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System Theory Techniques for Function Theory on Bergman Spaces

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***Abstract*— The Sz.-Nagy-Foias model theory for $C_{\cdot 0}$ contraction operators combined with the Beurling-Lax theorem establishes a correspondence between any two of four kinds of objects: shift-invariant subspaces of a vector-valued Hardy space over the unit disk, operator-valued inner functions on the unit disk, conservative discrete-time input/state/output linear systems, and $C_{\cdot 0}$ Hilbert-space contraction operators. We discuss recent work exploring an extension of all these ideas where the Hardy space is replaced by a weighted Bergman space over the unit disk, the conservative linear system is replaced by a more general discrete-time time-varying linear system with system operators satisfying more elaborate metric constraints, and the resulting model theory is for the class of κ-hypercontraction operators on a Hilbert space (here κ is a positive integer). We also discuss multivariate extensions of these ideas to the case where the linear system evolves over a free semigroup rather than the lattice of nonnegative integers, in which case weighted Bergman-Fock spaces of noncommutative formal power series come into play**

I. INTRODUCTION

A. *The classical case*

Consider a discrete-time input/state/output linear system

$$\Sigma\colon \begin{cases} x(n+1) &= Ax(n)+Bu(n) \\ y(n) &= Cx(n)+Du(n) \end{cases} \tag{1}$$

where the block operator matrix

$$U=\begin{bmatrix} A & B \\ C & D\end{bmatrix}\colon \begin{bmatrix}\mathcal{H}\\ \mathcal{U}\end{bmatrix}\to\begin{bmatrix}\mathcal{H}\\ \mathcal{Y}\end{bmatrix} \tag{2}$$

is unitary (U is also called a unitary colligation), where the *state space* $\mathcal{H}$, the *input space* $\mathcal{U}$, and the *output space* $\mathcal{Y}$ are all assumed to be Hilbert spaces. We also assume that the operator A is *strongly stable* in the sense that

$$\lim_{n\to\infty}\|A^n x\|_{\mathcal{H}}=0 \quad \text{for each} \quad x\in\mathcal{H}. \tag{3}$$

Solving the recursion in (1) and applying the Z-transform $\{x(n)\}_{n\in\mathbb{Z}_+}\mapsto\sum_{n=0}^{\infty}x(n)z^n$ leads to the following relation between the Z-transform $\widehat{u}(z):=\sum_{n=0}^{\infty}u(n)z^n$ of the input signal and the Z-transform $\widehat{y}(z):=\sum_{n=0}^{\infty}y(n)z^n$ of the output signal:

$$\widehat{y}(z)=\mathcal{O}_{C,A}(z)x_0+T_{\Sigma}(z)\widehat{u}(z)$$

where we denote by $\mathcal{O}_{C,A}(z)$ the Z-transform of the observability operator

$$\mathcal{O}_{C,A}(z)\colon x\mapsto C(I-zA)^{-1}x \tag{4}$$

and where T_{Σ} is the *transfer function* of the system Σ (1):

$$T_{\Sigma}(z):=D+zC(I-zA)^{-1}B. \tag{5}$$

Under the assumptions imposed here (namely, that the colligation matrix U is unitary with the state operator A stable) it can be shown that the observability operator $\mathcal{O}_{C,A}$ maps the state space $\mathcal{H}$ isometrically into the Hardy spaces $H^2_{\mathcal{Y}}$ (the vector-valued Hardy space $H^2\otimes\mathcal{Y}$ of $\mathcal{Y}$-valued functions on the unit disk $\mathbb{D}$ of the form $f(z)=\sum_{n=0}^{\infty}f_nz^n$ with $f_n\in\mathcal{Y}$ and with $\sum_{n=0}^{\infty}\|f_n\|^2_{\mathcal{Y}}<\infty$) with image space $\mathrm{Ran}\mathcal{O}_{C,A}$ being invariant for the backward shift operator $S^*_{\mathcal{Y}}\colon f(z)\mapsto[f(z)-f(0)]/z$ on $H^2_{\mathcal{Y}}$, while the transfer function T_{Σ} is an *inner* operator-valued function on $\mathbb{D}$ (i.e., T_{Σ} assumes values in $\mathcal{L}(\mathcal{U},\mathcal{Y})$ (the space of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$) with boundary-value function on the unit circle $\mathbb{T}=\partial\mathbb{D}$ having isometric values a.e.). Furthermore, the multiplication operator $M_{T_{\Sigma}}\colon f(z)\mapsto T_{\Sigma}(z)f(z)$ is isometric from $H^2_{\mathcal{U}}$ into $H^2_{\mathcal{Y}}$ with image space $\mathcal{M}:=\mathrm{Ran}M_{T_{\Sigma}}$ equal to a subspace of $H^2_{\mathcal{Y}}$ which is invariant for the forward shift operator $S_{\mathcal{Y}}\colon f(z)\mapsto zf(z)$ on $H^2_{\mathcal{Y}}$, and in fact $\mathrm{Ran}\mathcal{O}_{C,A}=\mathcal{M}^{\perp}$. The so-called *model operator* associated with the shift-invariant subspace $\mathcal{M}$ is the compressed shift operator

$$T=P_{\mathcal{M}^{\perp}}M_z|_{\mathcal{M}^{\perp}}. \tag{6}$$

Conversely, given *any* shift-invariant subspace $\mathcal{M}$ of the Hardy space $H^2_{\mathcal{Y}}$, there is a conservative linear system (1) with stable state-dynamics operator A such that

$$\mathcal{M}^{\perp}=\mathrm{Ran}\mathcal{O}_{C,A} \quad \text{and} \quad \mathcal{M}=\mathrm{Ran}M_{T_{\Sigma}}.$$

More generally, any forward shift-invariant subspace of $H^2_{\mathcal{Y}}$ can be represented in the form $\Theta\cdot H^2_{\mathcal{U}}$ for an inner function Θ, a result known as the Beurling-Lax theorem (see [25]). Moreover *any* Hilbert space contraction operator T in the class $C_{\cdot 0}$ is unitarily equivalent to a compressed shift operator as in (6), or equivalently, to A^* where A is the state-dynamics operator for the conservative linear system Σ (1) associated with $\mathcal{M}$ as described above. This representation for a contraction operator T is also closely associated with the Sz.-Nagy dilation theorem, and is the beginning of the Sz.-Nagy-Foias model theory for a Hilbert space contraction operator (see [25]).

B. *The Fock-space case*

The classical results on the system (1) admit nice and meaningful extensions to a number of multivariable settings, both commutative and non-commutative. Here we recall one which is related to the Fock space $H^2_{\mathcal{Y}}(\mathcal{F}_d)$. Let $\mathcal{F}_d$ be the free semigroup generated by the set of d letters $\{1,\dots,d\}$. Elements of $\mathcal{F}_d$ are words of the form $i_N\cdots i_1$ where $i_\ell\in$

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$\{1,\dots,d\}$ with multiplication given by concatenation and the empty word $\emptyset$ as the unit element. For $v \in \mathcal{F}_d$, we let $|v|$ denote the number of letters in v. Let $z = (z_1,\dots,z_d)$ be a collection of d formal noncommuting variables and let

$$z^v = z_{i_N} z_{i_{N-1}} \cdots z_{i_1} \quad \text{if} \quad v = i_N i_{N-1} \cdots i_1.$$

The Fock space $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ is then defined as the set of formal noncommutative series $\sum_{v\in\mathcal{F}_d} f_v z^v$ with $f_v \in \mathcal{Y}$ and with $\sum_{v\in\mathcal{F}_d} \|f_v\|^2_{\mathcal{Y}} < \infty$. For $j = 1,\dots,d$, we let $S_{R,j}$ denote the shift operator

$$S_{R,j} \colon f(z) = \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto f(z)\cdot z_j = \sum_{v\in\mathcal{F}_d} f_v z^{v\cdot j} \qquad (7)$$

on $H^2_{\mathcal{Y}}(\mathcal{F}_d)$. The d-tuple $\mathbf{S}_R = (S_{R,1},\dots,S_{R,d})$ is called the forward (right) shift, while the tuple $\mathbf{S}^*_R = (S^*_{R,1},\dots,S^*_{R,d})$ is called the backward (right) shift.

We now recall the noncommutative Fornasini-Marchesini linear system (we refer to [11] where such systems were introduced, and to [12] and [13] for further elaboration)

$$\Sigma_{nc}: \quad \left\{ \begin{array}{ccc} x(1v) & = & A_1x(v) + B_1u(v) \\ \vdots & \vdots & \vdots \\ x(dv) & = & A_dx(v) + B_du(v) \\ y(v) & = & Cx(v) + Du(v) \end{array} \right. \qquad (8)$$

evolving along the free semigroup $\mathcal{F}_d$ and with the unitary system matrix

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}^d \\ \mathcal{Y} \end{bmatrix}. \qquad (9)$$

For the d-tuple $\mathbf{A} = (A_1,\dots,A_d)$ of state space operators, we use notation

$$\mathbf{A}^v = A_{i_N} A_{i_{N-1}} \cdots A_{i_1} \quad \text{if} \quad v = i_N i_{N-1} \cdots i_1 \in \mathcal{F}_d,$$

and we will assume that this tuple is *strongly stable* in the sense that

$$\lim_{N\to\infty} \sum_{v\in\mathcal{F}_d\colon |v|=N} \|\mathbf{A}^v x\|^2 \to 0 \quad \text{for all } x \in \mathcal{H}. \qquad (10)$$

Solving the system (8) and applying the noncommutative Z-transform $\{f_v\}_{v\in\mathcal{F}_d} \mapsto \widehat{f}(z) = \sum_{v\in\mathcal{F}_d} f_v z^v$ leads to the following relation between the Z-transforms of the input signal and the Z-transform of the output signal:

$$\widehat{y}(z) = \mathcal{O}_{C,\mathbf{A}}(z)x(\emptyset) + T_{\Sigma_{nc}}(z)\widehat{u}(z)$$

where

$$\mathcal{O}_{C,\mathbf{A}} : x \mapsto \sum_{v\in\mathcal{F}_d} (C\mathbf{A}^v x) z^v \qquad (11)$$

is the observability operator of $(C,\mathbf{A})$ and where

$$T_{\Sigma_{nc}}(z) = D + \sum_{v\in\mathcal{F}_d} \sum_{j=1}^{d} C\mathbf{A}^v B_j z^{vj} \qquad (12)$$

is the transfer formal power series of the system (8). Making use of notation from (9) and letting

$$Z(z) = \begin{bmatrix} z_1 & \cdots & z_d \end{bmatrix} \otimes I_{\mathcal{H}}, \qquad (13)$$

one observes that

$$(Z(z)A)^j = \left(\sum_{i=1}^{d} z_i A_i \right)^j = \sum_{v\in\mathcal{F}_d\colon |v|=j} \mathbf{A}^v z^v \qquad (14)$$

for all $j \geq 0$. Therefore,

$$(I - Z(z)A)^{-1} = \sum_{j=0}^{\infty} \sum_{v\in\mathcal{F}_d\colon |v|=j} \mathbf{A}^v z^v = \sum_{v\in\mathcal{F}_d} \mathbf{A}^v z^v,$$

which allows us to write formulas (11) and (12) in realization form very much similar to (4), (5):

$$\mathcal{O}_{C,\mathbf{A}} : x \mapsto C(I - Z(z)A)^{-1}x,$$
$$T_{\Sigma_{nc}}(z) = D + C(I - Z(z)A)^{-1}Z(z)B.$$

Under the assumptions that the colligation matrix (9) is unitary and the state space d-tuple $\mathbf{A}$ is strongly stable, the observability operator $\mathcal{O}_{C,\mathbf{A}}$ maps the state space $\mathcal{H}$ isometrically onto an $\mathbf{S}^*_R$-invariant (i.e., invariant for the backward shift operators $S^*_{R,j}$ for $j = 1,\dots,d$) subspace $\mathcal{N} \subset H^2_{\mathcal{Y}}(\mathcal{F}_d)$ and any $\mathbf{S}^*_R$-invariant subspace of the Fock space $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ arises in this way (see [7] for the proofs).

On the other hand, the transfer function $T_{\Sigma_{nc}}$ is *inner* in the sense that the multiplication operator

$$M_{T_{\Sigma_{nc}}} \colon f(z) \mapsto T_{\Sigma_{nc}}(z)f(z)$$

is isometric from $H^2_{\mathcal{U}}(\mathcal{F}_d)$ into $H^2_{\mathcal{Y}}(\mathcal{F}_d)$. Furthermore, the subspace $\mathcal{M} := \text{Ran} M_{T_{\Sigma_{nc}}} \subset H^2_{\mathcal{Y}}(\mathcal{F}_d)$ is invariant for the forward shift d-tuple $\mathbf{S}_R$, and any $\mathbf{S}_R$-invariant subspace of $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ arises in this way (we refer to [8] for the proofs). The model d-tuple associated with the $\mathbf{S}_R$-invariant subspace $\mathcal{M} \subset H^2_{\mathcal{Y}}(\mathcal{F}_d)$ is constructed as follows. Define the transpose operation on $\mathcal{F}_d$: $v^\top = i_1 \dots i_N$ if $v = i_n \dots i_1$. Define the involution τ on $H^2_{\mathcal{Y}}(\mathcal{F}_d)$:

$$\tau\colon \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto \sum_{v\in\mathcal{F}_d} f_{v^\top} z^v.$$

Finally, let $S_{L,j}$ denote the left shift operator

$$S^L_j \colon f(z) = \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto z_j \cdot f(z) = \sum_{v\in\mathcal{F}_d} f_v z^{j\cdot v} \qquad (15)$$

whose adjoint (as an operator on $H^2_{\mathcal{Y}}(\mathcal{F}_d)$) is given by

$$(S^L_j)^* \colon \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto \sum_{v\in\mathcal{F}_d} f_{j\cdot v} z^v.$$

Then, as was shown in [8], the model d-tuple $\mathbf{T} = (T_1,\dots,T_d)$ associated with the $\mathbf{S}_R$-invariant subspace $\mathcal{M} \subset H^2_{\mathcal{Y}}(\mathcal{F}_d)$ is given by

$$T_j = P_{\tau(\mathcal{M}^\perp)} S^*_{L,j}|_{\tau(\mathcal{M}^\perp)}.$$

II. Bergman-space system theory

We fix a positive integer κ and consider the space $\mathcal{A}_{\kappa,\mathcal{Y}}$ consisting of functions $f(z) = \sum_{n=0}^{\infty} f_n z^n$ on the unit disk $\mathbb{D}$ with $\mathcal{A}_{\kappa,\mathcal{Y}}$-norm

$$\|f\|^2_{\mathcal{A}_{\kappa,\mathcal{Y}}} := \sum_{n=0}^{\infty} \mu_{\kappa,n} \cdot \|f_n\|^2_{\mathcal{Y}}$$

finite, where $\mu_{\kappa,n}$ is the reciprocal binomial coefficient:

$$\mu_{\kappa,n} := \frac{1}{\binom{n+\kappa-1}{n}} = \frac{n!(\kappa-1)!}{(n+\kappa-1)!}. \tag{16}$$

Note that in case $\kappa = 1$ the space $\mathcal{A}_{1,\mathcal{Y}}$ reduces to the familiar Hardy space $H^2_{\mathcal{Y}}$ while in case $\kappa = 2$ we get the unweighted Bergman space $A_{2,\mathcal{Y}}$ consisting of functions $f(z) = \sum_{n=0}^{\infty} f_n z^n$ with norm given by the area integral $\|f\|^2_{\mathcal{A}_{2,\mathcal{Y}}} = \frac{1}{\pi}\int_{\mathbb{D}} \|f(z)\|^2 \mathrm{d}A(z)$. Starting in the 1970s with a spurt of progress in the 1990s, there has been a search for Bergman-space analogs of inner functions (particularly, of Blaschke products) and for Beurling-Lax-type representations for subspaces $\mathcal{M}$ of the Bergman space $\mathcal{A}_{\kappa,\mathcal{Y}}$ invariant under the Bergman forward shift operator $S_{\kappa,\mathcal{Y}}$ (or S_κ for short) given by $S_\kappa \colon f(z) \mapsto zf(z)$ for $f \in \mathcal{A}_{\kappa,\mathcal{Y}}$. Canonical divisors (the Bergman-space counterpart of Blaschke products) were constructed in [18], [19], [15] as solutions of certain extremal problems; Bergman-inner function were introduced in [16] in a more general L^p-setting. In the Hilbert-space setting of $\mathcal{A}_\kappa$, Bergman-inner functions appear as unit norm elements of a wandering subspace $\mathcal{E} = \mathcal{M} \ominus S_\kappa\mathcal{M}$ for some S_κ-invariant subspace $\mathcal{M} \in \mathcal{A}_\kappa$. In other words, G is $\mathcal{A}_\kappa$-inner if and only if $\|G\|_{\mathcal{A}_\kappa} = 1$ and $\langle S^n_\kappa G, G\rangle_{\mathcal{A}_\kappa} = 0$ for all $n \geq 1$. We refer to monographs [17], [22] leading the reader through many of these developments.

Even in the scalar-valued setting of the space A_2, operator-valued Bergman-inner functions appear as a natural object. It is known that S_2-invariant subspaces $\mathcal{M} \subset \mathcal{A}_2$ can have arbitrary index $\operatorname{ind}\mathcal{M} := \dim(\mathcal{M} \ominus z\mathcal{M})$ [3], [20], [21], [14]. Nevertheless, the seminal work of Aleman-Richter-Sundberg [1] with later extensions by Shimorin [29], [30] showed that in all cases we recover $\mathcal{M}$ as the closed linear span $\bigvee_{k\geq 0} S^k_\kappa(\mathcal{M} \ominus S_\kappa\mathcal{M})$. Moreover, it was shown that for any shift-invariant subspace $\mathcal{M} \in \mathcal{A}_\kappa$ with $\operatorname{ind}\mathcal{M} = 1$, there exists a Bergman-inner function θ such that $\mathcal{M} \ominus S_\kappa\mathcal{M} = \theta\mathbb{C}$. A similar representation holds for the general case if one replaces $\mathbb{C}$ by an appropriate coefficient Hilbert space $\mathcal{U}$ and takes θ to be operator-valued rather than scalar-valued.

It was only recently that Olofsson (see [26], [27], [28]) introduced the notion of operator-valued Bergman inner function as an object of independent interest for study and started the investigation of a more system-theoretic framework for Bergman-space function theory. We present here the system-theoretic approach to Bergman-space function theory from [5], [6].

For κ a fixed positive integer, we consider the time-varying input/state/output linear system

$$\Sigma_\kappa \colon \begin{cases} x(n+1) = \frac{n+\kappa}{n+1} Ax(n) + \binom{n+\kappa}{n+1} B_n u(n) \\ y(n) = Cx(n) + \binom{n+\kappa-1}{n} D_n u(n) \end{cases} \tag{17}$$

where the embedded colligation matrices $U_n := \left[\begin{smallmatrix} A & B_n \\ C & D_n \end{smallmatrix}\right]$ are assumed to have the form

$$\begin{bmatrix} A & B_n \\ C & D_n \end{bmatrix} \colon \begin{bmatrix} \mathcal{H} \\ \mathcal{U}_n \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix}$$

where $\mathcal{H}$ is a state space, $\mathcal{Y}$ is an output space, and $\mathcal{U}_n$ is the input space at time n ($n = 0, 1, 2, \dots$), all taken to be Hilbert spaces. If we impose an initial condition $x(0) = x_0$ and apply the Z-transform to the system equations, a calculation analogous to the standard calculation for the time-invariant case as described in the Introduction gives us

$$\widehat{y}(z) = \mathcal{O}_{\kappa,C,A}(z)x_0 + \sum_{n=0}^{\infty} T_{\Sigma_\kappa,n}(z)u(n)$$

where the *κ-observability operator* $\mathcal{O}_{\kappa,C,A}$ is given by

$$\mathcal{O}_{\kappa,C,A}(z) = \sum_{n=0}^{\infty} \left(\binom{\kappa+n-1}{n} CA^n \right) z^n = C(I - zA)^{-\kappa}$$

and where the n-th transfer function of the *κ-family of transfer functions* $\{T_{\Sigma_\kappa,n}\}_{n\in\mathbb{Z}_+}$ is given by

$$T_{\Sigma_\kappa,n}(z) = \binom{n+\kappa-1}{n} D_n + zCR_{\kappa,n+1}(zA)B_n,$$

where the *shifted resolvent function* $R_{\kappa,k}$ is given by

$$\begin{aligned} R_{\kappa,k}(z) &= \sum_{j=0}^{\infty} \binom{\kappa+j+k-1}{j+k} z^j \\ &= \sum_{\ell=1}^{n} \binom{\ell+k-2}{\ell-1} (1-z)^{\kappa-\ell+1}, \end{aligned} \tag{18}$$

where the second formula follows from the first with the aid of the Chu-Vandermonde identity for binomial coefficients

$$\binom{\kappa+j+k-1}{j+k} = \sum_{\ell=1}^{\kappa} \binom{\ell+k-2}{\ell-1} \cdot \binom{\kappa+j-\ell}{j}.$$

More symbolically, the shifted resolvent function $R_{\kappa,k}(z)$ is obtained from the unshifted resolvent function $R_\kappa(z)$ formally by applying the 1-backward-shift operator $S_1^* \colon \sum_{n=0}^{\infty} f_n z^n \mapsto \sum_{n=0}^{\infty} f_{n+1} z^n$ to $R_\kappa(z) := (1-z)^{-\kappa} = \sum_{n=0}^{\infty} \binom{\kappa+n-1}{n} z^n$ k times: $R_{\kappa,k}(z) = (S_1^*)^k[R_\kappa(z)]$.

We shall need a similar calculus applied to gramian operators as follows. The *κ-observability gramian* associated with the output pair (C, A) is defined to be

$$\begin{aligned} \mathcal{G}_{\kappa,C,A} &= (\mathcal{O}_{\kappa,C,A})^* \mathcal{O}_{\kappa,C,A} \\ &= \sum_{j=0}^{\infty} \binom{j+\kappa-1}{j} A^{*j} C^* C A^j. \end{aligned}$$

If we introduce the completely positive map B_A by $B_A[X] = A^*XA$, then we may write $\mathcal{G}_{\kappa,C,A} = R_\kappa(B_A)[C^*C]$. We shall have use for the shifted versions of this object, namely, the k-*shifted* κ-*Gramian*

$$\begin{aligned}\mathfrak{G}_{\kappa,k,C,A} &= R_{\kappa,k}(B_A)[C^*C] \\ &= \sum_{j=0}^{\infty} \binom{\kappa+j+k-1}{j+k} A^{*j}C^*CA^j.\end{aligned}$$

We now impose metric constraints on the colligation matrices $\left[\begin{smallmatrix} A & B_k \\ C & D_k \end{smallmatrix}\right]$ appearing in the system equations (17):

$$\begin{aligned}&\begin{bmatrix} A^* & C^* \\ B_k^* & D_k^* \end{bmatrix} \begin{bmatrix} \mathfrak{G}_{\kappa,k+1,C,A} & 0 \\ 0 & \binom{\kappa+k-1}{k} \cdot I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B_k \\ C & D_k \end{bmatrix} \\ &\quad = \begin{bmatrix} \mathfrak{G}_{\kappa,k,C,A} & 0 \\ 0 & I_{\mathcal{U}_k} \end{bmatrix}, \end{aligned} \tag{19}$$

as well as

$$\begin{aligned}&\begin{bmatrix} A & B_k \\ C & D_k \end{bmatrix} \begin{bmatrix} \mathfrak{G}_{\kappa,k,C,A}^{-1} & 0 \\ 0 & I_{\mathcal{U}_k} \end{bmatrix} \begin{bmatrix} A^* & C^* \\ B_k^* & D_k^* \end{bmatrix} \\ &\quad = \begin{bmatrix} \mathfrak{G}_{\kappa,k+1,C,A}^{-1} & 0 \\ 0 & \mu_{\kappa,j} I_{\mathcal{Y}} \end{bmatrix}. \end{aligned} \tag{20}$$

We can now state the following theorem, one of the central results from [5].

Theorem 1. *Assume that the collection of colligation matrices* $\left\{\left[\begin{smallmatrix} A & B_k \\ C & D_k \end{smallmatrix}\right]\right\}_{k=0,1,2,\dots}$ *satisfies the metric conditions* (19) *and* (20) *with* A *is strongly stable in the sense of* (3). *Then:*

(i) *The* κ*-observability operator* $\mathcal{O}_{\kappa,C,A}$ *maps the state space* $\mathcal{H}$ *isometrically onto an* $(S_\kappa)^*$*-invariant subspace* $\mathcal{N} \subset \mathcal{A}_{\kappa,\mathcal{Y}}$ *and any* $(S_\kappa)^*$*-invariant subspace of* $\mathcal{A}_{\kappa,\mathcal{Y}}$ *arises in this way.*

(ii) *The input-output map of the time-varying system* (17) *maps the time-varying Hardy space* $\bigoplus_{n=0}^{\infty} z^n \mathcal{U}_n$ *isometrically onto an* S_κ*-invariant subspace* $\mathcal{M} \subset \mathcal{A}_{\kappa,\mathcal{Y}}$*, i.e.,*

$$\mathcal{M} = \left\{ \sum_{n=0}^{\infty} T_{\Sigma_\kappa,n}(z) z^n u_n \colon u_n \in \mathcal{U}_n \right. \left. \text{with } \sum_{n=0}^{\infty} \|u_n\|^2_{\mathcal{U}_n} < \infty \right\}$$

and furthermore

$$\left\| \sum_{n=0}^{\infty} T_{\Sigma_\kappa,n}(z) z^n u_n \right\|^2_{\mathcal{A}_{\kappa,\mathcal{Y}}} = \sum_{n=0}^{\infty} \|u(n)\|^2_{\mathcal{U}_n}.$$

Moreover, every S_κ*-invariant subspace* $\mathcal{M} \subset \mathcal{A}_{\kappa,\mathcal{Y}}$ *arises in this way.*

(iii) *If* $\mathcal{N}$ *is as in part (i) and* $\mathcal{M}$ *is as in part (ii), then* $\mathcal{N} = \mathcal{M}^\perp$ *and* $\mathcal{A}_{\kappa,\mathcal{Y}} = \mathcal{N} \oplus \mathcal{M}$.

As a corollary we get a new type of Beurling-Lax theorem for the weighted Bergman spaces. Let us say that a collection of functions $\{\Theta_k\}_{k\in\mathbb{Z}_+}$ is an *inner function family* if the operator $\begin{bmatrix} M_{\Theta_0} & M_{\Theta_1} & \cdots & M_{\Theta_n} & \cdots \end{bmatrix}$ is an isometry from the time-varying Hardy space $\bigoplus_{n=1}^{\infty} z^n \mathcal{U}_n$ into the Bergman space $\mathcal{A}_{\kappa,\mathcal{Y}}$.

Corollary 2. *Let* $\mathcal{M}$ *be an* S_κ*-invariant subspace of* $\mathcal{A}_{\kappa,\mathcal{Y}}$*. Then there is an inner function family* $\{\Theta_{\kappa,k}\}_{k\in\mathbb{Z}_+}$ *so that* $\mathcal{M} = M_\Theta \cdot H^2(\{\mathcal{U}_k\}_{k\in\mathbb{Z}_+})$.

It is also possible to develop an analogue of the Sz.-Nagy-Foias operator model theory for the class of κ-hypercontractions on a Hilbert space. We say that the operator T on a Hilbert space $\mathcal{H}$ is $*$-κ-*hypercontractive* if $(I - B_{T^*})^k[I] \geq 0$ for $1 \leq k \leq \kappa$ (or equivalently, as it turns out, just for $k = 1$ and $k = \kappa$). For the following result we also impose the hypothesis that T is in the Sz.-Nagy-Foias class $C_{\cdot 0}$ (i.e., T^* is stable). Then it has been known for some time (see e.g. [2], [4]) that it follows that T is unitarily equivalent to $P_{\mathcal{N}} S_\kappa|_{\mathcal{N}}$ where $\mathcal{N}$ is an $(S_\kappa)^*$-invariant subspace of $\mathcal{A}_{\kappa,\mathcal{Y}}$ with the coefficient space $\mathcal{Y}$ having dimension equal to $\operatorname{rank}(I - B_{T^*})^\kappa[I]$. The search for a Sz.-Nagy-Foias-type characteristic function for a $*$-κ-hypercontraction was begun in [26], [28] and continued in [6]. Here we simplify the formula from [6] as follows. Given a $*$-κ-hypercontraction operator T, denote by D_{κ,T^*} the operator

$$D_{\kappa,T^*} = ((I - B_{T^*})^\kappa[I])^{1/2}.$$

We let $\mathcal{D}_{\kappa,T^*}$ denote the closure of the range space $\mathcal{D}_{\kappa,T^*} = \overline{\operatorname{Ran}} D_{\kappa,T^*}$ and view D_{κ,T^*} as an operator from $\mathcal{H}$ into $\mathcal{D}_{\kappa,T^*}$. Then it can be shown that, for each $k = 0, 1, 2, \dots$, the operator

$$\begin{bmatrix} (\mathfrak{G}_{\kappa,k+1,D_{\kappa,T^*},T^*})^{-1} & 0 \\ 0 & \mu_{n,k} I_{\mathcal{D}_{\kappa,T^*}} \end{bmatrix} - \begin{bmatrix} T^* \\ D_{\kappa,T^*} \end{bmatrix} \mathfrak{G}^{-1}_{\kappa,k,D_{\kappa,T^*},T^*} \begin{bmatrix} T & (D_{\kappa,T^*})^* \end{bmatrix}$$

is positive semidefinite and hence has a positive semidefinite square root which we denote by $\mathbf{D}_{\kappa,T}$. We denote the closure of the range of $\mathbf{D}_{\kappa,T}$ by $\boldsymbol{\mathcal{D}}_{\kappa,T}$ and view $\mathbf{D}_{\kappa,T}$ as an operator from $\boldsymbol{\mathcal{D}}_{\kappa,k,T}$ into $\left[\begin{smallmatrix} \mathcal{H} \\ \mathcal{D}_{\kappa,T^*} \end{smallmatrix}\right]$. We then decompose $\mathbf{D}_{\kappa,k,T}$ as

$$\mathbf{D}_{\kappa,k,T} = \left[\begin{smallmatrix} B_k \\ D_k \end{smallmatrix}\right] : \boldsymbol{\mathcal{D}}_{\kappa,k,T} \to \left[\begin{smallmatrix} \mathcal{H} \\ \mathcal{D}_{\kappa,T^*} \end{smallmatrix}\right]$$

and we set

$$\mathbf{U}_k = \left[\begin{smallmatrix} T^* & B_k \\ D_{\kappa,T^*} & D_k \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{H} \\ \boldsymbol{\mathcal{D}}_{\kappa,k,T} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{H} \\ \mathcal{D}_{\kappa,T^*} \end{smallmatrix}\right].$$

We then define

$$\Theta_{T,k}(z) = \binom{k+\kappa-1}{k} D_k + z D_{\kappa,T^*} R_{n,k+1}(zT^*) B_k$$

and declare the collection $\{\Theta_{T,k}(z)\}_{k\in\mathbb{Z}_+}$ to be the *characteristic function family for the* $*$-κ-*hypercontraction* T. We can now state the following result, one of the main results from [6].

Theorem 3.

1) *The characteristic family for a* $C_{\cdot 0}$ $*$-κ-*hypercontraction is an inner function family.*
2) *The characteristic function-family as defined above is a complete unitary invariant for the class of* $C_{\cdot 0}$ $*$-κ-*hypercontractions in the following sense: if* T *and* T' *are two* $C_{\cdot 0}$ $*$-κ-*hypercontractions, then* T *and* T' *are unitarily equivalent if and only if the characteristic*

function families $\{\Theta_{T,k}\}_{k\in\mathbb{Z}_+}$ and $\{\Theta_{T',k}\}_{k\in\mathbb{Z}_+}$ coincide in the following sense: there are unitary operators $i_k\colon \boldsymbol{\mathcal{D}}_{\kappa,k,T} \to \boldsymbol{\mathcal{D}}_{\kappa,k,T'}$ and $j\colon \mathcal{D}_{\kappa,T^*} \to \mathcal{D}_{\kappa,T'^*}$ so that

$$\Theta_{T',k}(z)i_k = j\Theta_{T,k}(z) \quad \text{for} \quad k = 0, 1, 2, \ldots.$$

3) *There is a canonical functional model for the operator T constructed from its characteristic function family as follows. Set $\mathcal{M} = \bigoplus_{n=0}^{\infty}\Theta_{\kappa,n}(z)z^n\boldsymbol{\mathcal{D}}_{\kappa,n,T} \subset \mathcal{A}_{\kappa,\mathcal{D}_{\kappa,T^*}}$ (where the orthogonal direct sum is taken in the metric of the weighted Bergman space $\mathcal{A}_{\kappa,\mathcal{D}_{\kappa,T^*}}$). Set $\mathbf{T}$ equal to $\mathbf{T} = P_{\mathcal{M}^\perp}S_\kappa|_{\mathcal{M}^\perp}$. Then the original $C_{\cdot 0}$ $*$-κ-hypercontraction operator T is unitarily equivalent to the model $C_{\cdot 0}$ $*$-κ-hypercontraction operator $\mathbf{T}$.*

III. NONCOMMUTATIVE BERGMAN-FOCK SPACE SETTING

We now discuss a simultaneous generalization of all the settings discussed so far, namely, a weighted Bergman-space generalization of the Drury-Arveson noncommutative setting discussed in the Introduction which can also be seen as the free noncommutative extension of the single-variable weighted Bergman-space setting discussed above. These results are new; however full details of the proofs are rather lengthy and technical, and so will be worked out in a planned journal article to come; here we only survey the needed background so as to be able to give complete statements (without proofs) of the results and highlight the parallels with the results for the special cases discussed above.

Given a positive integer κ, the free semigroup $\mathcal{F}_d$ and the coefficient Hilbert space $\mathcal{Y}$, we define the Bergman-Fock space $\mathcal{A}_{\kappa,\mathcal{Y}}(\mathcal{F}_d)$ as the set of $\mathcal{Y}$-valued formal power series $f(z) = \sum_{v\in\mathcal{F}_d} f_v z^v$ in d noncommuting variables with $\mathcal{A}_{\kappa,\mathcal{Y}}(\mathcal{F}_d)$-norm

$$\|f\|^2_{\mathcal{A}_{\kappa,\mathcal{Y}}(\mathcal{F}_d)} = \sum_{v\in\mathcal{F}_d}\mu_{n,|v|}\cdot\|f_v\|^2_{\mathcal{Y}} \tag{21}$$

finite, where according to (16), $\mu_{\kappa,|v|} = \frac{|v|!(\kappa-1)!}{(\kappa+|v|-1)!}$. In case $\kappa = 1$, the space $\mathcal{A}_{\kappa,\mathcal{Y}}(\mathcal{F}_d)$ reduces to the Hardy-Fock space recalled in the introduction. The shift d-tuples $\mathbf{S}_{\kappa,R} = (S_{\kappa,R,1},\ldots,S_{\kappa,R,d})$ and $\mathbf{S}_{\kappa,L} = (S_{\kappa,L,1},\ldots,S_{\kappa,L,d})$ on $\mathcal{A}_{\kappa,\mathcal{Y}}(\mathcal{F}_d)$ are defined as in (7) and (15) and their adjoints, as simple inner-product calculations show, are given by

$$S^*_{\kappa,R,j}\colon \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto \sum_{v\in\mathcal{F}_d}\frac{\mu_{\kappa,|v|+1}}{\mu_{\kappa,|v|}}f_{vj}z^v,$$
$$S^*_{\kappa,L,j}\colon \sum_{v\in\mathcal{F}_d} f_v z^v \mapsto \sum_{v\in\mathcal{F}_d}\frac{\mu_{\kappa,|v|+1}}{\mu_{\kappa,|v|}}f_{jv}z^v.$$

We now consider the multidimensional linear system with evolution along the free semigroup $\mathcal{F}_d$:

$$\left\{\begin{array}{ccc} x(1v) & = & \frac{n+|v|}{|v|+1}A_1x(v) + \binom{n+|v|}{|v|+1}B_{1,v}u(v) \\ \vdots & \vdots & \vdots \\ x(dv) & = & \frac{n+|v|}{|v|+1}A_dx(v) + \binom{n+|v|}{|v|+1}B_{d,v}u(v) \\ y(v) & = & Cx(v) + \binom{n+|v|-1}{|v|}D_vu(v) \end{array}\right. \tag{22}$$

with the d-tuple of state space operators $\mathbf{A} = (A_1,\ldots,A_d)$ and the state-output operator $C \in \mathcal{L}(\mathcal{H},\mathcal{Y})$. Here we have a family of colligation matrices and the family of input spaces indexed by $v \in \mathcal{F}_d$:

$$\mathbf{U}_v = \begin{bmatrix} A & \widehat{B}_v \\ C & D_v\end{bmatrix}\colon \begin{bmatrix}\mathcal{H} \\ \mathcal{U}_v\end{bmatrix} \to \begin{bmatrix}\mathcal{H}^d \\ \mathcal{Y}\end{bmatrix},$$

where

$$A = \begin{bmatrix}A_1 \\ \vdots \\ A_d\end{bmatrix},\ \widehat{B}_v = \begin{bmatrix}B_{1v} \\ \vdots \\ B_{dv}\end{bmatrix}.$$

Making use of notation (13) and equality (14) we observe that

$$\begin{aligned}(I - Z(z)A)^{-\kappa} &= \sum_{j=0}^{\infty}\mu^{-1}_{\kappa,j}\cdot\sum_{v\in\mathcal{F}_d\colon |v|=j}\mathbf{A}^v z^v \\ &= \sum_{v\in\mathcal{F}_d}\mu^{-1}_{\kappa,|v|}\mathbf{A}^v z^v.\end{aligned}$$

Now we may define $R_{\kappa,k}(Z(z)A)$ using identity (18) or equivalently, via formal power series

$$R_{\kappa,k}(Z(z)A) = \sum_{v\in\mathcal{F}_d}\mu^{-1}_{\kappa,|v|+k}\mathbf{A}^v z^v.$$

Upon running the system (22) with a fixed initial condition $x(\emptyset) = x \in \mathcal{H}$ we get

$$y(v) = \mu^{-1}_{\kappa,|v|}(C\mathbf{A}^v x + \sum_{v''jv'=v} C\mathbf{A}^{v''}B_{jv'}u(v') + D_v u(v)).$$

Applying the noncommutative Z-transform to the output signal $\{y(v)\}_{v\in\mathcal{F}_d}$ then gives

$$\widehat{y}(z) = \mathcal{O}_{\kappa,C,\mathbf{A}}x + \sum_{v\in\mathcal{F}_d}\Theta_{\kappa,v}(z)z^v u(v),$$

where

$$\mathcal{O}_{\kappa,C,\mathbf{A}}\colon\ x \mapsto \sum_{v\in\mathcal{F}_d}(\mu^{-1}_{\kappa,|v|}C\mathbf{A}^v x)z^v = (I - Z(z)A)^{-\kappa}x \tag{23}$$

is the noncommutative κ-observability operator and where

$$\Theta_{\kappa,v}(z) = \mu^{-1}_{\kappa,|v|}D_v + CR_{\kappa,|v|+1}(Z(z)A)Z(z)\widehat{B}_v \tag{24}$$

is the family of transfer functions indexed by $v \in \mathcal{F}_d$.

The system (22) is called *κ-output-stable* (and in this case we will say that the pair $(C,\mathbf{A})$ is κ-output-stable) if $\mathcal{O}_{\kappa,C,\mathbf{A}}$ maps $\mathcal{H}$ into $\mathcal{A}_{\kappa,\mathcal{Y}}(\mathcal{F}_d)$ and is bounded. If $(C,\mathbf{A})$ is κ-output-stable, one may introduce the *κ-observability gramian* $\mathcal{G}_{\kappa,C,\mathbf{A}} := \mathcal{O}^*_{\kappa,C,\mathbf{A}}\mathcal{O}_{\kappa,C,\mathbf{A}}$ and its representation in terms of strongly converging series

$$\mathcal{G}_{\kappa,C,\mathbf{A}} = \sum_{v\in\mathcal{F}_d}\mu_{\kappa,|v|}\mathbf{A}^{*v^\top}C^*C\mathbf{A}^v \tag{25}$$

follows from definition (23) of $\mathcal{O}_{C,\mathbf{A}}$ and formula (21) for the norm in $\mathcal{A}_{n,\mathcal{Y}}(\mathcal{F}_d)$. We now introduce the completely positive map B_A by

$$B_{\mathbf{A}}[X] = \sum_{i=1}^{d}A_i^*XA_i$$

and record several results on κ-output stability.

Theorem 4. *Let $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ and $\mathbf{A} = (A_1, \dots, A_d) \in \mathcal{L}(\mathcal{H})^d$. Then:*

1) *The pair $(C, \mathbf{A})$ is κ-output-stable if and only if there exists an operator $H \in \mathcal{L}(\mathcal{H})$ such that*
$$H \geq B_{\mathbf{A}}[H] \geq 0, \quad (I - B_{\mathbf{A}})^{\kappa}[H] \geq C^*C. \tag{26}$$
2) *If $(C, \mathbf{A})$ is κ-output-stable, then the observability gramian $H = \mathcal{G}_{\kappa,C,\mathbf{A}}$ satisfies*
$$H \geq B_{\mathbf{A}}[H] \geq 0, \quad (I - B_{\mathbf{A}})^{\kappa}[H] = C^*C \tag{27}$$
and is the minimal positive semidefinite solution of the system (26).
3) *There is a unique solution H of the system* (27) *with $H = \mathcal{G}_{\kappa,C,\mathbf{A}}$ if $\mathbf{A}$ is strongly stable. Moreover, in case $\mathbf{A}$ is contractive in the sense that*
$$A_1^*A_1 + \cdots + A_d^*A_d \leq I_{\mathcal{H}}, \tag{28}$$
the solution of the equation in (27) *is unique if and only if $\mathbf{A}$ is strongly stable.*

The first statement in the latter theorem is supplemented by the following result.

Lemma 5. *Let $H, A_1, \dots, A_d \in \mathcal{L}(\mathcal{H})$ be such that*
$$H \geq \sum_{i=1}^{d} A_i^* H A_i \geq 0 \quad \text{and} \quad (I - B_{\mathbf{A}})^{\kappa}[H] \geq 0$$
for some integer $\kappa \geq 3$. Then $(I - B_{\mathbf{A}})^m[H] \geq 0$ for all $m = 1, \dots, \kappa - 1$.

Let us recall that the tuple $\mathbf{A} = (A_1, \dots, A_d)$ is *κ-hypercontractive* if $(I - B_{\mathbf{A}})^m[I_{\mathcal{H}}] \geq 0$ for $0 \leq m \leq \kappa$. Specializing Lemma 5 to the case $H = I_{\mathcal{H}}$ we recover a result from [2] (see also [23] as well as [24] for a multivariable version): *if a κ-contractive tuple $\mathbf{A} = (A_1, \dots, A_d)$ is contractive in the sense of* (28)*, then it is κ-hypercontractive*.

Let us say that a pair $(C, \mathbf{A})$ is *κ-isometric* if relations (27) hold with $H = I_{\mathcal{H}}$. The next result is the noncommutative extension of the first part of Theorem 1.

Theorem 6. *If the pair $(C, \mathbf{A})$ is κ-isometric and the d-tuple $\mathbf{A}$ is strongly stable, then the κ-observability operator $\mathcal{O}_{\kappa,C,\mathbf{A}}$ maps the state space $\mathcal{H}$ isometrically onto an $(S_{\kappa,R})^*$-invariant subspace $\mathcal{N} \subset \mathcal{A}_{\kappa,\mathcal{Y}}(\mathcal{F}_d)$ and any $(S_{\kappa,R})^*$-invariant subspace of $\mathcal{A}_{\kappa,\mathcal{Y}}$ arises in this way.*

Making use of shifted resolvent functions (18) we introduce the noncommutative *k-shifted κ-gramians*
$$\begin{aligned}\mathfrak{G}_{\kappa,k,C,\mathbf{A}} &= R_{\kappa,k}(B_{\mathbf{A}})[C^*C] \\ &= \sum_{v \in \mathcal{F}_d}^{\infty} \mu_{\kappa,k+|v|}^{-1} \mathbf{A}^{*v^{\top}} C^*C \mathbf{A}^v.\end{aligned} \tag{29}$$
Since
$$\mu_{\kappa,k+|v|}^{-1} = \mu_{\kappa,|v|}^{-1} \cdot \frac{\mu_{\kappa,|v|}}{\mu_{\kappa,k+|v|}} \leq \mu_{\kappa,|v|}^{-1} \kappa^k,$$
we conclude from (29) and (25) that $\mathfrak{G}_{\kappa,k,C,\mathbf{A}} \leq \kappa^k \mathcal{G}_{\kappa,C,\mathbf{A}}$ so that the series in (29) indeed converges whenever the pair $(C, \mathbf{A})$ is κ-output stable. Making use if the k-shifted κ-gramians (29) we impose metric constraints on the colligation matrices $\left[\begin{smallmatrix} A & \widehat{B}_v \\ C & D_v \end{smallmatrix}\right]$ appearing in the system equations (22):
$$\begin{aligned}&\begin{bmatrix} A^* & C^* \\ \widehat{B}_v^* & D_v^* \end{bmatrix} \begin{bmatrix} \mathfrak{G}_{\kappa,|v|+1,C,\mathbf{A}} \otimes I_d & 0 \\ 0 & \mu_{\kappa,|v|}^{-1} \cdot I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & \widehat{B}_v \\ C & D_v \end{bmatrix} \\ &\qquad = \begin{bmatrix} \mathfrak{G}_{\kappa,|v|,C,\mathbf{A}} & 0 \\ 0 & I_{\mathcal{U}_v} \end{bmatrix},\end{aligned} \tag{30}$$
$$\begin{aligned}&\begin{bmatrix} A & \widehat{B}_v \\ C & D_v \end{bmatrix} \begin{bmatrix} \mathfrak{G}_{\kappa,|v|,C,\mathbf{A}}^{-1} & 0 \\ 0 & I_{\mathcal{U}_v} \end{bmatrix} \begin{bmatrix} A^* & C^* \\ \widehat{B}_v^* & D_v^* \end{bmatrix} \\ &\qquad = \begin{bmatrix} \mathfrak{G}_{\kappa,|v|+1,C,\mathbf{A}}^{-1} \otimes I_d & 0 \\ 0 & \mu_{n,|v|} I_{\mathcal{Y}} \end{bmatrix}.\end{aligned} \tag{31}$$
We can now state the following noncommutative analog of the second part of Theorem 1.

Theorem 7. *Assume that the collection of colligation matrices $\left\{\left[\begin{smallmatrix} A & \widehat{B}_v \\ C & D_v \end{smallmatrix}\right]\right\}_{v \in \mathcal{F}_d}$ satisfies the metric conditions* (30) *and* (31) *with A strongly stable. Then the input-output map of the time-varying system* (22) *maps the time-varying Hardy space $\bigoplus_{v \in \mathcal{F}_d} z^v \mathcal{U}_v$ isometrically onto an $\mathbf{S}_{\kappa,R}$-invariant subspace $\mathcal{M} \subset \mathcal{A}_{\kappa,\mathcal{Y}}(\mathcal{F}_d)$, i.e.,*
$$\mathcal{M} = \left\{ \sum_{n=0}^{\infty} \Theta_{\kappa,v}(z) z^v u_v \colon u_v \in \mathcal{U}_v \text{ s.t.} \sum_{v \in \mathcal{F}_d} \|u_v\|_{\mathcal{U}_v}^2 < \infty \right\}$$
where $\Theta_{\kappa,v}$ are given in (24)*, and moreover*
$$\left\| \sum_{v \in \mathcal{F}_d} \Theta_{\kappa,v}(z) z^v u_v \right\|_{\mathcal{A}_{n,\mathcal{Y}}(\mathcal{F}_d)}^2 = \sum_{v \in \mathcal{F}_d} \|u_v\|_{\mathcal{U}_v}^2.$$
Moreover, every $\mathbf{S}_{\kappa,R}$-invariant subspace $\mathcal{M} \subset \mathcal{A}_{n,\mathcal{Y}}(\mathcal{F}_d)$ arises in this way.

IV. CONCLUSIONS

The results presented above recover only a subset of the results known for the classical case. There are many questions which remain to be resolved. We mention just a couple:

1) It remains to remove the $C_{\cdot 0}$ hypothesis in Theorems 1 and 3. There already exist results in this direction for the dilation-theoretic component of the theory (see [2], [4]).
2) The notion of inner function family defined above provides a good analogue of isometric multiplier, including a good time-varying realization theory. It is unclear what should be the proper notion of *contractive multiplier*. In particular what class of functions results if the relations (19) and (20) are relaxed to contractive inequalities?
3) As is to be expected there are more issues to be resolved for the multivariable setting. It is also of interest to explore commutative multivariable Bergman spaces (e.g., over the unit polydisk or over the unit ball in $\mathbb{C}^d$) using this system-theoretic approach, similar to what was done in our earlier work [7], [9], [10] for the

special case of the Drury-Arveson space. We plan to pursue this direction once the freely noncommutative setting is better understood.

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