

The Schwarz lemma and the Schur-Agler class*

Anatolii Grinshpan, Dmitry S. Kaliuzhnyi-Verbovetskyi, and Hugo J. Woerdeman

Abstract—We show that multiplication by a one of the variables can lower the Agler norm of a multivariable function that is analytic on the open d -disk. This is in contrast to the H^∞ -norm, where multiplication by a one of the variables leaves the norm unchanged, a consequence of the Schwarz lemma. To show this failure of the Schwarz lemma for the Schur-Agler class the theory of determinantal representations is used.

I. NOTATION AND BACKGROUND

We start with some notation. \mathbb{D}^d and \mathbb{T}^d denote the open unit d -disk and unit d -torus. The inequality $n \geq m$ for $n, m \in \mathbb{Z}^d$ is meant in the componentwise sense. Let $\mathbb{N}_0^d = \{n = (n_1, \dots, n_d) \in \mathbb{Z}^d: n \geq 0\}$. For $n \in \mathbb{N}_0^d$ and $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, we write $|n| = n_1 + \dots + n_d$, $\bar{z} = (\bar{z}_1, \dots, \bar{z}_d)$, $1/z = (1/z_1, \dots, 1/z_d)$, $z^n = z_1^{n_1} \dots z_d^{n_d}$, and $Z_n = \bigoplus_{i=1}^d z_i I_{n_i}$. The multi-degree of a polynomial $p \in \mathbb{C}[z_1, \dots, z_d]$ is the tuple $\text{deg } p = (\text{deg}_1 p, \dots, \text{deg}_d p)$ of its partial degrees, and the total degree, $\text{tdeg } p$, is the maximum of $|n|$ over all of its multi-powers n . We write \bar{p} for the polynomial with conjugated coefficients, and $\overleftarrow{p}(z) := z^m \bar{p}(1/z)$ when $m = \text{deg } p$.

The Schur class $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ consists of analytic functions f on \mathbb{D}^d with values in the space $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ of bounded linear operators from a Hilbert space \mathcal{U} to a Hilbert space \mathcal{Y} such that

$$\|f\|_\infty := \sup_{z \in \mathbb{D}^d} \|f(z)\| \leq 1.$$

The Schur-Agler class $\mathcal{SA}_d(\mathcal{U}, \mathcal{Y})$ [1] consists of functions $f \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ satisfying

$$\|f\|_{\mathcal{A}} := \sup_T \|f(T)\| \leq 1,$$

where the supremum is taken over all d -tuples $T = (T_1, \dots, T_d)$ of commuting strict contractions on a (separable infinite-dimensional) Hilbert space, and

$$f(T) = \sum_{n \geq 0} f_n \otimes T^n.$$

Here f_n are the Taylor coefficients of f , and $T^n = T_1^{n_1} \dots T_d^{n_d}$. If $\mathcal{U} = \mathcal{Y} = \mathbb{C}$ or $\mathcal{U} = \mathcal{Y}$, we will use respective shortcuts \mathcal{S}_d , \mathcal{SA}_d , and $\mathcal{S}_d(\mathcal{U})$, $\mathcal{SA}_d(\mathcal{U})$.

Let $f: \mathbb{D}^d \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$ be a bounded analytic function and T a d -tuple of commuting strict contractions. The von Neumann inequality,

$$\|f(T)\| \leq \|f\|_\infty, \quad (I.1)$$

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All three authors are with the Department of Mathematics, Drexel University, 3141 Chestnut Street, Philadelphia, PA 19104, USA {tolya, dmitryk, hugo}@math.drexel.edu

is valid when $d = 1$ [21] and when $d = 2$ [3], and not always valid when $d \geq 3$ [20], [7], [13]. In fact, (I.1) holds for all T as above if and only if the Schur and Agler norms coincide,

$$\|f\|_{\mathcal{A}} = \|f\|_\infty. \quad (I.2)$$

Thus a Schur function f of norm 1 is Schur-Agler if and only if (I.2) holds. For more on the multivariable von Neumann inequality, see [9].

A rational function in $\mathcal{S}_d(\mathbb{C}^N)$ is said to be inner if its radial limits are unitary a.e. on the d -torus \mathbb{T}^d . A scalar-valued rational inner function f is said to have a conservative transfer-function realization of order $m \in \mathbb{N}_0^d$ if there exists a unitary colligation matrix

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{(1+|m|) \times (1+|m|)}$$

so that

$$f(z) = A + BZ_m(I_{|m|} - DZ_m)^{-1}C. \quad (I.3)$$

Such a realization of f exists (for some m) if and only if $f \in \mathcal{SA}_d$ [1],[14, Theorem 2.9]. We notice that this statement is also true for $N \times N$ matrix-valued functions, where the colligation matrix $U \in \mathbb{C}^{(N+|m|) \times (N+|m|)}$ is partitioned accordingly [4].

Rudin's theorem [19, Theorem 5.2.5] states that every scalar-valued rational inner function has the form

$$f(z) = z^n \frac{\bar{p}(1/z)}{p(z)} = z^{n-\text{deg } p} \frac{\overleftarrow{p}(z)}{p(z)}$$

for some $n \geq \text{deg } p$ and p a stable¹ polynomial, that is, having no zeros in \mathbb{D}^d . Such functions are dense in \mathcal{S}_d , in the topology of uniform convergence on compact subsets of \mathbb{D}^d [19, Theorem 5.5.1]. Following [17], we will say that a stable polynomial p is an Agler denominator if the rational inner function \overleftarrow{p}/p is Schur-Agler. We call a stable polynomial p an eventual Agler denominator of order $n \in \mathbb{N}_0^d$ if $z^n \bar{p}(1/z)/p(z) \in \mathcal{SA}_d$ [11]. The eventual Agler denominators are simply the denominators of inner functions from the Schur-Agler class. There do exist stable polynomials which are not Agler denominators; see Example 2.1. In this paper we shall see that a non Agler denominator may be an eventual Agler denominator; see the proof of Theorem 2.3.

¹In [11] stable polynomials are referred to as semi-stable.

II. STABLE POLYNOMIALS, EVENTUAL AGLER DENOMINATORS, AND THE FAILURE OF A SCHWARZ LEMMA ON THE SCHUR–AGLER CLASS

The concept of polynomial stability arises naturally in a variety of disciplines, such as Analysis, Electrical Engineering, and Control Theory; see, e.g., [6], [22], [5], [18], [8], [12].

Every polynomial $p \in \mathbb{C}[z_1, \dots, z_d]$ with no zeros in the closed polydisk \mathbb{D}^d is a denominator of some rational function f with $\|f\|_{\mathcal{A}} = 1$. Indeed, one can take f as an appropriate scalar multiple of $\sqrt[p]{p}/p$, because $\|\sqrt[p]{p}/p\|_{\mathcal{A}} < \infty$. For $d = 1, 2$, since the Schur and Agler norms coincide, every stable polynomial p is an Agler denominator. For $d > 2$, this is no longer true, which is seen from the following construction of a scalar-valued rational inner function $f \in \mathcal{S}_d \setminus \mathcal{S}\mathcal{A}_d$.

Example 2.1: [11, Example 5.1] Let $p \in \mathbb{C}[z_1, \dots, z_d]$, with $\|p\|_{\infty} = 1$ and $\deg p = m$, fail the von Neumann inequality (I.1) for a tuple $T = (T_1, \dots, T_d)$ of commuting contractions such that $T^k = 0$, for some $k \in \mathbb{N}_0^d$. Examples of such a scenario can be found in [20], [13], [7], [11]. Then, for $0 < r < 1$, the polynomial $q(z) = 1 + rz^k \sqrt[p]{p}(z)$ has no zeros in \mathbb{D}^d , and the function

$$f(z) = \frac{z^{k+m} + rp(z)}{1 + rz^k \sqrt[p]{p}(z)}$$

is inner. Since $f(T) = rp(T)$, f is not in $\mathcal{S}\mathcal{A}_d$ whenever $r > 1/\|p(T)\|$. Hence, q is not an eventual Agler denominator of order $m - \deg \sqrt[p]{p}$. In particular, if $\deg \sqrt[p]{p} = m$, then $f = \sqrt[q]{q}/q$, so that q is not an Agler denominator. For instance, the Kaijser–Varopoulos polynomial

$$p(z_1, z_2, z_3) = \frac{1}{5}(z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_2z_3 - 2z_3z_1)$$

satisfies $\|p\|_{\infty} = 1$, $\deg p = \deg \sqrt[p]{p} = (2, 2, 2)$, and there exist commuting contractions T_1, T_2, T_3 such that $\|p(T_1, T_2, T_3)\| = 6/5$ and $T_1T_2T_3 = 0$ (see [13]).

An important question is the following.

Problem 2.2: Is every stable polynomial in $d > 2$ variables an eventual Agler denominator?

The answer is positive for every trivariate polynomial of multi-degree $n = (n_1, n_2, 1)$ with no zeros in the \mathbb{D}^3 [15].

The distinction between the notions of Agler and eventual Agler denominators is highlighted by the following theorem.

Theorem 2.3: When $d > 2$, the multiplication by an independent variable, e.g. z_1 , may diminish the Agler norm.

A version of the Schwarz lemma on the class $\mathcal{S}\mathcal{A}_d$, $d > 2$, thus fails²: the condition $z_1 f \in \mathcal{S}\mathcal{A}_d$ for an analytic function f on \mathbb{D}^d does not force $f \in \mathcal{S}\mathcal{A}_d$. In fact, the inequality $\|z_1 f\|_{\mathcal{A}} < \|f\|_{\mathcal{A}}$ can be realized for f a rational inner function or even a polynomial.

Proof of Theorem 2.3. We now show that the polynomial q from Example 2.1, which corresponds to the Kaijser–Varopoulos polynomial, with an additional restriction on the parameter r , admits a representation (II.1) with a contractive

matrix K and thus by Theorem 5.2 in [11] is an eventual Agler denominator. Since q is not an Agler denominator, this example proves Theorem 2.3.

Write

$$\begin{aligned} q(z) &= 1 + \frac{r}{5} z_1 z_2 z_3 (z_2^2 z_3^2 + z_1^2 z_3^2 \\ &\quad + z_1^2 z_2^2 - 2z_1^2 z_2 z_3 - 2z_1 z_2^2 z_3 - 2z_1 z_2 z_3^2) \\ &= 1 + \frac{r}{5} z_1 z_2 z_3 v^T X v, \end{aligned}$$

where

$$v = \begin{bmatrix} z_1^2 \\ z_2^2 \\ z_3^2 \\ z_2 z_3 \\ z_1 z_3 \\ z_1 z_2 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & b & b & c & 0 & 0 \\ b & 0 & b & 0 & c & 0 \\ b & b & 0 & 0 & 0 & c \\ c & 0 & 0 & a & d & d \\ 0 & c & 0 & d & a & d \\ 0 & 0 & c & d & d & a \end{bmatrix}$$

and a, b, c, d satisfy $a + 2b = 1$, $c + d = -1$. Then

$$\|1 - q\|_{\mathcal{A}} \leq \frac{6r}{5} \|X\|.$$

Observe that if $\frac{5}{6} \leq \|X\| < 1$, then for every $r: \frac{5}{6} < r < \frac{5}{6\|X\|}$ one has $\|1 - q\|_{\mathcal{A}} < 1$. Then by Theorem 3.2 in [11] q admits a determinantal representation (II.1) with a contractive matrix K , and Theorem 5.2 in [11] applies. The condition on X can be met: minimization of $\|X\|$ over all admissible a, b, c, d yields $\min \|X\| \approx 0.92$. The choice

$$a = \frac{20534}{90000}, b = \frac{34733}{90000}, c = d = -\frac{1}{2},$$

provides an X close to the minimal one. \square

Observe that if an analytic function f satisfies the von Neumann inequality, i.e., (I.2) holds, then $\|z^n f\|_{\mathcal{A}} = \|f\|_{\mathcal{A}}$ for every $n \in \mathbb{N}_0^d$. It is natural to explore the converse.

Problem 2.4: If a polynomial p violates (I.2), is it true that p then satisfies the inequality $\|z^n p\|_{\mathcal{A}} < \|p\|_{\mathcal{A}}$, for some $n \in \mathbb{N}_0^d$?

Consider representations of polynomials $p \in \mathbb{C}[z_1, \dots, z_d]$, with $p(0) = 1$, of the form

$$p(z) = \det(I_{|n|} - KZ_n). \tag{II.1}$$

When K is a contractive matrix, every polynomial defined by (II.1) is stable. Thus determinantal representation (II.1) provides one with a certificate for polynomial stability. The converse question is open for $d > 2$.

Problem 2.5: Does every multivariable stable polynomial p , with $p(0) = 1$, admit a representation (II.1) with a contractive matrix K ?

In the univariate and bivariate cases, the answer is affirmative; see [10].

We end this paper with a strengthening of [11, Theorem 5.6]. A stable polynomial p is *scattering Schur* [5] if p and $\sqrt[p]{p}$ are coprime (i.e., have no common factors).

Theorem 2.6: Let p be a scattering Schur polynomial, with $p(0) = 1$. Then the following statements are equivalent:

- p is an eventual Agler denominator of order k such that $f(z) = z^k \sqrt[p]{p}(1/z)/p(z)$ has a conservative transfer-function realization of order m .

²See [2], [16] for another Schwarz lemma on $\mathcal{S}\mathcal{A}_d$.

- There exists a self-reversive polynomial a of multi-degree $m - k$ such that $p(z)a(z) = a(0) \det(I_{|m|} - KZ_m)$, with $\|K\| \leq 1$.

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