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Infinite-dimensional Lur'e systems: the circle criterion, input-to-state stability and the converging-input-converging-state property*

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***Abstract*— In this talk we consider forced infinite-dimensional Lur'e systems. We investigate input-to-state stability (ISS) and convergent-input-convergent-state (CICS) properties. In particular, it is shown that, under the conditions of the classical circle criterion, the Lur'e system is ISS.**

I. INTRODUCTION

In this talk we consider the Lur'e system shown in Figure 1, where Σ is an infinite-dimensional well-posed linear system [10], [12], [13] and f is a static nonlinearity. We will investigate input-to-state stability (ISS) and convergent-input-convergent-state (CICS) properties of the system shown in Figure 1.

For real or complex Hilbert spaces U and Y, let $\mathcal{B}(U,Y)$ denote the space of all linear bounded operators mapping U to Y. For $Z \in \mathcal{B}(U,Y)$ and $r > 0$, define

$$\mathbb{B}(Z,r) := \{T \in \mathcal{B}(U,Y) : \|T - Z\| < r\},$$

the open ball in $\mathcal{B}(U.Y)$, with centre Z and radius r. We write $\mathcal{B}(U)$ for $\mathcal{B}(U,U)$. For $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \mathrm{Re}\, s > \alpha\}$. The space of all holomorphic and bounded functions $\mathbb{C}_\alpha \to \mathcal{B}(U,Y)$ is denoted by $H^\infty_\alpha(\mathcal{B}(U,Y))$. We write $H^\infty(\mathcal{B}(U,Y))$ for $H^\infty_0(\mathcal{B}(U,Y))$. If U is a real Hilbert space, then $U_\mathbb{C}$ denotes its complexification.

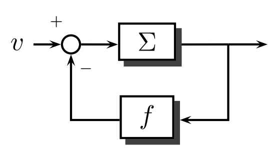

Fig. 1. Lur'e system: feedback interconnection of linear system Σ and nonlinearity f

Throughout, we shall be considering a well-posed system Σ with state-space X, input space U and output space Y, generating operators (A, B, C), input-output operator G and transfer function $\mathbf{G}$. Here X, U and Y are separable *complex* Hilbert spaces, A is the generator of a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ on X and $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, Y)$, respectively, are admissible control and observations for $\mathbf{T}$. The spaces X_1 and X_{-1}, respectively, are the usual interpolation and extrapolation spaces associated with A and X.

The transfer function $\mathbf{G}$ has the property that $\mathbf{G} \in H^\infty_\alpha(\mathcal{B}(U,Y))$ for every $\alpha > \beta(\mathbf{T})$, where $\beta(\mathbf{T})$ denotes the exponential growth constant of $\mathbf{T}$. In the following, let $s_0 \in \mathbb{C}_{\beta(\mathbf{T})}$, let C_Λ denote the so-called Λ-*extension* of C, let $f : Y \to U$ be continuous and let $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$.

Formally, the feedback system shown in Figure 1 is given by

$$\left.\begin{aligned} &\dot{x} = Ax + Bu, \quad x(0) = x^0 \in X,\\ &y = C_\Lambda\big(x - (s_0 I - A)^{-1}Bu\big) + \mathbf{G}(s_0)u,\\ &u = v - f(y), \end{aligned}\right\} \qquad (1)$$

A *solution* of (1) on $[0,\omega)$, where $0 < \omega \leq \infty$, is a pair $(x,y) \in C([0,\omega), X) \times L^2_{\mathrm{loc}}([0,\omega), Y)$ such that $f \circ y \in L^2_{\mathrm{loc}}([0,\omega), U)$,

$$x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B\big(v(\tau) - f(y(\tau))\big)\mathrm{d}\tau$$

for all $t \in [0,\omega)$, and, on $[0,\omega)$,

$$y = C_\Lambda\big(x - (s_0 I - A)^{-1}B(v - f \circ y)\big) + \mathbf{G}(s_0)(v - f \circ y).$$

If $\omega = \infty$, the solution is called *global*. The set of all global solutions of (1) is denoted by $\mathcal{S}(x^0, v)$.

It can be shown (by invoking Zorn's lemma) that, for every solution of (1), there exists a *maximally defined* solution which cannot be extended any further. System (1) is said to have the *blow-up property* if, for every maximally defined solution (x,y) with finite interval of existence $[0,\omega)$,

$$\max\left\{\limsup_{t\uparrow\omega}\|x(t)\|,\ \lim_{t\uparrow\omega}\int_0^t \|y(\tau)\|^2\mathrm{d}\tau\right\} = \infty.$$

We are mainly concerned with stability and convergence properties of (1): existence and/or uniqueness of solutions is not the main concern here. The question of existence requires addressing on a less general basis, taking into account relevant features of the particular system or subclass of systems under consideration.

II. THE CIRCLE CRITERION AND ISS

We say that $K \in \mathcal{B}(Y,U)$ is an *admissible feedback operator* for Σ if there exists $\alpha \in \mathbb{R}$ such that

$$\mathbf{G}(I + K\mathbf{G})^{-1} \in H^\infty_\alpha(\mathcal{B}(U,Y)).$$

An operator $K \in \mathcal{B}(Y,U)$ is called a *stabilizing feedback operator* for Σ if

$$\mathbf{G}(I + K\mathbf{G})^{-1} \in H^\infty(\mathcal{B}(U,Y)).$$

The set of all stabilizing feedback operators is denoted by $\mathbb{S}(\mathbf{G})$.

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Theorem II.1. *Let $K \in \mathcal{B}(Y,U)$ and $r > 0$. Assume that Σ is optimizable and estimatable and $\mathbb{B}(K,r) \subset \mathbb{S}(\mathbf{G})$. If*

$$\sup_{z\neq 0} \frac{\|f(z) - Kz\|}{\|z\|} < r,$$

then there exist positive γ and Γ such that, for all $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, U)$, all $(x,y) \in \mathcal{S}(x^0, v)$, and all $t \geq 0$

$$\|x(t)\| \leq \Gamma\left(\exp(-\gamma t)\|x^0\| + \|v\|_{L^\infty(0,t)}\right). \qquad (2)$$

In particular, system (1) *is ISS.*

It can be shown that, under the assumptions in Theorem II.1, maximally defined solutions are global, provided that (1) has the blow-up property. Theorem II.1 guarantees a strong form of ISS: the contribution of the initial state x^0 to the RHS of (2) is decaying exponentially fast, whilst the effect of $\|v\|_{L^\infty(0,t)}$ on $\|x(t)\|$ is linearly bounded. For an overview of ISS theory (in a finite-dimensional context), see [9]. Furthermore, we note that Theorem II.1 is somewhat reminiscent of the finite-dimensional complex Aizerman conjecture, see [1], [2]. Theorem II.1 can be proved by combining small-gain, loop-shifting exponential weighting techniques with results from the theory of well-posed linear systems. The following lemma plays a key role in the proof.

Lemma II.2. *For $K \in \mathcal{B}(Y,U)$ and $r > 0$, $\mathbb{B}(K,r) \subset \mathbb{S}(\mathbf{G})$ if, and only if, $\|\mathbf{G}(I + K\mathbf{G})^{-1}\|_{H^\infty} \leq 1/r$.*

We provide a (very) brief sketch of the proof of Theorem II.1.

Key ideas of the proof of Theorem II.1. Application of the linear feedback $u = v - Ky$ to the well-posed linear system Σ results in a well-posed linear closed-loop system Σ_K with transfer function

$$\mathbf{G}_K := \mathbf{G}(I + K\mathbf{G})^{-1}.$$

The corresponding input-output operator is denoted by G_K. Let $(x,y) \in \mathcal{S}(x^0, v)$. Loop shifting shows that

$$y = y^{\mathrm{free}}_{x^0} + G_K(v - f_K \circ y), \qquad (3)$$

where $y^{\mathrm{free}}_{x^0}$ is the output of the free motion of Σ_K with initial state x^0 and f_K is given by

$$f_K(z) := f(z) - Kz \quad \forall\, z \in Y.$$

For real ρ, set $\mathrm{e}_\rho(t) := e^{\rho t}$. Let $\varepsilon > 0$. It follows from (3) that

$$y\,\mathrm{e}_\varepsilon = y^{\mathrm{free}}_{x^0}\mathrm{e}_\varepsilon + G^\varepsilon_K\big(v\,\mathrm{e}_\varepsilon - f^\varepsilon_K(\cdot\,, y\,\mathrm{e}_\varepsilon)\big), \qquad (4)$$

where $G^\varepsilon_K : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ is defined by

$$G^\varepsilon_K w = \mathrm{e}_\varepsilon G_K(\mathrm{e}_{-\varepsilon} w),$$

and $f^\varepsilon_K : \mathbb{R}_+ \times Y \to U$ is given by

$$f^\varepsilon_K(t,z) = \mathrm{e}_\varepsilon(t) f_K(\mathrm{e}_{-\varepsilon}(t)z).$$

Note that the transfer function $\mathbf{G}^\varepsilon_K$ of G^ε_K satisfies

$$\mathbf{G}^\varepsilon_K(s) = \mathbf{G}_K(s - \varepsilon).$$

By the hypothesis on $\mathbf{G}$ and Lemma II.2,

$$\|G_K\| = \|\mathbf{G}_K\|_{H^\infty} \leq \frac{1}{r}, \qquad (5)$$

where $\|G_K\|$ denotes the L^2-induced operator norm of G_K. Moreover, exploiting the hypothesis on f, there exists $r_0 \in (0,r)$ such that

$$\|f^\varepsilon_K(t,z)\| \leq r_0\|z\| \quad \forall\,(t,z) \in \mathbb{R}_+ \times Y. \qquad (6)$$

Let $r_1 \in (r_0, r)$. Then, appealing to (5), there exists $\varepsilon > 0$ such that

$$\|G^\varepsilon_K\| = \|\mathbf{G}^\varepsilon_K\|_{H^\infty} \leq \frac{1}{r_1}. \qquad (7)$$

Invoking (4), (6) and (7), we conclude that there exists $M > 0$ such that, for all $t \in \mathbb{R}_+$ and all $x^0 \in X$,

$$\|y\,\mathrm{e}_\varepsilon\|_{L^2(0,t)} \leq M\big(\|x^0\| + \|v\,\mathrm{e}_\varepsilon\|_{L^2(0,t)}\big).$$

Consequently, since,

$$\|v\,\mathrm{e}_\varepsilon\|_{L^2(0,t)} \leq (2\varepsilon)^{-1/2}\mathrm{e}_\varepsilon(t)\|v\|_{L^\infty(0,t)} \quad \forall\, t \in \mathbb{R}_+,$$

we obtain that there exists a positive constant N such that, for all $t \in \mathbb{R}_+$, $x^0 \in X$ and $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, U)$,

$$\|y\,\mathrm{e}_\varepsilon\|_{L^2(0,t)} \leq N\big(\|x^0\| + \mathrm{e}_\varepsilon(t)\|v\|_{L^\infty(0,t)}\big).$$

This estimate, in combination with suitable results from the theory of well-posed linear systems (see [10], [12], [13]), can then be used to derive the ISS estimate (2). □

Remark. Whilst, in Lemma II.2, necessity does in general not hold in the *real* case (wherein X, U and Y are real Hilbert spaces), Theorem II.1 and Lemma II.2 remain true in the real case, provided that the condition $\mathbb{B}(K,r) \subset \mathbb{S}(\mathbf{G})$ is replaced by $\mathbb{B}_{\mathbb{C}}(K,r) \subset \mathbb{S}_{\mathbb{C}}(\mathbf{G})$, where $\mathbb{B}_{\mathbb{C}}(K,r) := \{T \in \mathcal{B}(U_{\mathbb{C}}, Y_{\mathbb{C}}) : \|T - Z\| < r\}$ and $\mathbb{S}_{\mathbb{C}}(\mathbf{G}) := \{K \in \mathcal{B}(Y_{\mathbb{C}}, U_{\mathbb{C}}) : \mathbf{G}(I + K\mathbf{G})^{-1} \in H^\infty(\mathcal{B}(U_{\mathbb{C}}, Y_{\mathbb{C}}))\}$.

The simple example below provides an illustration of Theorem II.1.

Example. Consider the heat equation on the spatial interval $[0,1]$ with point control and point observation:

$$\begin{aligned}\theta_t(\xi,t) &= \theta_{\xi\xi}(\xi,t) + \delta_{\xi_1}u(t),\\ \theta_\xi(0,t) &= \theta_\xi(1,t) = 0;\\ y(t) &= \theta(\xi_2,t),\end{aligned}$$

where $\xi_1, \xi_2 \in (0,1)$. This PDE system can be written in form of a well-posed linear system and its transfer function is given by

$$\mathbf{G}(s) = \frac{1}{s} + 2\sum_{n=1}^{\infty} \frac{\cos(n\pi\xi_1)\cos(n\pi\xi_2)}{s + n^2\pi^2}.$$

It can be shown that

$$\|\mathbf{G}(1 + (3/2)\mathbf{G})^{-1}\|_{H^\infty} = 2/3,$$

and hence, invoking Lemma II.2 together with the subsequent Remark, we obtain that $\mathbb{B}_{\mathbb{C}}(3/2, 3/2) \subset \mathbb{S}_{\mathbb{C}}(\mathbf{G})$. If f is a locally Lipschitz nonlinearity, it is not difficult to prove that the corresponding Lur'e system has the blow-up property and, consequently, by Theorem II.1, the Lur'e system is ISS if there exists $\varepsilon > 0$ such that

$$|f(z) - (3/2)z|/|z| \leq 3/2 - \varepsilon \quad \forall\, z \in \mathbb{R},\ z \neq 0,$$

or, equivalently,

$$\varepsilon z^2 \leq f(z)z \leq (3-\varepsilon)z^2 \quad \forall\, z \in \mathbb{R}.$$

The following corollary of Theorem II.1 shows that, under the assumptions of the well-known circle criterion (see, for example, [11]), the infinite-dimensional Lur'e system (1) is ISS. The corollary can be proved by a suitable modification of the proof of the analogous finite-dimensional discrete-time result in [8].

Corollary II.3. *Let $K_1, K_2 \in \mathcal{B}(Y,U)$ and let Σ be optimizable and estimatable. Assume that K_1 is an admissible feedback operator, and $(I + K_2\mathbf{G})(I + K_1\mathbf{G})^{-1}$ is positive real. Moreover, assume that there exists $\delta > 0$ such that the sector condition*

$$\mathrm{Re}\,\langle f(z) - K_1 z, f(z) - K_2 z\rangle \leq -\delta\|z\|^2 \quad \forall\, z \in Y$$

holds. Then there exist positive γ and Γ such that, for all $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, U)$, all $(x,y) \in \mathcal{S}(x^0, v)$, and all $t \geq 0$,

$$\|x(t)\| \leq \Gamma\left(\exp(-\gamma t)\|x^0\| + \|v\|_{L^\infty(0,t)}\right).$$

In particular, system (1) is ISS.

Finite-dimensional results of a similar nature can be found in [4], [5]. We emphasize that Corollary II.3 represents a considerable improvement as compared to related results in [3]: whilst Corollary II.3 provides a clear-cut generalization of the multivariable circle criterion to an infinite-dimensional ISS setting, [3] contains only partial results of circle criterion type.

III. The circle criterion and CICS

The feedback system (1) is said to have the 0-*converging-input-converging-state* (0-CICS) property if, for every $x^0 \in X$ and every $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, U)$ with $\lim_{t\to\infty} v(t) = 0$, we have

$$\lim_{t\to\infty} x(t) = 0$$

for every $x \in C(\mathbb{R}_+, X)$ such that $(x,y) \in \mathcal{S}(x^0, v)$ for some function $y \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$. It is well known that, in a finite-dimensional context, ISS implies the 0-CICS property. It is not difficult to show that the same is true for infinite-dimensional Lur'e systems of the form (1). This observation combined with Theorem II.1 leads to the following corollary.

Corollary III.1. *Assume that Σ is optimizable and estimatable and let $K \in S(\mathbf{G})$. If*

$$\sup_{z\neq 0} \frac{\|f(z) - Kz\|}{\|z\|} < \frac{1}{\|\mathbf{G}(I + K\mathbf{G})^{-1}\|_{H^\infty}},$$

then (1) has the 0-CICS property.

We say that (1) has the *converging-input-converging-state* (CICS) property if, for every $v^\infty \in U$, there exists $x^\infty \in X$ such that, for every $x^0 \in X$ and every $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, U)$ with $\lim_{t\to\infty} v(t) = v^\infty$, we have

$$\lim_{t\to\infty} x(t) = x^\infty,$$

for every $x \in C(\mathbb{R}_+, X)$ such that $(x,y) \in \mathcal{S}(x^0, v)$ for some function $y \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$.

If $K \in \mathcal{B}(Y,U)$ is an admissible feedback operator for Σ, then, in the following, (A_K, B_K, C_K) denote the generating operators of the well-posed system Σ_K obtained by applying linear feedback of the form $u = v - Ky$ to the underlying well-posed system Σ. The transfer function $\mathbf{G}_K$ of Σ_K is given by $\mathbf{G}_K = \mathbf{G}(I + K\mathbf{G})^{-1}$.

The next result, a corollary of Theorem II.1 and Corollary III.1, provides a sufficient condition for (1) to have the CICS-property.

Corollary III.2. *Assume that Σ is optimizable and estimatable and let $K \in S(\mathbf{G})$. If*

$$\sup_{z_1\neq z_2} \frac{\|f(z_1) - f(z_2) - K(z_1 - z_2)\|}{\|z_1 - z_2\|} < \frac{1}{\|\mathbf{G}_K\|_{H^\infty}},$$

then (1) has the CICS property. Moreover:

(a) *for every $x^0 \in X$ and every $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, (1) has a unique global solution $x(\cdot\,; x^0, v)$;*

(b) *the map $h := I + \mathbf{G}_K(0)(f - K)$ is a bijection and its inverse is globally Lipschitz;*

(c) *given $v^\infty \in U$, we have that for every $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, U)$ with $\lim_{t\to\infty} v(t) = v^\infty$ and every $x^0 \in X$,*

$$x(t; x^0, v) \to -A_K^{-1} B_K\left(v^\infty + Ky^\infty - f(y^\infty)\right),$$

as $t \to \infty$, where $y^\infty = h^{-1}(\mathbf{G}_K(0)v^\infty)$.

Related results in a finite-dimensional SISO context can be found in [6], [7]. Finally, we state a circle criterion version of Corollary III.2.

For a map $g : Y \to U$ define $\Delta_g(z_1, z_2) := g(z_1) - g(z_2)$ for all $z_1, z_2 \in Y$.

Corollary III.3. *Let $K_1, K_2 \in \mathcal{B}(Y,U)$ and let Σ be optimizable and estimatable. Assume that K_1 is an admissible feedback operator and that $(I + K_2\mathbf{G})(I + K_1\mathbf{G})^{-1}$ is positive real. Moreover, assume that there exists $\delta > 0$ such that, for all $z_1, z_2 \in Y$,*

$$\begin{aligned}\mathrm{Re}\,\big\langle &\Delta_f(z_1,z_2) - \Delta_{K_1}(z_1,z_2),\\ &\Delta_f(z_1,z_2) - \Delta_{K_2}(z_1,z_2)\big\rangle \leq -\delta\|z_1 - z_2\|^2\end{aligned}$$

Then (1) has the CICS property.

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