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# Regular Positive-Real Functions and a New Look at Reichert's Theorem

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*Abstract*— **The first part of the talk will recall the results of [1] relating to the Ladenheim catalogue. In particular it will be seen that all but two of the 108 Ladenheim networks realise regular biquadratics. The remaining two (#70 and #95) are capable of realising some but not all non-regular biquadratics, as shown by considering a canonical form for biquadratics. Reichert's theorem shows that additional resistors do not expand the Ladenheim class [2]. Reichert's original proof contained a complex topological argument. The proof was reworked and amplified in [3]. This talk will discuss an alternative approach to proving Reichert's theorem which dispenses with the topological argument.**

**Key words.** Circuit synthesis, passivity, realisation, inerter

**AMS subject classification.** 70Q05, 93C05, 94C99

## I. Ladenheim's Catalog and Regular Positive-Real Functions

The first systematic attempt to classify simple RLC networks by exhaustive enumeration appears to be the Master's thesis of Ladenheim [4]. All networks with at most five elements and at most two reactive elements are considered and reduced to 108 networks by various transformations. Realisability conditions on the parameters $A, B, \ldots, F$ in the biquadratic function

$$Z(s) = \frac{As^2 + Bs + C}{Ds^2 + Es + F}, \tag{1}$$

are listed for each network. However, the totality of biquadratics which may be realised by some network in the set is not obvious from the results.

In this talk, we will first explain the concept of a regular positive-real function, which was introduced in [1].

*Definition 1:* A positive-real function $Z(s)$ is defined to be *regular* if the smallest value of $\mathrm{Re}\,(Z(j\omega))$ or $\mathrm{Re}\,\left(Z^{-1}(j\omega)\right)$ occurs at $\omega = 0$ or $\omega = \infty$.

Some of the properties of regular positive-real functions were also summarised in [1]. By analysing all possible network structures in Ladenheim's catalog, the following two theorems were obtained for series-parallel networks and bridge networks, respectively.

*Theorem 2 ([1], Theorem 1):* A biquadratic impedance (1) can be realised by series-parallel five-element networks with two reactive elements if and only if it is regular.

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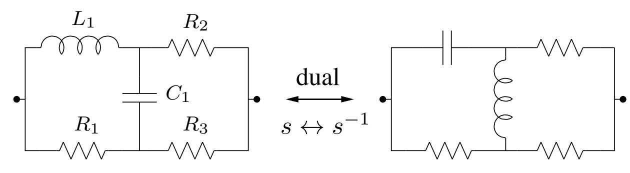


(a) (b)

Fig. 1. The two-reactive five-element bridge network quartet (which reduces to two distinct networks) that can realise non-regular impedances.

*Theorem 3 ([1], Theorem 2):* Bridge networks with two reactive and three resistive elements can only realise regular immittances except for the network quartet of Fig. 1.

The two networks shown in Fig. 1 are #70 and #95 in Ladenheim's catalog.

## II. Reichert's Theorem and an Alternative Approach

Reichert's theorem [2] completely characterises all biquadratic impedances (1) which may be realised by RLC-circuits with at most two reactive elements. It is shown that no such function requires more than three resistors for its realisation. The crucial part of Reichert's proof is a topological argument that a certain system of polynomial equations has no solution in some region of its variable parameters. In [3], a complete reworking of Reichert's proof was presented. In particular, some new lemmas were provided to clarify the main topological argument.

In the terminology of this talk, the main result of [2] can be stated as follows.

*Theorem 4 ([5], Theorem 3):* A biquadratic impedance (1) can be realised with at most two reactive elements if and only if it satisfies one of the following conditions:

1) $Z(s)$ is regular,
2) $Z(s)$ is the driving-point impedance of the networks shown in Fig. 1(a) or Fig. 1(b).

It has been pointed out in [6] that at most four resistors are sufficient to synthesize any biquadratic function which is realisable with two reactive elements. Using the concept of regularity we can deduce here a stronger result.

*Theorem 5 ([5], Theorem 4):* A biquadratic impedance (1) can be realised with at most two reactive elements if and only if it satisfies one of the following conditions:

1) $Z(s)$ is regular,
2) $Z(s)$ is the driving-point impedance of the networks shown in Fig. 2(a) or Fig. 2(b).

Based on Theorems 4 and 5, we can see that an alternative proof of Reichert's theorem is to show there are no extra

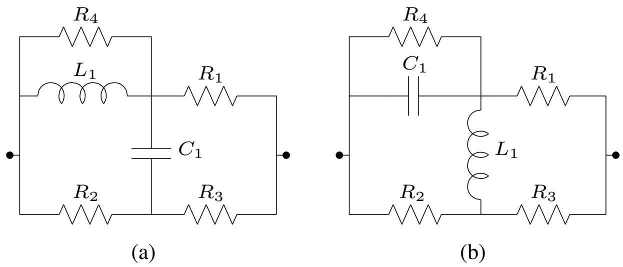


Fig. 2. The two networks in [6] that can realise non-regular biquadratics.

non-regular biquadratic impedances (1) that can be realised by the networks in Fig. 2 compared with Fig. 1. This paper will discuss a direct approach to proving this latter fact.

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