $t \in \mathbf{Z}$, $\tilde{X}(\cdot)$ being the unique stabilizing and θ -periodic solution of (1).

Sketch of the proof. Plugging (14) written for k replaced by $k-1$ in (11) one obtains

$$\begin{aligned} F_1^{(k)}(t) &= (I_{m_1} \quad 0) F^{(k)}(t) \\ &= (V_{11}^{(k)}(t))^{-2} [(\hat{B}_1^{(k)}(t))^T X^{(k)}(t+1)A(t) + \hat{D}_1^T(t)C(t)] \end{aligned}$$

and

$$\begin{aligned} F_2^{(k)}(t) &= (0 \quad I_{m_2}) F^{(k)}(t) = -(V_{22}^{(k)}(t))^{-1} V_{21}^{(k)}(t) F_1^{(k)}(t) \\ &\quad - (V_{22}^{(k)}(t))^{-2} [B_2^T(t)X^{(k)}(t+1)A(t) + D_2^T(t)C(t)]. \end{aligned}$$

After several algebraic manipulations involving the last two equalities from (14) written for k replaced by $k-1$, we deduce:

$$\begin{aligned} F_2^{(k)}(t) &= -(D_2^T(t)D_2(t) + B_2^T(t)X^{(k)}(t+1)B_2(t))^{-1} \\ &\quad \times [B_2^T(t)X^{(k)}(t+1)(A(t) + B_1(t)F_1^{(k)}(t)) \\ &\quad + D_2^T(t)(C(t) + D_1(t)F_1^{(k)}(t))]. \end{aligned} \quad (17)$$

One sees that (17) has the structure of the optimal feedback gain of a linear quadratic optimization problem with the dynamic described by the matrix $A(t) + B_1(t)F_1^{(k)}(t)$ and the regulated output of the form

$$y(t) = (C(t) + D_1(t)F_1^{(k)}(t))x(t) + D_2(t)u(t).$$

This fact suggests us to consider an optimal control problem described by the system (6) and the cost (7) with the input $w(t)$ replaced by $F_1^{(k)}(t)x(t)$. So, we want to minimize the cost functional

$$\begin{aligned} \mathbb{J}(x_0, u(\cdot)) &= \sum_{t=0}^{\infty} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \\ &\quad \times \begin{pmatrix} \mathbf{M}^{(k)}(t) & \mathbf{L}^{(k)}(t) \\ \star & D_2^T(t)D_2(t) \end{pmatrix} \star \end{aligned} \quad (18)$$

subject to the trajectories of the system

$$\begin{cases} x(t+1) = (A(t) + B_1(t)F_1^{(k)}(t))x(t) + B_2(t)u(t) \\ x(0) = x_0 \end{cases} \quad (19)$$

The class of admissible controls $\mathcal{U}(x_0)$ consists of all sequences $\{u(t)\}_{t \geq 0} \in \ell^2(\mathbf{Z}_+, \mathbb{R}^{m_2})$ with the property that the corresponding trajectories $x_u(t, x_0)$ of (19) satisfy

$$\lim_{t \rightarrow \infty} x_u(t, x_0) = 0.$$

It is known that in this case the optimal control is in a state feedback form $u(t) = \tilde{\Gamma}^{(k)}(t)x(t)$ where

$$\begin{aligned} \tilde{\Gamma}^{(k)}(t) &= -(D_2^T(t)D_2(t) + B_2^T(t)Y^{(k)}(t+1)B_2(t))^{-1} \\ &\quad \times [B_2^T(t)Y^{(k)}(t+1)(A(t) + B_1(t)F_1^{(k)}(t)) \\ &\quad + (\mathbf{L}^{(k)}(t))^T], \end{aligned} \quad (20)$$

$Y^{(k)}(\cdot)$ being the unique bounded and stabilizing solution of the following discrete-time Riccati equation:

$$\begin{aligned} Y^{(k)}(t) &= (A(t) + B_1(t)F_1^{(k)}(t))^T Y^{(k)}(t+1) \star \\ &\quad - [(A(t) + B_1(t)F_1^{(k)}(t))^T Y^{(k)}(t+1)B_2(t) \\ &\quad + \mathbf{L}^{(k)}(t)] \\ &\quad \times (D_2^T(t)D_2(t) + B_2^T(t)Y^{(k)}(t+1)B_2(t))^{-1} \star \\ &\quad + \mathbf{M}^{(k)}(t) \end{aligned} \quad (21)$$

We recall that $Y^{(k)}(\cdot)$ is stabilizing solution of (21) if the zero solution of the linear equation

$$x(t+1) = (A(t) + B_1(t)F_1^{(k)}(t) + B_2(t)\tilde{\Gamma}^{(k)}(t))x(t) \quad (22)$$

is exponentially stable, $\tilde{\Gamma}^{(k)}(t)$ being introduced via (20).

In order to find sufficient conditions which guarantee the existence of the bounded and stabilizing solution of (21) we follow an idea from [13].

Let $Y(t)$, $t_0 \leq t \leq t_1$ be a solution of (21) and we define $P(t) = Y(t) - X^{(k)}(t)$. We prove that if $k \geq 1$ then the following items hold:

- a) $Y(t)$, $t_0 \leq t \leq t_1$ is a solution of (21) if and only if $P(t)$ solves (10).

- b) $Y^{(k)}(t), t \geq 0$ is the bounded and stabilizing solution of (21) if and only if $P^{(k)}(t) := Y^{(k)}(t) - X^{(k)}(t), t \geq 0$ is the bounded and stabilizing solution of (10).

To show that a) holds, we firstly remark that (21) satisfied by $X^{(k)}(t) + P(t)$ may be rewritten in the equivalent form:

$$\begin{pmatrix} P(t) + X^{(k)}(t) \\ 0 \end{pmatrix} = \begin{pmatrix} \Theta^{(k)}(t) & \star \\ \Psi^{(k)}(t) & \Pi^{(k)}(t) \end{pmatrix} \begin{pmatrix} I_n \\ \tilde{K}^{(k)}(t) \end{pmatrix} \quad (23)$$

where:

$$\begin{cases} \Theta^{(k)}(t) = (A(t) + B_1(t)F_1^{(k)}(t))^T P(t+1) \star + \tilde{M}^{(k)}(t) \\ \Psi^{(k)}(t) = B_2^T(t)P(t+1)(A(t) + B_1(t)F_1^{(k)}(t)) \\ \quad + (\tilde{L}^{(k)}(t))^T \\ \Pi^{(k)}(t) = R_2^{(k)}(t) + B_2^T(t)P(t+1)B_2(t) \end{cases}$$

and:

$$\tilde{M}^{(k)}(t) = (A(t) + B_1(t)F_1^{(k)}(t))^T X^{(k)}(t+1) \star + \mathbf{M}^{(k)}(t) \quad (24)$$

$$\tilde{L}^{(k)}(t) = (A(t) + B_1(t)F_1^{(k)}(t))^T X^{(k)}(t+1)B_2(t) + \mathbf{L}^{(k)}(t) \quad (25)$$

$$\tilde{K}^{(k)}(t) = -(\Pi^{(k)}(t))^{-1}[B_2^T(t)P(t+1) \times (A(t) + B_1(t)F_1^{(k)}(t)) + (\tilde{L}^{(k)}(t))^T] \quad (26)$$

$R_2^{(k)}(t)$ being introduced via (12).

Let us consider the invertible matrix $\Theta^{(k)}(t) = \begin{pmatrix} I_n & 0 \\ -(R_2^{(k)}(t))^{-1}(\tilde{L}^{(k)}(t))^T & I_{m_2} \end{pmatrix}$. One sees that the equation (23) is equivalent to

$$\begin{pmatrix} P(t) + X^{(k)}(t) \\ 0 \end{pmatrix} = (\Theta^{(k)}(t))^T \begin{pmatrix} \Theta^{(k)}(t) & \star \\ \Psi^{(k)}(t) & \Pi^{(k)}(t) \end{pmatrix} \times \Theta^{(k)}(t)(\Theta^{(k)}(t))^{-1} \begin{pmatrix} I_n \\ \tilde{K}^{(k)}(t) \end{pmatrix} \quad (27)$$

Following the line of the proof of Lemma A.4 in [7] we deduce that (27) is equivalent to:

$$\begin{aligned} X^{(k)}(t) + P(t) = & [A(t) + B_1^{(k)}(t)F_1^{(k)}(t) - (R_2^{(k)}(t))^{-1}(\tilde{L}^{(k)}(t))^T] P(t+1) \star \\ & - [(R_2^{(k)}(t))^{-1}(\tilde{L}^{(k)}(t))^T + \tilde{K}^{(k)}(t)]^T \\ & \times [R_2^{(k)}(t) + B_2^T(t)P(t+1)B_2(t)] \star \\ & + \tilde{M}^{(k)}(t) - \tilde{L}^{(k)}(t)(R_2^{(k)}(t))^{-1} \star \end{aligned} \quad (28)$$

Employing (16), (17), (25) we deduce that:

$$(R_2^{(k)}(t))^{-1}(\tilde{L}^{(k)}(t))^T = -F_2^{(k)}(t). \quad (29)$$

Hence,

$$\begin{aligned} A(t) + B_1^{(k)}(t)F_1^{(k)}(t) - (R_2^{(k)}(t))^{-1}(\tilde{L}^{(k)}(t))^T \\ = A(t) + B(t)F^{(k)}(t). \end{aligned} \quad (30)$$

On the other hand, (26) allows us to write:

$$\begin{aligned} (R_2^{(k)}(t))^{-1}(\tilde{L}^{(k)}(t))^T + \tilde{K}^{(k)}(t) = & [(R_2^{(k)}(t))^{-1} - (R_2^{(k)}(t) + B_2^T(t)P(t+1)B_2(t))^{-1}] \\ & \times (\tilde{L}^{(k)}(t))^T - (R_2^{(k)}(t) + B_2^T(t)P(t+1)B_2(t))^{-1} \\ & \times (A(t) + B_1(t)F_1^{(k)}(t)) \\ = & (R_2^{(k)}(t) + B_2^T(t)P(t+1)B_2(t))^{-1} \\ & \times B_2^T(t)P(t+1)B_2(t)(R_2^{(k)}(t))^{-1}(\tilde{L}^{(k)}(t))^T \\ & - (R_2^{(k)}(t) + B_2^T(t)P(t+1)B_2(t))^{-1}B_2^T(t)P(t+1) \\ & \times (A(t) + B_1(t)F_1^{(k)}(t)) \\ = & -(R_2^{(k)}(t) + B_2^T(t)P(t+1)B_2(t))^{-1}B_2^T(t)P(t+1) \\ & \times (A(t) + B(t)F^{(k)}(t)) \end{aligned} \quad (31)$$

Thus we obtain:

$$\begin{aligned} [(R_2^{(k)}(t))^{-1}(\tilde{L}^{(k)}(t))^T + \tilde{K}^{(k)}(t)]^T \\ \times [R_2^{(k)}(t) + B_2^T(t)P(t+1)B_2(t)] \star = \\ [A(t) + B(t)F^{(k)}(t)]^T P(t+1) \\ \times B_2(t)[R_2^{(k)}(t) + B_2^T(t)P(t+1)B_2(t)]^{-1} \star \end{aligned} \quad (32)$$

Applying Lemma A.2 in [7] for $(t, X^{(k)}(t+1)) \in \text{Dom}(\mathcal{G})$ and $\Gamma_1 = F_1^{(k)}(t), \Gamma_2 = 0$ and using again (29) we deduce

$$\tilde{M}^{(k)}(t) - \tilde{L}^{(k)}(t)(R_2^{(k)}(t))^{-1}(\tilde{L}^{(k)}(t))^T = \mathcal{G}(t, X^{(k)}(t)) \quad (33)$$

Finally, combining (13) and (30)-(33) we conclude that $Y(t)$ is a solution of (21) if and only if $P(t) := Y(t) - X^{(k)}(t), t_0 \leq t \leq t_1$ satisfies the Riccati equation

$$\begin{aligned} P(t) = & (A(t) + B(t)F^{(k)}(t))^T P(t+1)(A(t) + B(t)F^{(k)}(t)) \\ & - (A(t) + B(t)F^{(k)}(t))^T P(t+1)B_2(t) \\ & \times (R_2^{(k)}(t) + B_2^T(t)P(t+1)B_2(t))^{-1} \star \\ & + M^{(k)}(t) \end{aligned} \quad (34)$$

Comparing (34) with (10) we deduce that a) holds.

Let $\{Y^{(k)}(t)\}_{t \geq 0}$ be a global solution of (21). Set $P^{(k)}(t) = Y^{(k)}(t) - X^{(k)}(t), t \geq 0$. If $\tilde{\Gamma}^{(k)}(t)$ is a matrix gain associated to $Y^{(k)}(\cdot)$ via (20) then we have:

$$\begin{aligned} A(t) + B_1(t)F_1^{(k)}(t) + B_2(t)\tilde{\Gamma}^{(k)}(t) = & A(t) + B_1(t)F_1^{(k)}(t) \\ & - B_2(t)(R_2^{(k)}(t) + B_2^T(t)Y^{(k)}(t+1)B_2(t))^{-1} \\ & \times [B_2^T(t)P^{(k)}(t+1)(A(t) + B_1(t)F_1^{(k)}(t)) + (\tilde{L}^{(k)}(t))^T] \\ = & A(t) + B(t)F^{(k)}(t) \\ & - B_2(t)(R_2^{(k)}(t) + B_2^T(t)P^{(k)}(t+1)B_2(t))^{-1} \\ & \times B_2^T(t)P^{(k)}(t+1)(A(t) + B_1(t)F_1^{(k)}(t)) \\ & + B_2(t)[(R_2^{(k)}(t))^{-1} \\ & - (R_2^{(k)}(t) + B_2^T(t)P^{(k)}(t+1)B_2(t))^{-1}] \\ & \times (\tilde{L}^{(k)}(t))^T \end{aligned} \quad (35)$$

Invoking again (29) we obtain

$$\begin{aligned} & [(R_2^{(k)}(t))^{-1} - (R_2^{(k)}(t) + B_2^T(t)P^{(k)}(t+1)B_2(t))^{-1}] \\ & \times (\tilde{\Gamma}^{(k)}(t))^T = \\ & [R_2^{(k)}(t) + B_2^T(t)P^{(k)}(t+1)B_2(t)]^{-1}B_2^T(t) \\ & \times P^{(k)}(t+1)B_2(t)F_2^{(k)}(t) \end{aligned} \quad (36)$$

Thus we have obtained the equality:

$$\begin{aligned} & A(t) + B_1(t)F_1^{(k)}(t) + B_2(t)\tilde{\Gamma}^{(k)}(t) = \\ & A(t) + B(t)F^{(k)}(t) - B_2(t)(R_2^{(k)}(t) \\ & + B_2^T(t)P^{(k)}(t+1)B_2(t))^{-1}B_2^T(t)P^{(k)}(t+1) \\ & \times (A(t) + B(t)F^{(k)}(t)) \end{aligned}$$

This equality shows that the equation (22) coincides with:

$$\begin{aligned} x(t+1) = & [A(t) + B(t)F^{(k)}(t) - B_2(t)(R_2^{(k)}(t) + \\ & + B_2^T(t)P^{(k)}(t+1)B_2(t))^{-1}B_2^T(t) \\ & \times P^{(k)}(t+1)(A(t) + B(t)F^{(k)}(t))]x(t) \end{aligned} \quad (37)$$

According with the item a) we deduce that $P^{(k)}(\cdot)$ is a global bounded solution of the Riccati equation (34). If the zero solution of the equation (37) is exponentially stable, then, $P^{(k)}(\cdot)$ is just the bounded and stabilizing solution of (34). Since (37) coincides with (22), then $P^{(k)}(\cdot)$ is the bounded and stabilizing solution of (34) if and only if $Y^{(k)}(\cdot)$ defined by $Y^{(k)}(t) = X^{(k)}(t) + P^{(k)}(t)$ is the bounded and stabilizing solution of (21). On the other hand, since (34) coincides with (10) we deduce that under the assumptions of Theorem 3.1 the discrete-time Riccati equation (34) has a unique bounded and stabilizing solution $P^{(k)}(\cdot)$. From the uniqueness of the bounded and stabilizing solution of a discrete-time Riccati equation we deduce that the bounded and stabilizing solution of (34) is just $Z^{(k)}(\cdot)$. Hence, for $k \geq 1$ the unique, bounded and stabilizing solution of the discrete-time Riccati equation (21) is:

$$Y^{(k)}(t) = X^{(k)}(t) + Z^{(k)}(t) = X^{(k+1)}(t), t \in \mathbb{Z}. \quad (38)$$

In the case $k = 0$ $X^{(0+1)}(t) = Z^{(0)}(t)$, $t \in \mathbb{Z}$ is the unique bounded and stabilizing solution of the Riccati equation (9). Reasoning as in Remark 2.2 a) from [7], we may conclude that under the assumption H1 the unique, bounded and stabilizing solutions of (9), (10) and consequently (34) and (21) are θ -periodic sequences. ■

Remark 3.2: The developments from this subsection shows that the unique, stabilizing and θ -periodic solution $\tilde{X}(\cdot)$ of the game theoretic Riccati equation (2.1) may be obtained as a limit of the nondecreasing sequence $0 \leq X^{(1)}(t) \leq X^{(2)}(t) \leq \dots \leq X^{(k)}(t) \leq \dots \tilde{X}(t)$, $t \in \mathbb{Z}$, where $X^{(1)}(\cdot)$ is the unique, stabilizing and θ -periodic solution of the discrete-time Riccati equation with defined sign of its quadratic part (9), while, for $k \geq 2$ $X^{(k)}(\cdot)$ is the unique, stabilizing and θ -periodic solution of discrete-time Riccati equation with definite sign (15).

B. Procedural issues

In order to obtain the approximations $X^{(k)}(t)$, $k = 1, 2, \dots$ of the stabilizing and θ -periodic solution $\tilde{X}(t)$ of (1), at each step k we have to solve a discrete-time Riccati equation of the form:

$$\begin{aligned} Y(t) = & \hat{A}^T(t)Y(t+1)\hat{A}(t) \\ & - [\hat{A}^T(t)Y(t+1)B_2(t) + \hat{L}(t)][\hat{R}(t) \\ & + B_2^T(t)Y(t+1)B_2(t)]^{-1} \star + \hat{M}(t) \end{aligned} \quad (39)$$

To compute the stabilizing and θ -periodic solution $\tilde{Y}(\cdot)$ of (39) we propose the following iterative procedure, which is the discrete-time version of the Kleinman method [12]:

For each $j = 1, 2, \dots$ compute $Y^{(j)}(\cdot)$ as the unique θ -periodic solution of the backward affine equation:

$$\begin{aligned} Y^{(j)}(t) = & [\hat{A}(t) + B_2(t)K^{(j-1)}(t)]^T Y^{(j)}(t+1) \star \\ & + Q^{(j)}(t) \end{aligned} \quad (40)$$

where

$$Q^{(j)}(t) = \begin{pmatrix} I_n \\ K^{(j-1)}(t) \end{pmatrix}^T \begin{pmatrix} \hat{M}(t) & \hat{L}(t) \\ \hat{L}^T(t) & \hat{R}(t) \end{pmatrix} \star \quad (41)$$

and

$$\begin{aligned} K^{(j)}(t) = & - [\hat{R}(t) + B_2^T(t)Y^{(j)}(t+1)B_2(t)]^{-1} \\ & \times [B_2^T(t)Y^{(j)}(t+1)\hat{A}(t) + \hat{L}^T(t)] \end{aligned} \quad (42)$$

$j \geq 1$, while $\{K^{(0)}(t)\}_{t \in \mathbb{Z}} \subset \mathbb{R}^{m_2 \times n}$ is a θ -periodic and stabilizing sequence of the pair $(\hat{A}(\cdot), B_2(\cdot))$. This means that the zero solution of the linear equation $x(t+1) = [\hat{A}(t) + B_2(t)K^{(0)}(t)]x(t)$ is exponentially stable. Inductively one shows that for each $j = 1, 2, \dots$, the zero solution of the linear equation

$$x(t+1) = [\hat{A}(t) + B_2(t)K^{(j-1)}(t)]x(t) \quad (43)$$

is exponentially stable. This guarantee the fact that the equation (40) has a unique θ -periodic solution. We have $Y^{(1)}(t) \geq \dots \geq Y^{(j)}(t) \geq \dots \geq \tilde{Y}(t)$ and

$$\lim_{j \rightarrow \infty} Y^{(j)}(t) = \tilde{Y}(t), t \in \mathbb{Z}.$$

The algorithm stops if for some $j_0 \geq 1$ we have $\lambda_{\max}[Y^{(j_0)} - \mathcal{R}(t, Y^{(j_0)}(t+1))] < \eta$ where $\eta > 0$ is prefixed, $\mathcal{R}(t, Y^{(j_0)}(t+1))$ coincides with the right hand side of (39) with $Y(t+1)$ replaced by $Y^{(j_0)}(t+1)$.

Let us shown how we can compute the final value $Y^{(j)}(\theta)$ of the θ -periodic solution of (40) which is not a priori known. Set

$$\begin{aligned} \mathbb{T}^{(j)}(t, s) = & [\hat{A}(t-1) + B_2(t-1)K^{(j-1)}(t-1)] \\ & \times [\hat{A}(t-2) + B_2(t-2)K^{(j-1)}(t-2)] \dots \\ & \times [\hat{A}(s) + B_2(s)K^{(j-1)}(s)] \end{aligned}$$

if $t > s$ and

$$\mathbb{T}^{(j)}(t, s) = I_n$$

if $t = s$. For each $j \geq 1$ $\mathbb{T}^{(j)}(\theta, 0)$ is the monodromy matrix of equation (43). It is known (see e.g. [3]), that the zero solution of (43) is exponentially stable if and only if the eigenvalues of $\mathbb{T}^{(j)}(\theta, 0)$ are in the inside of the unit disk. The solutions of (40) have the representation

$$Y^{(j)}(t) = (\mathbb{T}^{(j)}(\theta, t))^T Y^{(j)}(\theta) \mathbb{T}^{(j)}(\theta, t) + \sum_{s=t}^{\theta-1} (\mathbb{T}^{(j)}(s, t))^T Q^{(j)}(s) \mathbb{T}^{(j)}(s, t), t \leq \theta - 1. \quad (44)$$

The periodicity condition $Y^{(j)}(0) = Y^{(j)}(\theta)$ yields:

$$Y^{(j)}(\theta) = [\mathbb{T}^{(j)}(\theta, 0)]^T Y^{(j)}(\theta) \mathbb{T}^{(j)}(\theta, 0) + H^{(j)} \quad (45)$$

where $H^{(j)} = \sum_{s=0}^{\theta-1} [\mathbb{T}^{(j)}(s, 0)]^T Q^{(j)}(s) \mathbb{T}^{(j)}(s, 0)$. Therefore, the value $Y^{(j)}(\theta)$ of the periodic solution (40) is obtained solving the Stein equation (45). This equation has a unique solution because the spectrum of the matrix $\mathbb{T}^{(j)}(\theta, 0)$ is in the inside of the unit disk. The other values $Y^{(j)}(t), 1 \leq t \leq \theta - 1$ of the periodic solution may be computed recursively from (40). From the proof of Theorem 3.1, we may identify the following procedure to design the stabilizing feedback gain $K^{(0)}(t)$.

Let $(K(t), W(t)) \in \mathcal{A}_\gamma$. We may chose:

- $K^{(0)}(t) = K(t)$ if the algorithm described by (40)-(42) is applied to compute the stabilizing and θ -periodic solution $X^{(1)}(t)$ of the Riccati equation (9).
- $K^{(0)}(t) = (W(t) \quad 0) F^{(k-1)}(t) + K(t)$ when the algorithm described by (40)-(42) is used for numerical computation of the stabilizing and θ -periodic solution of Riccati equation (15) for $k \geq 2$.

In both cases the implementation of the algorithm described by (40)-(42) is mainly based on the solution of Stein equation (45).

IV. NUMERICAL EXPERIMENTS

Example 1. The objective in this example is to show the efficiency and accuracy of the proposed algorithm compared with the periodic QZ algorithm developed in [10]. The periodic QZ algorithm is a numerically backward-stable algorithm which relies on an extension of the generalized Schur method. For this purpose, we use a similar procedure as the one followed in [13] (Example 2). We have generated a random test including 100 samples for the specified level of accuracy $\epsilon = 10^{-4}$. The simulation procedure is resumed as follows:

- Fix $\gamma = 10$;
- Set the example number $i = 0$;
- Choose n, m_1, m_2, p randomly and uniformly among the integers from 1 to 10 and θ randomly and uniformly among the integers from 2 to 5;
- Generate randomly the corresponding system matrices;
- If the assumptions in Theorem 3.2 are not verified, go back to step 3;

- Use the periodic QZ algorithm to solve the corresponding PRDE. If there does not exist a stabilizing solution to the PRDE, go back to step 3;
- Use the proposed algorithm to solve this PRDE. The iteration of the algorithm will be stopped if for some $k_0 \geq 0$ we have $0 \leq \lambda_{\max}[M^{k_0}(t)] < \epsilon$;
- Let the stabilizing solution of the PRDE obtained using the periodic QZ algorithm be $X_{\text{QZ}}(t)$ and the solution obtained by our algorithm be $X_1(t), 0 \leq t \leq \theta - 1$;
- Set $i = i + 1$, and compute the relative error with respect to the reference solution $X_{\text{QZ}}(t)$ as:

$$R_i = \frac{1}{\theta} \sum_{t=0}^{\theta-1} \frac{\|X_1(t) - X_{\text{QZ}}(t)\|}{\|X_{\text{QZ}}(t)\|}$$

- Repeat the steps 3-9 until $i = 100$.

Iterations \ $O(R_i)$	2	3	4	5
10^{-6}	2	5		
10^{-7}	3	17	5	
10^{-8}	1	17	4	
10^{-9}		26	6	1
10^{-10}		3	8	
10^{-11}			2	
Number of examples	6	68	25	1

TABLE I

ACCURACY COMPARISON WITH THE PERIODIC QZ ALGORITHM FOR 100 RANDOM EXAMPLES ($\epsilon = 10^{-4}$)

The obtained results are resumed in Table 1. In this two table, **Iterations** indicates the number of necessary iterations to obtain the specified level of accuracy; **$O(R_i)$** is the order of magnitude of R_i ; **Number of examples** indicates the number of examples requiring the same number of iterations for convergence. It follows from the obtained results that the proposed algorithm works well in most cases with good efficiency (only 2 – 4 iterations for convergence in most examples) and accuracy ($R_i < 10^{-5}$ for $\epsilon = 10^{-4}$) when compared with the the periodic QZ algorithm.

Example 2. The following example illustrates that the proposed algorithm works well when the periodic QZ algorithm fails. Consider a discrete-time 2-periodic linear system described by:

$$A(0) = \begin{pmatrix} -0.3374 & 0.6996 & -0.3129 \\ 1.1031 & -1.1283 & -1.5469 \\ 0.9397 & 1.1847 & 0.4700 \end{pmatrix},$$

$$A(1) = \begin{pmatrix} 0.5306 & -0.5703 & 0.7749 \\ -0.4615 & 0.2818 & -1.1079 \\ -0.6835 & 0.0082 & 0.0132 \end{pmatrix}$$

$$B_1(0) = \begin{pmatrix} 1.8048 & 0.5877 \\ -0.3916 & -0.0175 \\ 1.2778 & 0.5082 \end{pmatrix}$$

$$B_1(1) = \begin{pmatrix} -0.4959 & 0.1848 \\ 2.4629 & -1.4624 \\ -0.9885 & 1.3702 \end{pmatrix}$$

$$B_2(0) = \begin{pmatrix} 0.4089 & -1.0077 \\ -0.1932 & -0.4189 \\ 0.4144 & -1.1287 \end{pmatrix}$$

$$B_2(1) = \begin{pmatrix} -0.7202 & 0.5488 \\ 0.5385 & 0.1610 \\ -0.4652 & -1.0513 \end{pmatrix}$$

$$C(0) = \begin{pmatrix} 0.3857 & 0.1667 & -0.5188 \\ 0.3608 & 0.2532 & -0.1670 \end{pmatrix}$$

$$C(1) = \begin{pmatrix} -0.2178 & -0.3821 & -1.2258 \\ -0.6205 & 0.1863 & -0.2550 \end{pmatrix}$$

$$D_1(0) = \begin{pmatrix} 0.6794 & -0.2595 \\ -0.3521 & -0.4750 \end{pmatrix}$$

$$D_1(1) = \begin{pmatrix} -0.8972 & 1.2238 \\ -0.9517 & -1.0165 \end{pmatrix}$$

$$D_2(0) = \begin{pmatrix} -0.6039 & -1.4918 \\ -0.8539 & -0.0919 \end{pmatrix}$$

$$D_2(1) = \begin{pmatrix} -1.2021 & 0.1257 \\ -0.5596 & 0.3733 \end{pmatrix}$$

We set $\gamma^2 = 10$. For this example, the QZ algorithm does not produce an accurate result while the proposed algorithm gives a stabilizing solution with the specified accuracy ($\epsilon = 10^{-4}$) after 3 main iterations. The stabilizing solution is given by:

$$\tilde{X}(0) = \begin{pmatrix} 0.3040 & -0.0355 & -0.0837 \\ -0.0355 & 0.9107 & 0.4405 \\ -0.0837 & 0.4405 & 0.2277 \end{pmatrix}$$

$$\tilde{X}(1) = \begin{pmatrix} 1.2263 & -0.5188 & 0.9522 \\ -0.5188 & 0.2492 & -0.3346 \\ 0.9522 & -0.3346 & 0.8964 \end{pmatrix}$$

V. CONCLUSIONS

In this paper, we have addressed the problem of computing the stabilizing solution of a class of H_∞ -type periodic discrete-time Riccati equations. For the computation of the sequence of approximations of the stabilizing solution, we have proposed an alternative iterative algorithm to the one proposed in [7]. The main advantage of this new formulation is that the sequence of approximations of the stabilizing solution are computed directly without involving the computation of some intermediate vanishing matrix sequence. This improves the reliability of the method when facing ill conditioned problems.

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