

A Low-Rank Reduced Basis Method for Parameter-Dependent Lyapunov Equations

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Abstract—The need for solving parameter-dependent Lyapunov matrix equations arises when applying model reduction to parameterized control systems. Although reduced basis methods have been shown to deal quite effectively with parameter dependencies, the dimensionality of the solution space for Lyapunov equations provides an additional challenge. We show how this challenge can be addressed by incorporating low-rank techniques into the reduced basis method.

Keywords. Large Scale Systems, Computations in Systems Theory, Computational Control, Parametrized Linear Systems, Model Reduction

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I. INTRODUCTION

Suppose we are given a large-scale control system

$$\begin{aligned} E(p)\dot{x}(t) &= A(p)x(t) + B(p)u(t), \\ y(t) &= C(p)x(t), \end{aligned} \quad (1)$$

where the system matrices $E, A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times m}$ and $C \in \mathbb{R}^{l \times N}$ depend on a parameter $p \in \mathbb{P} \subset \mathbb{R}^d$. Such systems frequently arise from the spatial discretization of partial differential equations (PDEs), in which the parameters are used to model the variation of geometries and material properties to be optimized. In order to enable fast simulations for the whole parameter domain, we aim to find a reduced-order model

$$\begin{aligned} \tilde{E}(p)\dot{\tilde{x}}(t) &= \tilde{A}(p)\tilde{x}(t) + \tilde{B}(p)u(t), \\ \tilde{y}(t) &= \tilde{C}(p)\tilde{x}(t), \end{aligned} \quad (2)$$

with $\tilde{E}, \tilde{A} \in \mathbb{R}^{r \times r}$, $\tilde{B} \in \mathbb{R}^{r \times m}$, $\tilde{C} \in \mathbb{R}^{l \times r}$ and $r \ll N$. In the last two decades, several model reduction approaches have been developed for parameterized systems. These include multivariate moment matching based on (rational) Krylov subspaces, interpolation-based techniques and the reduced basis method. None of these methods can be considered as optimal. Our model reduction approach is based on balanced truncation [4], which preserves stability in the reduced-order model and provides computable error bounds. Balanced truncation requires the computation of the controllability Gramian $P(p)$ and the observability Gramian $Q(p)$, which

are defined as unique symmetric positive semidefinite solutions of the Lyapunov equations

$$A(p)P(p)E(p)^T + E(p)P(p)A(p)^T = -B(p)B(p)^T, \quad (3)$$

$$A(p)^T Q(p)E(p) + E(p)^T Q(p)A(p) = -C(p)^T C(p), \quad (4)$$

respectively, provided that $E(p)$ is nonsingular and all eigenvalues of the matrix pencil $A(p) - \lambda E(p)$ have negative real part. The reduced-order model (2) can then be computed by projection onto a dominant eigensubspace of the product $P(p)E(p)^T Q(p)E(p)$. To determine the reduced-order model for many parameter values, we have to approximate the solutions of the Lyapunov equations (3) and (4) for all these parameters. In this paper, we present a low-rank reduced basis method for approximating solutions to parameter-dependent Lyapunov equations on the whole parameter domain.

II. REDUCED BASIS METHODS

The reduced basis (RB) method provides a framework for the solution of parametrized PDEs [5]. It consists of an *offline* stage, where solutions of the PDEs are solved for suitably chosen parameters and their solutions are collected in a (low-dimensional) subspace. The subsequent *online* stage then uses a Galerkin projection approach to compute an approximate solution for any parameter which belongs to this subspace. This may speed up the solution process dramatically, especially if the PDE needs to be solved for many parameter values. *A posteriori* error analysis is an important part of the RB method to ensure its reliability.

In this section, we will sketch how the RB method applies to Lyapunov equations. Necessarily our explanations will be rather brief and we refer to, e.g., [5] for more details. From now on, we consider Lyapunov equations of the form

$$-A(p)X(p)E - EX(p)A(p) = BB^T, \quad (5)$$

where we assume that

- 1) neither E nor B depend on the parameters;
- 2) E is symmetric, positive definite and $A(p)$ is symmetric, negative definite for all $p \in \mathbb{P}$.

The purpose of the first assumption is mainly to simplify the discussion. Our algorithms can be trivially extended to parameter-dependent E and B . On the other hand, the second assumption is central to the error estimators we are using; dropping this assumption would require the use of different estimators [1]. The Lyapunov equation (5) can be equivalently written as the $N^2 \times N^2$ linear system

$$\mathcal{A}(p)x(p) = b \quad (6)$$

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with $\mathcal{A}(p) = -E \otimes A(p) - A(p) \otimes E$, $b = \text{vec}(BB^T)$ and $x(p) = \text{vec}(X(p))$. We now recall the RB method for solving this system without exploiting the particular structure of b and $\mathcal{A}(p)$.

A. Offline Phase

In the offline phase, we subsequently select n_{\max} parameter samples $p_1, p_2, \dots, p_{n_{\max}} \in \Xi$, where $\Xi \subset \mathbb{P}$ is a training set of finitely many (typically a few thousand) parameter samples, and construct the subspaces $\mathcal{V}_1 \subseteq \dots \subseteq \mathcal{V}_{n_{\max}} \subset \mathbb{R}^{N^2}$. The parameter samples are selected by a greedy strategy aiming at to minimizing an estimate of the error. Assuming that the first k samples have been processed, the $(k+1)$ -th step of this procedure consists of selecting

$$p_{k+1} = \text{argmax}\{\Delta_k(p) : p \in \Xi\}, \quad (7)$$

where $\Delta_k(p)$ is an error estimator, see Section II-C below. Then (6) with $p = p_{k+1}$ is solved for $x(p_{k+1})$ and the subspace \mathcal{V}_k is extended to a new subspace

$$\begin{aligned} \mathcal{V}_{k+1} &= \mathcal{V}_k + \text{span}\{x(p_{k+1})\} \\ &= \text{span}\{x(p_1), x(p_2), \dots, x(p_{k+1})\}. \end{aligned}$$

B. Online Phase

In the online phase, having an orthonormal basis $V \in \mathbb{R}^{N^2 \times n_{\max}}$ of $\mathcal{V}_{n_{\max}}$ available, we compute an approximation to the solution of the linear system (6) at an arbitrary parameter $p \in \mathbb{P}$ as $\tilde{x}(p) = Vy(p)$, where $y(p)$ solves the compressed linear system

$$(V^T \mathcal{A}(p)V)y(p) = V^T b. \quad (8)$$

Since n_{\max} is usually small, a standard direct solver for linear systems can be used to solve (8). Reshaping $\tilde{x}(p)$ back into an $N \times N$ matrix $\tilde{X}(p)$ yields an approximate solution to the Lyapunov equation (5).

To setup the linear system (8) efficiently, it is assumed that

$$\mathcal{A}(p) = \theta_1(p)\mathcal{A}_1 + \dots + \theta_q(p)\mathcal{A}_q \quad (9)$$

for some functions $\theta_j : \mathbb{P} \rightarrow \mathbb{R}$ and constant symmetric matrices \mathcal{A}_j . Such an affine decomposition often arises naturally or can be attained by techniques described in [8]. Thus,

$$V^T \mathcal{A}(p)V = \theta_1(p)V^T \mathcal{A}_1 V + \dots + \theta_q(p)V^T \mathcal{A}_q V.$$

Having precomputed $V^T \mathcal{A}_j V$ for $j = 1, \dots, q$ then allows us to attain a complexity of $O(qn_{\max}^2 + n_{\max}^3)$ for solving (8).

The accuracy of $\tilde{x}(p)$ can be quantified using the error estimator described in the next subsection.

C. Error Estimator

Given an approximate solution of the form $\tilde{x}(p) = Vy(p)$ for a fixed parameter sample $p \in \mathbb{P}$, the norm of the residual $r(p) = b - \mathcal{A}(p)\tilde{x}(p)$ can be computed from

$$\begin{aligned} \|r(p)\|_2^2 &= (b - \mathcal{A}(p)Vy(p))^T (b - \mathcal{A}(p)Vy(p)) \\ &= b^T b - 2b^T \mathcal{A}(p)Vy(p) + y(p)^T V^T \mathcal{A}(p)Vy(p). \end{aligned}$$

For $\mathcal{A}(p)$ as in (9), we have

$$\begin{aligned} \|r(p)\|_2^2 &= b^T b - 2 \sum_{j=1}^q \theta_j(p) b^T \mathcal{A}_j V y(p) \\ &\quad + \sum_{i,j=1}^q \theta_i(p) \theta_j(p) y(p)^T V^T \mathcal{A}_i \mathcal{A}_j V y(p). \end{aligned} \quad (10)$$

If we precompute and store the parameter-independent quantities $b^T \mathcal{A}_j V$ and $V^T \mathcal{A}_i \mathcal{A}_j V$ for $i, j = 1, \dots, q$, then the computation of $\|r(p)\|_2$ costs $O(q^2 n_{\max}^2)$ operations. The difference to the true solution $x(p)$ can then be estimated as

$$\|x(p) - \tilde{x}(p)\|_2 \leq \frac{\|r(p)\|_2}{\lambda_{\min}(\mathcal{A}(p))} \leq \frac{\|r(p)\|_2}{\alpha_{\text{LB}}(p)} =: \Delta(p),$$

where λ_{\min} denotes the smallest eigenvalue of a symmetric matrix and $\alpha_{\text{LB}}(p)$ is a lower bound for $\lambda_{\min}(\mathcal{A}(p))$.

Effective and reliable lower bounds for the smallest eigenvalue of $\mathcal{A}(p)$ can be determined a priori by the successive constraint method (SCM) [2]. Using the Rayleigh quotient characterization of the smallest eigenvalue and the assumed affine decomposition (9) of $\mathcal{A}(p)$, the SCM is based on the following relation:

$$\begin{aligned} \lambda_{\min}(\mathcal{A}(p)) &= \min_{v \in \mathbb{R}^{N^2}} \frac{v^T \mathcal{A}(p)v}{v^T v} \\ &= \min_{v \in \mathbb{R}^{N^2}} \sum_{i=1}^q \theta_i(p) \frac{v^T \mathcal{A}_i v}{v^T v} \\ &= \min_{v \in \mathbb{R}^{N^2}} \Theta(p) \left[\frac{v^T \mathcal{A}_1 v}{v^T v}, \dots, \frac{v^T \mathcal{A}_q v}{v^T v} \right]^T \\ &= \min_{y \in \mathcal{Y}} \Theta(p)y, \end{aligned} \quad (11)$$

where

$$\mathcal{Y} = \left\{ \left[\frac{v^T \mathcal{A}_1 v}{v^T v}, \dots, \frac{v^T \mathcal{A}_q v}{v^T v} \right]^T, v \in \mathbb{R}^{N^2} \right\} \subset \mathbb{R}^q$$

is the joint numerical range of $\mathcal{A}_1, \dots, \mathcal{A}_q$ and $\Theta(p) = [\theta_1(p), \dots, \theta_q(p)]$. In principle, this shows that we can compute $\lambda_{\min}(\mathcal{A}(p))$ by solving the optimization problem (11). However, the structure of \mathcal{Y} does not allow for an efficient optimization procedure and therefore \mathcal{Y} is approximated by a convex polyhedron \mathcal{Y}_{LB} that encloses \mathcal{Y} . The set \mathcal{Y}_{LB} is constructed in the offline phase from the smallest eigenvalue and an associated eigenvector of $\mathcal{A}(\hat{p})$ for a few parameter samples \hat{p} , as explained in [5]. In the online phase, $\alpha_{\text{LB}}(p)$ is obtained from solving the (small-scale) linear program $\min_{y \in \mathcal{Y}_{\text{LB}}} \langle \Theta(p), y \rangle$.

The error estimator $\Delta_k(p)$ used in (7) to guide the sampling strategy in the offline phase is defined in an analogous way, with V replaced by a basis V_k of \mathcal{V}_k .

D. Limitations for Lyapunov equations

When applying the plain RB method to the $N^2 \times N^2$ linear system (6), the high dimensionality N^2 of the solution space leads to inefficiencies. For example, the complexity for orthonormalizing and storing V is $O(N^2 n_{\max}^2)$ and $O(N^2 n_{\max})$, respectively. This puts a limitation on the size

N that can be handled; N can be at most a few thousand. Another major disadvantage is that the approximate solution $\tilde{X}(p)$ obtained in the online phase is *not* guaranteed to be positive semidefinite, a property that is highly desirable in model reduction applications.

III. A LOW-RANK REDUCED BASIS METHODS

In this section, we show how the RB method can be accelerated for the Lyapunov equation (5) by making use of low-rank properties.

A. Low-rank solution for a fixed parameter sample

For $m \ll N$, it is known [3], [7] that the singular values of the solution $X(p)$ to (5) decay very quickly. Hence, $X(p)$ can be well approximated by a low-rank matrix of the form $L(p)L(p)^T$, where the Cholesky factor $L(p)$ has $m_p \ll N$ columns. Using the ADI method [6] or the extended Arnoldi method [9], such an approximation can be computed efficiently, that is, at the cost of solving a few linear systems with shifted matrices $A(p) - \tau E$.

B. Low-rank structure in the offline phase

The offline phase proceeds as described in Section II-A, with the notable difference that instead of the $N \times N$ solutions $X(p_j)$ we collect the $N \times m_j$ Cholesky factors $L(p_j) \in \mathbb{R}^{N \times m_j}$ in a subspace

$$\mathcal{U}_{n_{\max}} = \text{span}\{L(p_1), L(p_2), \dots, L(p_{n_{\max}})\} \subset \mathbb{R}^N.$$

In the absence of approximation error, we have $\mathcal{V}_{n_{\max}} \subset \mathcal{U}_{n_{\max}} \otimes \mathcal{U}_{n_{\max}}$. For a basis U of $\mathcal{U}_{n_{\max}}$, this means that any vector $v \in \mathcal{V}_{n_{\max}}$ can be represented as

$$v = (U \otimes U)\text{vec}(Y) = \text{vec}(UYU^T) \quad (12)$$

for some (small) matrix Y . The dimension of $\mathcal{U}_{n_{\max}}$ is given by

$$M = m_1 + \dots + m_{n_{\max}},$$

which can be – depending on m – significantly larger than n_{\max} , the dimension of $\mathcal{V}_{n_{\max}}$. To attain a smaller value for M , we repeatedly apply column compressions while building the basis U , using the truncated singular value decomposition (SVD) with relative truncation tolerance $\text{tol}_{\text{compr}}$.

C. Low-rank structure in the online phase

The online phase proceeds as described in Section II-B, with the subspace $\mathcal{V}_{n_{\max}}$ replaced by $\mathcal{U}_{n_{\max}} \otimes \mathcal{U}_{n_{\max}}$. In view of (8) and (12), this means that we have to solve the compressed linear system

$$((U \otimes U)^T \mathcal{A}(p)(U \otimes U))y(p) = (U \otimes U)^T b,$$

which is equivalent to solving the compressed Lyapunov equation

$$-\tilde{A}(p)Y(p)\tilde{E} - \tilde{E}Y(p)\tilde{A}(p) = \tilde{B}\tilde{B}^T \quad (13)$$

with $\tilde{A}(p) = U^T A(p)U$, $\tilde{E} = U^T E U$ and $\tilde{B} = U^T B$ for the unknown $Y(p) \in \mathbb{R}^{M \times M}$. The approximate solution of (5) is then obtained as

$$\tilde{X}(p) = UY(p)U^T. \quad (14)$$

The compressed matrices $\tilde{A}(p)$ and \tilde{E} inherit the negative/positive definiteness from $A(p)$ and E , respectively. Hence, both $Y(p)$ and $\tilde{X}(p)$ are guaranteed to stay positive semidefinite, in contrast to the plain RB method.

To setup the Lyapunov equation (13) efficiently, we again assume an affine decomposition

$$A(p) = \theta_1(p)A_1 + \dots + \theta_q(p)A_q, \quad (15)$$

where A_j are parameter-independent symmetric matrices. Then we have

$$\tilde{A}(p) = U^T A(p)U = \theta_1(p)U^T A_1 U + \dots + \theta_q(p)U^T A_q U.$$

Again, the precomputation of $U^T A_j U$ for $j = 1, \dots, q$ allows us to reduce the complexity to $O(qM^2)$ for constructing the coefficients and $O(M^3)$ for solving (13).

D. Low-rank structure in the error estimator

The computation of the error estimates proceeds as described in Section II-C, with the approximate solution $\tilde{x}(p) = Vy(p)$ replaced by $\tilde{x}(p) = (U \otimes U)\text{vec}(Y(p))$ which can be reshaped as $\tilde{X}(p)$ in (14). For a fixed parameter sample $p \in \mathbb{P}$, the Frobenius norm of the residual

$$R(p) = BB^T + A(p)\tilde{X}(p)E + E\tilde{X}(p)A(p)$$

can be computed as

$$\begin{aligned} \|R(p)\|_F^2 &= (b - \mathcal{A}(p)\tilde{x}(p))^T (b - \mathcal{A}(p)\tilde{x}(p)) \\ &= b^T b - 2b^T \mathcal{A}(p)(U \otimes U)\text{vec}(Y(p)) \\ &\quad + \text{vec}(Y(p))^T (U \otimes U)^T \mathcal{A}(p)\mathcal{A}(p)(U \otimes U)\text{vec}(Y(p)). \end{aligned}$$

Taking (15) into account we obtain

$$\begin{aligned} \|R(p)\|_F^2 &= \text{tr}((B^T B)(B^T B)) \\ &\quad + 4 \sum_{j=1}^q \theta_j(p) \text{tr}((B^T E U)Y(p)(U^T A_j B)) \\ &\quad + 2 \sum_{i,j=1}^q \theta_i(p)\theta_j(p) \text{tr}(Y(p)(U^T A_j A_i U)Y(p)(U^T E E U)) \\ &\quad + 2 \sum_{i,j=1}^q \theta_i(p)\theta_j(p) \text{tr}(Y(p)(U^T A_i E U)Y(p)(U^T A_j E U)), \end{aligned}$$

where $\text{tr}(\cdot)$ denotes the trace of the corresponding matrix. Having precomputed $B^T E U$, $U^T A_j B$, $U^T A_j E U$, $U^T E E U$ and $U^T A_j A_i U$ for $i, j = 1, \dots, q$ then allows us to attain a complexity of $O(q^2 M^3)$ for computing $\|R(p)\|_F$.

The difference to the true solution $X(p)$ of the Lyapunov equation (5) can be estimated as

$$\|X(p) - \tilde{X}(p)\|_F \leq \frac{\|R(p)\|_F}{\lambda_{\min}(\mathcal{A}(p))} \leq \frac{\|R(p)\|_F}{\alpha_{\text{LB}}(p)} =: \Delta(p),$$

where $\alpha_{\text{LB}}(p) = 2\lambda_{\min}(E)\lambda_{\min}(A(p))$ is a lower bound for $\lambda_{\min}(\mathcal{A}(p))$. This bound immediately follows from the properties of eigenvalues of the Kronecker product and the Courant-Fisher theorem. Thus, it turns out to be sufficient to compute a lower bound for the smallest eigenvalue of the $N \times N$ matrix $A(p)$. As in Section II-C, we apply SCM to compute lower bounds for $\lambda_{\min}(A(p))$.

E. Summary

In the offline phase, the computational cost of this approach is

$$O(n_{\max}qMN^2 + n_{\max}q^2M^2N + n_{\max}n_{\Xi}(M^3 + q^2M^3))$$

for calculating approximate solutions and error estimates at the training set Ξ , where n_{Ξ} is the number of samples in Ξ . The computational cost for solving n_{\max} Lyapunov equations for sampled parameters is $O(n_{\max}k_LmN)$, where k_L is the number of iterations in the Lyapunov solver. It is important to note here that the contribution to the obtained computational cost of N and n_{Ξ} is independent and both can be chosen large if necessary.

In the online phase, the computational cost for calculating the approximate solution of the Lyapunov equation and the accompanying error estimator is $O(qM^2 + M^3)$ and $O(q^2M^3)$, respectively, and thus independent of N .

The computational cost of the SCM during the offline phase reduces significantly, as we are only required to compute the smallest eigenvalues of $N \times N$ matrices instead of $N^2 \times N^2$ matrices.

IV. NUMERICAL EXPERIMENTS

In this section, we report on preliminary numerical experiments to illustrate the properties of the low-rank RB method described in Section III.

We consider the parameter-dependent Lyapunov equation associated with an example from [3, Section 4]. The matrices $E, A(p) \in \mathbb{R}^{N \times N}$, with $N = 1580$, result from the finite element discretization of the stationary heat equation on a domain $[0, 4]^2$ containing four disjoint discs. The heat conductivity coefficient in each of these discs is governed by a parameter $p^{(j)}$; thus $A(p)$ depends on 4 parameters and can be written as $A(p) = A_0 + \sum_{j=1}^4 p^{(j)}A_j$. Each of the parameters $p^{(j)}$ is assumed to be in the interval $[0.1, 10]$, resulting in $\mathbb{P} = [0.1, 10]^4$. The training set Ξ is chosen as a random subset of \mathbb{P} consisting of 1000 parameter samples. We have set the truncation tolerance for low-rank approximation to $\text{tol}_{\text{compr}} = 10^{-8}$.

Figure 1 shows the convergence of the greedy algorithm used in the offline phase. It turns out that one iteration of SCM already provides reliable lower bounds for the smallest eigenvalue of $A(p)$. Rank of the exact Cholesky factor $L(p)$ is 25 for all sampled p_i , giving $m_i = 25$. After $n_{\max} = 15$ iterations, the dimension of the resulting subspace (after truncation) $\mathcal{U}_{n_{\max}}$ is $M = 181$. The error estimates and the exact error for the computed approximate solutions of the Lyapunov equation (5) at 100 randomly selected parameter values that do not belong to Ξ can be seen in Figure 2.

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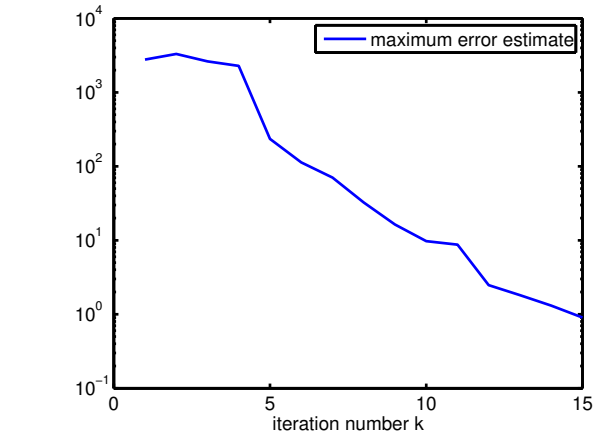


Fig. 1: Maximal error estimate $\max_{p \in \Xi} \Delta_k(p)$ at iterations $k = 1, 2, \dots, 15$ of the offline phase for the example from Section IV.

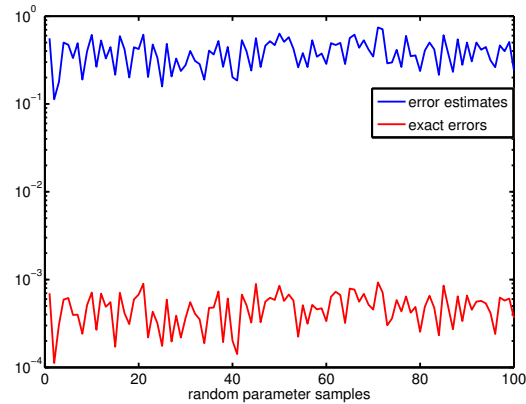


Fig. 2: Error estimates $\Delta(p)$ and exact error $\|\tilde{X}(p) - X(p)\|_F$ after 15 iterations of the offline phase vs. random parameter samples $p \in \mathbb{P}$ for the example from Section IV.

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