

Construction of Polyhedral Lyapunov Functions for Discrete-Time Systems

Roman Geiselhart¹, Mircea Lazar², Fabian R. Wirth³

Abstract—In this paper we make use of the alternative converse Lyapunov theorem presented in [1] for specific classes of systems. We show that the developed converse Lyapunov theorem can be used to establish non-conservatism of a particular type of Lyapunov functions. Most notably, a proof that the existence of conewise linear Lyapunov functions are non-conservative for globally exponentially stable (GES) conewise linear systems is given and, as a by-product, tractable construction of polyhedral Lyapunov functions for linear systems is attained.

keyword: Conewise linear systems, Polyhedral Lyapunov functions

I. INTRODUCTION

In [1] we present an alternative converse Lyapunov function for time-invariant discrete-time systems. The approach essentially requires an assumption such that a global Lyapunov function can be constructed as a finite sum of trajectory pieces. In particular, for the class of discrete-time system with globally exponentially stable (GES) origin, the required assumption is always satisfied. Hence, the construction of the global Lyapunov function can be performed. Note that the Lyapunov function construction hinges of finding a suitable natural number a priori. In this paper, we discuss several possibilities to find such a suitable number in a systematic way for certain classes of systems. Of course, finding such a suitable number may be undecidable or computationally intractable, as it is well known that stability analysis and hence, also construction of Lyapunov functions, is an NP-hard problem in general [2].

For certain classes of dynamical systems Lyapunov functions are guaranteed to exist in classes of functions that are computationally easy to describe; e.g. quadratic Lyapunov functions [3] for linear difference equations and polyhedral Lyapunov functions [4], [5], [6] for linear difference inclusions. But for most nonlinear systems we only know that C^∞ Lyapunov functions exist, which in contrast is a class of functions that is not computationally easily described. However, for the case of polyhedral Lyapunov functions, developing tractable constructive methods is still an open problem.

¹R. Geiselhart is with the Institute for Mathematics, University of Würzburg, Germany, roman.geiselhart@mathematik.uni-wuerzburg.de

²M. Lazar is with the Department of Electrical Engineering, Eindhoven University of Technology, The Netherlands, m.lazar@tue.nl

³F. Wirth is with IBM Research Ireland, Damastown Industrial Estate, Mulhuddart, Dublin 15, Ireland, fabwirth@ie.ibm.com

The main contribution of this paper is to establish non-conservatism of specific types of Lyapunov functions *via* the developed converse Lyapunov theorem given in [1]. Most notably, it is established that the existence of conewise linear Lyapunov functions are non-conservative for GES conewise linear systems. The latter result further yields, as a by-product, a new method to construct polyhedral Lyapunov functions for linear systems, which is tractable even in state spaces of high dimension.

The remainder of this paper is structured as follows. The necessary preliminaries are given in Section II. In Section III we give the system description, and summarize the construction of the global Lyapunov function obtained in [1]. Section IV contains the main results of this paper. In particular, non-conservatism of the existence of a conewise linear Lyapunov function for GES conewise linear systems is derived in Section IV-A, while non-conservatism of the existence of a polyhedral Lyapunov function for GES linear systems is derived in Section IV-B. In both sections, we give an explicit construction of the Lyapunov functions, and show its applicability in several examples.

Note that [1] contains a preliminary version of the results, which is improved and corrected in this paper.

II. PRELIMINARIES

By \mathbb{N} we denote the natural numbers and we assume $0 \in \mathbb{N}$. Let \mathbb{R} denote the field of real numbers, \mathbb{R}_+ the set of non-negative real numbers and \mathbb{R}^n the vector space of real column vectors of length n . For matrices $A_1, \dots, A_N \in \mathbb{R}^{n \times m}$ we use the abbreviation $(A_1; \dots; A_N) := (A_1^\top \dots A_N^\top)^\top$.

By $\|\cdot\|$ we denote any arbitrary norm on \mathbb{R}^n . In particular, we use the 1-norm $\|x\|_1 = \sum_{i=1}^n |[x]_i|$ and the infinity norm $\|x\|_\infty = \max_{i \in \{1, \dots, n\}} |[x]_i|$. We call a function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a *function of class \mathcal{K}* (denoted by $\alpha \in \mathcal{K}$), if it is strictly increasing, continuous, and satisfies $\alpha(0) = 0$. A function $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called *\mathcal{K} -bounded*, if there exists an $\omega \in \mathcal{K}$ such that

$$\|G(x)\| \leq \omega(\|x\|), \quad \forall x \in \mathbb{R}^n.$$

III. PROBLEM STATEMENT

We consider discrete-time systems of the form

$$x(k+1) = G(x(k)), \quad k \in \mathbb{N}, \quad (1)$$

where $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to satisfy the following standing assumption.

Assumption 1: The function G in (1) is \mathcal{K} -bounded.

Note that Assumption 1 does not require continuity of the map $G(\cdot)$ (except at $x = 0$, which is a necessary

condition for Lyapunov stability). We refer to [1] for a more comprehensive discussion on Assumption 1.

By $x(k, \xi) \in \mathbb{R}^n$ we denote the solution of system (1) at time instance $k \in \mathbb{N}$ with initial condition $x(0) = \xi \in \mathbb{R}^n$.

Definition 2: The origin of system (1) is called *globally exponentially stable* (GES) if there exist $C \geq 1$ and $\mu \in [0, 1)$ such that for all $\xi \in \mathbb{R}^n$ and all $k \in \mathbb{N}$

$$\|x(k, \xi)\| \leq C\mu^k \|\xi\|. \quad (2)$$

Note that for a GES system the \mathcal{K} -bound ω on G in (1) can always be chosen to be linear. This follows directly, since $\|G(\xi)\| = \|x(1, \xi)\| \leq C\mu^1 \|\xi\|$ for all $\xi \in \mathbb{R}^n$.

To show that the origin of system (1) is GES the concept of a Lyapunov function is often used.

Lemma 3 ([1, Corollary 8]): Let the \mathcal{K} -bound on G be $\omega(s) = ws$ for all $s \geq 0$ and $w > 0$. Then the existence of a function $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$, real numbers $0 < a_1 \leq a_2$, $\lambda > 0$ and $c \in [0, 1)$ satisfying

$$\begin{aligned} a_1 \|\xi\|^\lambda &\leq W(\xi) \leq a_2 \|\xi\|^\lambda, \\ W(G(\xi)) &\leq cW(\xi). \end{aligned} \quad (3)$$

implies that the origin of system (1) is GES.

The function W in Lemma 3 is called a *global Lyapunov function* for system (1). Note that Lemma 3 is a sufficient condition to conclude GES of the origin of system (1). From the alternative converse Lyapunov theorems derived in [1] we obtain the following necessary condition.

Lemma 4: If the origin of system (1) is GES, then there exist numbers $M \in \mathbb{N}$ and $\tilde{c} \in [0, 1)$ such that for all $\xi \in \mathbb{R}^n$ we have

$$\|x(M, \xi)\| \leq \tilde{c} \|\xi\|. \quad (4)$$

In particular, we can choose

$$M := \min\{k \in \mathbb{N} : C\mu^k < 1\}, \quad \tilde{c} := C\mu^M \in [0, 1), \quad (5)$$

where $C \geq 1$ and $\mu \in [0, 1)$ come from (2).

Theorem 5 ([1, Corollary 21 and 22]): If the origin of system (1) is GES then with $M \in \mathbb{N}$, and $\tilde{c} \in [0, 1)$ given by (5) the function $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined by

$$W(\xi) := \sum_{j=0}^{M-1} \|x(j, \xi)\| \quad (6)$$

resp.

$$W(\xi) := \max_{j \in \{0, \dots, M-1\}} \tilde{c}^{j/M} \|x(M-1-j, \xi)\| \quad (7)$$

satisfies (3). Hence, W is a global Lyapunov function for system (1).

In the remainder of this paper we will exploit Theorem 5 for certain classes of discrete-time systems. Note that the global Lyapunov functions in (6) and (7) are continuous if the underlying dynamics G in (1) is continuous.

IV. MAIN RESULTS

This section exploits the developed converse theorems to obtain relevant implications for several classes of dynamical systems. Namely, we consider the cases of conewise linear dynamical systems (Section IV-A) and linear dynamical systems (Section IV-B) in more detail.

A. Conewise linear dynamical systems

In this section we focus on conewise linear systems, see, e.g., the survey [8] or [9], for which the results in this paper turn out to be quite useful. In [7] it was shown that conewise linear Lyapunov functions are sufficient for establishing GES for conewise linear systems and that such functions can be computed by linear programming. See also [10], which focuses on the discrete-time setting. The open question that remains to be answered is whether the existence of conewise linear Lyapunov functions are also necessary for GES conewise linear systems. In what follows we make use of the Theorem 5 to answer this question affirmatively, within the discrete-time setting.

To this end, firstly, a formal characterization of conewise linear dynamics is given. We need the following notion. A nonempty set $C \subset \mathbb{R}^n$ is convex if for any two points $\xi_1, \xi_2 \in C$ and $\lambda \in (0, 1)$ we have $\lambda\xi_1 + (1-\lambda)\xi_2 \in C$. The *dimension* $\dim(C)$ of a convex set C is equal to the dimension of the smallest affine subspace $U \subset \mathbb{R}^n$ containing C . We define the *relative interior* of a convex set C (denoted by $\text{relint}(C)$) as its interior relative to the smallest affine subspace $U \subset \mathbb{R}^n$ containing C . This is equivalent to the definition $\text{relint}(C) := \{\xi \in C : \forall \tilde{\xi} \in C \exists \lambda > 1 \text{ such that } \lambda\xi + (1-\lambda)\tilde{\xi} \in C\}$. The *convex hull* $\text{co}\{S\}$ of a set $S \subset \mathbb{R}^n$ is the smallest convex set containing S , and $\text{cl}\{S\}$ denotes the *closure* of S . A *ray* induced by a vector $v \in \mathbb{R}^n$ is the set $\langle v \rangle := \{cv : c \in \mathbb{R}_+\}$.

In the following definition we define convex polyhedral cones. As this is the only type of cones considered in this paper, we will for the sake of simplicity only speak of cones.

Definition 6: A nonempty set $\mathcal{C} \subset \mathbb{R}^n$ is a (*convex polyhedral*) *cone* if \mathcal{C} is the convex hull of a finite number of rays, i.e., $\mathcal{C} := \text{co}\{\langle v_1 \rangle, \dots, \langle v_r \rangle\}$. Thus $\dim(\mathcal{C})$ is the number of linearly independent vectors v_1, \dots, v_r . If \mathcal{S}, \mathcal{C} are cones with $\mathcal{S} \subset \mathcal{C}$, then \mathcal{S} is called a *subcone* of \mathcal{C} . If additionally $\dim(\mathcal{S}) < \dim(\mathcal{C})$, then \mathcal{S} is a *lower dimensional subcone* of \mathcal{C} .

A finite set of cones $\{\mathcal{C}_i \subset \mathbb{R}^n\}_{i \in \{1, \dots, l\}}$ defines an *l-conic partition* of \mathbb{R}^n , if the following holds for $i, j \in \{1, \dots, l\}$

- (i) $\bigcup_i \text{relint}(\mathcal{C}_i) = \mathbb{R}^n$;
- (ii) for $i \neq j$ we have $\text{relint}(\mathcal{C}_i) \cap \text{relint}(\mathcal{C}_j) = \emptyset$;

Note that by definition of an *l-conic partition* two cones can only intersect on the boundaries, and for any point $\xi \in \mathbb{R}^n$ there exists a unique cone \mathcal{C}_i such that ξ is contained in the relative interior of \mathcal{C}_i . In particular, the cone $\{0\}$ must be contained in the *l-conic partition*.

Next, consider the class of *conewise linear dynamics*, i.e.

$$G(x) := A_i x \quad \text{if } x \in \text{relint}(\mathcal{C}_i); \quad i \in \{1, \dots, N\}, \quad (8)$$

where $N \in \mathbb{N}$, $A_i \in \mathbb{R}^{n \times n}$, and the finite set of cones $\{\mathcal{C}_i\}_{i \in \{1, \dots, N\}}$ defines an *N-conic partition* of \mathbb{R}^n . By the above considerations, the map G in (8) is well-defined. Observe that G in (8) satisfies Assumption 1 with $\omega(s) := \max_{i \in \{1, \dots, N\}} \|A_i\|s$.

Remark 7: (i) Note that G is continuous if and only if for any $\xi \in \mathcal{C}_i \cap \mathcal{C}_j$ it holds $A_i \xi = A_j \xi$, or, equivalently,

$\xi \in \ker(A_i - A_j)$. Nevertheless, continuity of G is not required in our next result.

(ii) The map G in (8) can also be defined on the closed cone \mathcal{C}_i . However, to guarantee well-posedness of G , a rule is needed to decide which map is applied in points that lie on the boundary of several cones. In the case that G is continuous this is not an issue, see (i).

For any solution $x(k, \xi)$ of the conewise linear system

$$x(k+1) = A_i x(k) \quad \text{if} \quad x(k) \in \text{relint}(\mathcal{C}_i) \quad (9)$$

with $x(0) = \xi \in \mathbb{R}^n$, we associate the k -tuple (j_1, \dots, j_k) , $j_i \in \{1, \dots, N\}$, if $x(l, \xi) \in \text{relint}(\mathcal{C}_{j_{l+1}})$ for $l \in \{0, \dots, k-1\}$. Note that since G is well-defined, the associated k -tuple (j_1, \dots, j_k) is uniquely determined. Unifying these sets we have

$$\begin{aligned} \mathcal{I}_k := & \{(j_1, \dots, j_k) \in \{1, \dots, N\}^k : \\ & A_{j_k} \{ \dots \{ A_{j_2} \{ A_{j_1} \mathcal{C}_{j_1} \cap \mathcal{C}_{j_2} \} \cap \mathcal{C}_{j_3} \} \cap \dots \cap \mathcal{C}_{j_k} \} \neq \emptyset \}. \end{aligned} \quad (10)$$

Note that the set \mathcal{I}_k can be computed by basic operations (image under A_{j_i} and intersection) involving cones and/or via reachability graphs. These operations can be performed efficiently for the case of convex polyhedral cones.

Furthermore, we define the set

$$A_k := \left\{ \left[\prod_{i=0}^{k-1} A_{j_{k-i}} \right] : (j_1, \dots, j_k) \in \mathcal{I}_k \right\},$$

where $\left[\prod_{i=0}^{k-1} A_{j_{k-i}} \right] := A_{j_k} A_{j_{k-1}} \dots A_{j_1}$, and for $k = 0$ this product is defined as the identity matrix I .

The following theorem states that GES of the origin of a conewise linear system (9) is equivalent to the existence of a conewise linear Lyapunov function. Notice also that conewise linear maps are positively homogeneous maps of degree one and, as such, global asymptotic stability (GAS) as defined in [1, Definition 2] is equivalent to GES by Corollary V.3 of [11]. As such, without any loss of generality we can state the following result in terms of GES.

Theorem 8: The origin of the conewise linear system (9) is GES, if and only if it admits a global conewise linear Lyapunov function.

Proof: In [10, Thorem 4.6] it is shown that the existence of a conewise linear Lyapunov function implies GAS of the origin of system (9), and hence even GES. So in this proof we consider the converse statement.

Let the origin of system (9) be GES. Then by Theorem 5 the function $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined by $W(\xi) := \sum_{k=0}^{M-1} \|x(k, \xi)\|_1$ is a global Lyapunov function for system (9), with $M \in \mathbb{N}$ given in (5).

For any $\xi \in \mathbb{R}^n$ let $\iota := (j_1, \dots, j_M) \in \mathcal{I}_M$ be associated to the solution $x(\cdot, \xi)$, i.e., $x(l, \xi) \in \text{relint}(\mathcal{C}_{j_{l+1}})$ for $l \in \{0, \dots, M-1\}$. Note that the number $\#\mathcal{I}_M$ of non-identical M -tuples in \mathcal{I}_M is at most N^M . Then for any

$\iota = (j_1, \dots, j_M) \in \mathcal{I}_M$ we define the set

$$\begin{aligned} \mathcal{D}_\iota := & \text{cl} \{ \xi \in \text{relint}(\mathcal{C}_{j_1}) : x(k, \xi) \in \text{relint}(\mathcal{C}_{j_{k+1}}) \\ & \forall k \in \{1, \dots, M-1\} \} \\ = & \text{cl} \{ \xi \in \text{relint}(\mathcal{C}_{j_1}) : \left[\prod_{j=0}^{k-1} A_{j_{k-j}} \right] \xi \in \text{relint}(\mathcal{C}_{j_{k+1}}) \\ & \forall k \in \{1, \dots, M-1\} \} \end{aligned}$$

Note that the sets \mathcal{D}_ι are cones in the sense of Definition 6, and form a conic partition of \mathbb{R}^n . We omit the proof of this observation here and refer to our upcoming journal version.

For any cone \mathcal{D}_ι there exists a matrix $P_\iota \in \mathbb{R}^{p \times n}$ with $p \geq n$ such that

$$\sum_{k=0}^{M-1} \left\| \left[\prod_{i=0}^{k-1} A_{j_{k-i}} \right] \xi \right\|_1 = \|P_\iota \xi\|_1$$

for all $\xi \in \text{relint}(\mathcal{D}_\iota)$. This matrix $P_\iota \in \mathbb{R}^{p \times n}$ can be chosen as

$$P_\iota = \left(\left[\prod_{i=0}^{M-2} A_{j_{M-1-i}} \right]; \dots; A_{j_2} A_{j_1}; A_{j_1}; I \right), \quad (11)$$

where I denotes the identity matrix. In particular, $p = Mn$. Then the global Lyapunov function takes the explicit form

$$\begin{aligned} W(\xi) &= \sum_{k=0}^{M-1} \|x(k, \xi)\|_1 = \sum_{k=0}^{M-1} \left\| \left[\prod_{i=0}^{k-1} A_{j_{k-i}} \right] \xi \right\|_1 \\ &= \|P_\iota \xi\|_1 \quad \text{if } \xi \in \text{relint}(\mathcal{D}_\iota). \end{aligned}$$

Since weighted 1-norms are conewise linear functions, see, for example, [6], and $\{\mathcal{D}_\iota\}_{\iota \in \mathcal{I}_M}$ defines a conic partition of \mathbb{R}^n we obtain that W is a conewise linear function, which concludes the proof. ■

The proof of Theorem 8 essentially relies on refining the conic partition $\{\mathcal{C}_i\}_{i \in \{1, \dots, l\}}$ to obtain the conic partition $\{\mathcal{D}_\iota\}_{\iota \in \mathcal{I}_M}$. In the next example we will indicate how this refinement is obtained.

Example 9: Consider the vector space \mathbb{R}^3 , and denote the i th unit vector in \mathbb{R}^3 by e_i , $i \in \{1, 2, 3\}$. Assume the partition of \mathbb{R}^3 into the 8 orthants given by

$$\begin{aligned} \mathcal{C}_1 &= \text{co}\{e_1, e_2, e_3\}, & \mathcal{C}_2 &= \text{co}\{e_1, e_2, -e_3\}, \\ \mathcal{C}_3 &= \text{co}\{e_1, -e_2, e_3\}, & \mathcal{C}_4 &= \text{co}\{e_1, -e_2, -e_3\}, \\ \mathcal{C}_5 &= \text{co}\{-e_1, e_2, e_3\}, & \mathcal{C}_6 &= \text{co}\{-e_1, e_2, -e_3\}, \\ \mathcal{C}_7 &= \text{co}\{-e_1, -e_2, e_3\}, & \mathcal{C}_8 &= \text{co}\{-e_1, -e_2, -e_3\}. \end{aligned}$$

Note that this partition is not a conic partition in the sense of Definition 6, since $\bigcup_{i \in \{1, \dots, 8\}} \text{relint} \mathcal{C}_i \neq \mathbb{R}^n$. To achieve this, we have to add the 2-dimensional cones

$$\begin{aligned} \mathcal{C}_9 &= \text{co}\{e_1, e_2\}, & \mathcal{C}_{10} &= \text{co}\{e_1, -e_2\} \\ \mathcal{C}_{11} &= \text{co}\{e_1, e_3\}, & \mathcal{C}_{12} &= \text{co}\{e_1, -e_3\}, \\ \mathcal{C}_{13} &= \text{co}\{-e_1, e_3\}, & \mathcal{C}_{14} &= \text{co}\{-e_1, -e_3\}, \\ \mathcal{C}_{15} &= \text{co}\{-e_1, -e_2\}, & \mathcal{C}_{16} &= \text{co}\{-e_1, e_2\} \\ \mathcal{C}_{17} &= \text{co}\{e_2, e_3\}, & \mathcal{C}_{18} &= \text{co}\{e_2, -e_3\}, \\ \mathcal{C}_{19} &= \text{co}\{-e_2, e_3\}, & \mathcal{C}_{20} &= \text{co}\{-e_2, -e_3\}, \end{aligned}$$

the 1-dimensional cones

$$\begin{aligned} \mathcal{C}_{21} &= \langle e_1 \rangle, & \mathcal{C}_{22} &= \langle -e_1 \rangle, \\ \mathcal{C}_{23} &= \langle e_2 \rangle, & \mathcal{C}_{24} &= \langle -e_2 \rangle, \\ \mathcal{C}_{25} &= \langle e_3 \rangle, & \mathcal{C}_{26} &= \langle -e_3 \rangle, \end{aligned}$$

and the 0-dimensional cone

$$\mathcal{C}_{27} = \{0\}.$$

To see how the cones \mathcal{D}_i are generated consider the cone \mathcal{C}_9 with corresponding linear map $A_9 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$. Thus, for any point $\xi \in \text{relint}(\mathcal{C}_9)$ we have $G(\xi) = A_9\xi$. We see that for all $\xi \in \text{relint}(\mathcal{C}_9)$ we have $G(\xi) \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_9$. In particular, we obtain the refinement of the cone \mathcal{C}_9 into the cones

$$\begin{aligned} \mathcal{D}_{(9,1)} &= \text{co} \left\{ \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \langle e_2 \rangle \right\}, \\ \mathcal{D}_{(9,2)} &= \text{co} \left\{ \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \langle e_1 \rangle \right\}, \\ \mathcal{D}_{(9,9)} &= \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle. \end{aligned}$$

The partition of the cone \mathcal{C}_9 into the cones $\mathcal{D}_{(9,1)}$, $\mathcal{D}_{(9,2)}$, and $\mathcal{D}_{(9,9)}$ is shown in Figure 1.

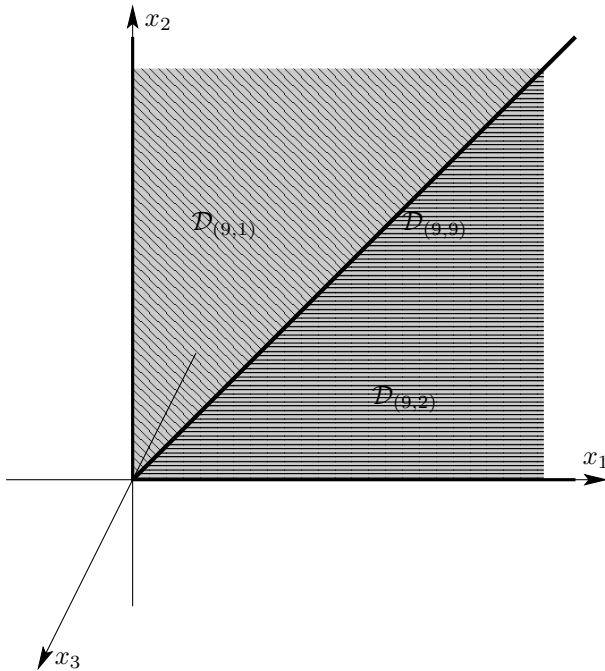


Fig. 1. The partition of the cone \mathcal{C}_9 into the cones $\mathcal{D}_{(9,1)}$, $\mathcal{D}_{(9,2)}$, and $\mathcal{D}_{(9,9)}$.

If the dynamics of the conewise linear system (9) are continuous then the conewise linear dynamics (8) are well-defined on the intersection of two cones, see Remark 7. Thus the conewise linear system (9) can be written as

$$x(k+1) = A_i x(k) \quad \text{if } x(k) \in \mathcal{C}_i, \quad (12)$$

where

- (i) $\bigcup_i \mathcal{C}_i = \mathbb{R}^n$;
- (ii) for $i \neq j$ we have $\text{relint}(\mathcal{C}_i) \cap \text{relint}(\mathcal{C}_j) = \emptyset$; and

- (iii) for $\xi \in \mathcal{C}_i \cap \mathcal{C}_j$ it holds $A_i \xi = A_j \xi$.

By continuity of the right-hand side of system (12), we don't have to worry about the points on the boundary of the cones \mathcal{C}_i . Hence, the proof of Theorem 8 can be simplified as outlined in Procedure 10. We emphasize that if the dynamics of the conewise linear system is continuous then the global conewise linear Lyapunov function constructed is linear.

Procedure 10: The following steps show in an algorithmic fashion how a continuous global conewise linear Lyapunov function can be obtained for a continuous conewise linear system (9).

- [1] Compute $M \in \mathbb{N}$ satisfying (4) for the 1-norm as follows:
 - (i) Set $k = 1$;
 - (ii) Compute $\tilde{c} := \max_{(j_1, \dots, j_k) \in \mathcal{A}_k} \left\| \prod_{i=1}^k A_{j_i} \right\|_1$;
 - (iii) If $\tilde{c} < 1$ set $M = k$ and go to step [2], else set $k = k + 1$ and go to step [1]-(i).
- [2] For any $\iota = (j_1, \dots, j_M) \in \mathcal{I}_M$ define the cones

$$\mathcal{D}_\iota := \left\{ \xi \in \mathcal{C}_i : \left[\prod_{i=0}^{k-1} A_{j_{k-i}} \right] \xi \in \mathcal{C}_{j_{k+1}} \right. \\ \left. \forall k \in \{1, \dots, M-1\} \right\}.$$

(Note that these cones are closed.)

- [3] Take those cones \mathcal{D}_{ι_1} that are not contained in another cone \mathcal{D}_{ι_2} , i.e.,

$$\mathcal{P}_M := \{ \mathcal{D}_{\iota_1} : \iota \in \mathcal{I}_M \text{ and } \forall \iota_2 \in \mathcal{I}_M \mathcal{D}_{\iota_1} \not\subset \mathcal{D}_{\iota_2} \}.$$

(Note that $\bigcup_{\mathcal{D}_i \in \mathcal{P}_M} \mathcal{D}_i = \mathbb{R}^n$ and $\text{relint}(\mathcal{D}_i) \cap \text{relint}(\mathcal{D}_j) = \emptyset$ if $i \neq j$.)

- [4] Define P_ι as in (11).
- [5] Then

$$W(\xi) := \|P_\iota \xi\|_1 \quad \text{if } \xi \in \mathcal{D}_\iota$$

is a continuous global conewise linear Lyapunov function.

Remark 11: The proof of Theorem 8 is constructive as it yields a global conewise linear Lyapunov function W . In this proof we use the *sum* formulation (6) of Theorem 5 for the 1-norm to construct the Lyapunov function W . An alternative is to use the *max* formulation (7) of Theorem 5 for the infinity norm as follows.

Taking $M \in \mathbb{N}$ and $\tilde{c} \in [0, 1)$ from (5), and using the same conic partition $\{\mathcal{D}_\iota\}_{\iota \in \mathcal{I}_M}$ as in the proof of Theorem 8, we have for $\xi \in \text{relint}(\mathcal{D}_\iota)$,

$$\begin{aligned} W(\xi) &= \max_{k \in \{0, \dots, M-1\}} \tilde{c}^{k/M} \|x(k, \xi)\|_\infty \\ &= \max_{k \in \{0, \dots, M-1\}} \left\| \tilde{c}^{k/M} \left[\prod_{i=0}^{k-1} A_{j_{k-i}} \right] \xi \right\|_\infty = \|P_\iota \xi\|_\infty \end{aligned}$$

with

$$P_\iota = \left(\tilde{c}^{\frac{M-1}{M}} \left[\prod_{i=0}^{M-2} A_{j_{M-1-i}} \right]; \dots; \tilde{c}^{\frac{2}{M}} A_{j_2} A_{j_1}; \tilde{c}^{\frac{1}{M}} A_{j_1}; I \right),$$

where $P_i \in \mathbb{R}^{nM \times n}$. Hence W is a global conewise linear Lyapunov function for system (9). This is the infinity norm analogue to the 1–norm construction in Theorem 8.

The above results, besides establishing non–conservatism of conewise linear Lyapunov functions for stability analysis of conewise linear systems, provide an explicit construction of such Lyapunov functions. The construction depends on finding an admissible value of the positive integer M , related to (5), which hinges on computing the set \mathcal{A}_k . In the case of polytopic cones, reachability analysis¹ for conewise linear systems can be performed efficiently, which yields \mathcal{A}_k . Hence, finding admissible values of M for conewise linear dynamics with polytopic conic partitions is tractable whenever P=NP for the corresponding stability analysis problem [2]. Otherwise, no other stability analysis method can provide a tractable test.

Example 12: To illustrate the above results, consider a discontinuous dynamics that corresponds to (8), but with $N = 9$, $A_i = \begin{pmatrix} 0.197 & -0.241 \\ 1.845 & 1.703 \end{pmatrix}$ for $i \in \{1, 3, 5, 7, 9\}$, and $A_i = \begin{pmatrix} -0.638 & -0.823 \\ 0.175 & 0.538 \end{pmatrix}$ for $i \in \{2, 4, 6, 8\}$, where A_1 is unstable. The corresponding conic partition is defined by $\{\mathcal{C}_i\}_{i \in \{1, \dots, 9\}}$, where $\mathcal{C}_i = \{x \in \mathbb{R}^2 : E_i x > 0\}$ for all $i \in \{1, \dots, 4\}$ with $E_1 = -E_3 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $E_2 = -E_4 = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$ are the 2-dimensional cones. Furthermore, the 1-dimensional cones (rays) are $\mathcal{C}_5 = \{x \in \mathbb{R}^2 : x_1 = x_2 \geq 0\}$, $\mathcal{C}_6 = \{x \in \mathbb{R}^2 : x_1 = x_2 \leq 0\}$, $\mathcal{C}_7 = \{x \in \mathbb{R}^2 : -x_1 = x_2 \geq 0\}$ and $\mathcal{C}_8 = \{x \in \mathbb{R}^2 : -x_1 = x_2 \leq 0\}$, and the 0-dimensional cone is $\mathcal{C}_9 = \{0\}$.

To make use of the results developed in this paper, we first indicate that (4) is satisfied for $M = 18$, which was established by computing $\|\prod_{i=1}^k A_{j_i}\|_1$ for all $(j_1, \dots, j_k) \in \mathcal{A}_k$ for $k = 1, \dots, 18$. Hence, the function $W(\xi) := \sum_{k=0}^{M-1} \|x(k, \xi)\|_1$ is a global Lyapunov function for this system. In Figure 2 we show a contour plot of the constructed non–convex conewise linear Lyapunov function.

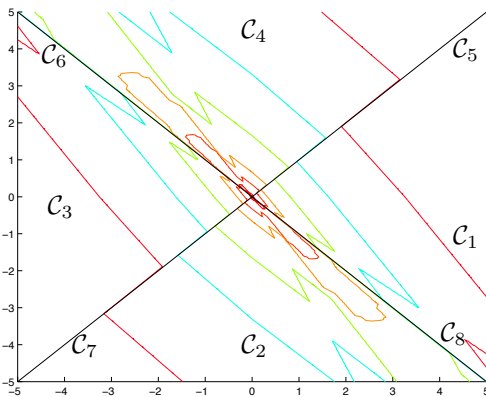


Fig. 2. Contour plot of the Lyapunov function $W(\xi) := \sum_{i=0}^{17} \|x(k, \xi)\|_1$ for Example 12.

Note that W is conewise with respect to a conic partition

¹As implied by the results in [2], this may be an NP–hard problem for conewise linear systems.

\mathcal{D}_v , which is finer than that given by $\mathcal{C}_1, \dots, \mathcal{C}_9$. This follows from the trajectorywise definition of W . The effect can be seen in Figure 2, where discontinuities of W occur along rays.

Example 13: Consider Example 6.1 in [8]. In this example the author showed the global exponential stability of a conewise linear system, where two out of three of the linear dynamics² are unstable. The system is given by

$$x(k+1) = A_i x(k), \quad k \in \mathbb{N}, \quad x(k) \in \mathcal{C}_i$$

with

$$\begin{aligned} A_1 &= \begin{bmatrix} -1.4078 & 0.1223 \\ 1.3846 & 0.4437 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.2405 & 0.1223 \\ -0.3420 & 0.4437 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} -0.5837 & -0.7019 \\ 0.5213 & 1.3070 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_1 &= \{x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} x \geq 0\}, \\ \mathcal{C}_2 &= \{x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x \geq 0\}, \\ \mathcal{C}_3 &= \{x \in \mathbb{R}^2 : \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x \geq 0\}. \end{aligned}$$

Accordingly we define

$$\begin{aligned} \mathcal{C}_4 &:= \mathcal{C}_1 \cap \mathcal{C}_2, & \mathcal{C}_5 &:= \mathcal{C}_2 \cap \mathcal{C}_3, \\ \mathcal{C}_6 &:= \mathcal{C}_1 \cap \mathcal{C}_3, & \mathcal{C}_7 &:= \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3 = \{0\}, \end{aligned}$$

and

$$A_4 = A_7 = A_1, \quad A_5 = A_2, \quad A_6 = A_3.$$

Similarly as done in the previous example, one can compute $M = 4$ for the infinity norm case, and $M = 3$ for the 1–norm case, such that (4) is met. Hence, by Theorem 5 the function $W(\xi) := \sum_{i=0}^2 \|x(k, \xi)\|_1$ is a global Lyapunov function for this system. A contour plot showing the sublevel sets of W is given in Figure 3.

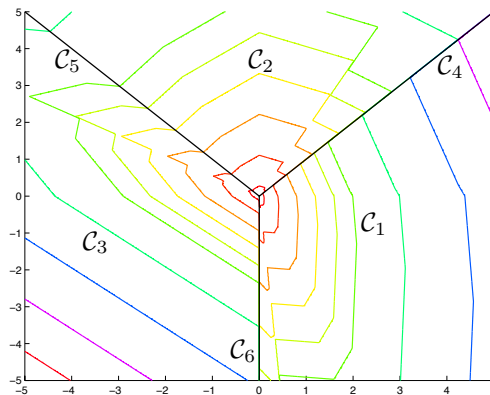


Fig. 3. A contour plot of the Lyapunov function $W(\xi) := \sum_{i=0}^2 \|x(k, \xi)\|_1$ for Example 13.

²Please note the typo in [8]: The matrices A_2 and A_3 have to be interchanged.

B. Linear dynamical systems

Interestingly, if the conewise linear dynamics reduces to the standard linear dynamics, with the convention that this dynamics is valid in \mathbb{R}^n , Theorem 8 implies that GES of the origin is equivalent to the existence of a global polyhedral³ Lyapunov function of the form $W(\xi) := \|P\xi\|_{1,\infty}$. In this case the Lyapunov weight matrix $P \in \mathbb{R}^{p \times n}$ with $p \geq n$ is not square in general, but of full column rank. Polyhedral Lyapunov functions [5], [6], [12] are in fact convex conewise linear functions, which can be expressed as the maximum over a finite number of linear functions (see also [13] for further insights).

The above observation is formally stated next.

Theorem 14: The origin of the linear system

$$x(k+1) = Ax(k) \tag{13}$$

with $k \in \mathbb{N}$, and $A \in \mathbb{R}^{n \times n}$ is GES if and only if there exists a global polyhedral Lyapunov function of the form $W(\xi) = \|P\xi\|_1$ with $P \in \mathbb{R}^{p \times n}$, $p \geq n$.

In particular, the number of rows of P can always be chosen as $p = Mn$, with $M \in \mathbb{N}$ given in (5).

We highlight that Theorem 14 explicitly gives an exact bound on the number of rows of P , which by the best of the authors' knowledge has not been solved elsewhere, see also Remark 16.

Proof: We omit the detailed proof here, but indicate that the global polyhedral Lyapunov function is constructed as

$$W(\xi) := \sum_{k=0}^{M-1} \|x(k, \xi)\|_1 = \sum_{k=0}^{M-1} \|A^k \xi\|_1 = \|P\xi\|_1$$

where

$$P := (A^{M-1}; A^{M-2}; \dots; A; I) \tag{14}$$

has $p = Mn$ number of rows. ■

Remark 15: Theorem 14 constructs a suitable global Lyapunov function W as a weighted 1-norm. Following Remark 11 it is easy to see that $W(\xi) := \|P\xi\|_\infty$, i.e., a weighted infinity norm, is a global Lyapunov function as well. In this case, however, the matrix P is defined as

$$P = \left((\tilde{c}^{1/M} A)^{M-1}; \dots; \tilde{c}^{2/M} A^2; \tilde{c}^{1/M} A; I \right), \tag{15}$$

where $\tilde{c} \in [0, 1)$ is given in (5).

Remark 16: In [5], [6] and, among several other works, [13], existence of a polyhedral Lyapunov function $W(\xi) = \|P\xi\|_{1,\infty}$ with $P \in \mathbb{R}^{p \times n}$ is established for GES linear systems. We stress that [5] treats the more general problem of difference inclusions. However, the proofs therein are rather complex and not constructive. In particular, no estimate of an upper bound on the number of rows p of the Lyapunov weight matrix P is given. This is in fact one of the non-trivial, open problems in construction of polyhedral Lyapunov functions for linear systems, see, e.g., [14], [15], [12]. In [14], [15] the problem is studied for continuous-time

³I.e., the sublevel sets of such a function are convex polyhedra with zero in their interior.

systems and lower bounds are given in terms of the geometry of the spectrum of A . Theorem 14 solves this problem by explicitly giving an admissible value of p for the 1-norm case, while an admissible value of p for the infinity norm case is given in Remark 15. In both cases $p = Mn$, where M is derived from (4).

Based on the above results and insights, we are in a position to provide a systematic procedure for constructing polyhedral Lyapunov functions for linear systems that is applicable in state spaces of high dimension. Note that this is attained without employing a (Jordan) decomposition of the A matrix or any further assumptions on the eigenvalues of A , as done in existing works on this topic, see, e.g., [13] and the references therein. To this end, in view of (5), it is possible to obtain an admissible value for M analytically, for linear systems. The procedure was obtained in [11], and is as follows.

Procedure 17: Take any norm $\|\cdot\|$. Then it holds $\|x(k, \xi)\| = \|A^k \xi\| \leq \|A^k\| \|\xi\|$. Hence, we have $M := \min\{k \in \mathbb{N} : \|A^k\| < 1\}$.

Example 18: To illustrate the results for linear dynamics, consider system (13) with $A = \begin{pmatrix} -0.1 & 0.4 \\ -0.2 & 0.9 \end{pmatrix}$. We construct Lyapunov functions both for the 1-norm case and for the infinity norm case.

(i) In the 1-norm case we obtain $\|A^{11}\|_1 < 1$. By Procedure 17, we get $M = 11$. Hence, the function $W_1(\xi) = \|P_1 \xi\|_1$ is a global polyhedral Lyapunov function for system (13), where $P_1 \in \mathbb{R}^{22 \times 2}$ can be computed in a straightforward manner by (14) as

$$P_1 = \begin{bmatrix} -0.8193 & 0.3726 \\ -0.1863 & -0.9125 \\ -0.6764 & 0.7147 \\ -0.3573 & -0.8551 \\ -0.4753 & 1.0053 \\ -0.5027 & -0.7267 \\ -0.2314 & 1.2199 \\ -0.6099 & -0.5363 \\ 0.0365 & 1.3392 \\ -0.6696 & -0.2983 \\ -0.3068 & 1.3516 \\ -0.6758 & -0.0311 \\ 0.5576 & 1.2540 \\ -0.6270 & 0.2441 \\ 0.7680 & 1.0520 \\ -0.5260 & 0.5050 \\ 0.9200 & 0.7600 \\ -0.3800 & 0.7300 \\ 1.0000 & 0.4000 \\ -0.2000 & 0.9000 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix}.$$

In Figure 4 we provide a contour and surface plot of the polyhedral Lyapunov function W_1 .

(ii) In the infinity norm case, again, we obtain $\|A^{11}\|_\infty < 1$. Hence for $M = 11$, the function $W_\infty(\xi) = \|P_\infty \xi\|_\infty$ is a global polyhedral Lyapunov function for system (13), where $P_\infty \in \mathbb{R}^{22 \times 2}$ can be computed in a straightforward manner

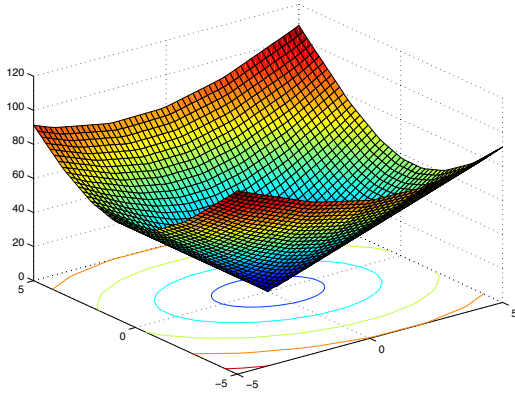


Fig. 4. Surface plot of the polyhedral Lyapunov function W_1 of Example 18.

by (15) as

$$P_\infty = \begin{bmatrix} -0.7520 & 0.3420 \\ -0.1710 & -0.8375 \\ -0.6262 & 0.6616 \\ -0.3308 & -0.7916 \\ -0.4438 & 0.9387 \\ -0.4693 & -0.6785 \\ -0.2179 & 1.1488 \\ -0.5744 & -0.5051 \\ 0.0346 & 1.2720 \\ -0.6360 & -0.2834 \\ 0.2939 & 1.2949 \\ -0.6475 & -0.0298 \\ 0.5388 & 1.2117 \\ -0.6059 & 0.2359 \\ 0.7485 & 1.0253 \\ -0.5126 & 0.4922 \\ 0.9044 & 0.7471 \\ -0.3735 & 0.7176 \\ 0.9915 & 0.3966 \\ -0.1983 & 0.8923 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix},$$

where we took $\tilde{c} = 0.91$.

Example 19: To illustrate the applicability of the developed methods in high dimension state spaces, consider the linear system (13) with

$$A = \begin{bmatrix} 0 & -0.3 & 0.1 & -0.1 & -0.3 & 0 & -0.1 & -0.1 & 0 & -0.4 \\ -0.4 & 0.4 & -0.3 & 0.1 & -0.2 & 0.1 & -0.4 & 0 & -0.2 & 0.4 \\ -0.4 & -0.1 & 0.1 & 0.2 & -0.3 & -0.3 & 0 & 0.2 & 0.3 & 0.3 \\ -0.4 & -0.3 & 0.2 & -0.1 & -0.1 & -0.1 & 0 & -0.1 & -0.2 & 0.4 \\ 0.3 & 0.2 & 0.4 & 0.2 & -0.2 & -0.4 & -0.3 & -0.4 & 0.1 & 0.2 \\ -0.4 & 0.4 & 0.2 & 0.3 & -0.2 & -0.1 & 0.4 & -0.2 & -0.3 & 0.4 \\ 0 & 0.2 & 0.2 & 0 & 0.2 & -0.3 & -0.3 & -0.4 & 0.2 & -0.1 \\ 0.4 & 0.3 & 0.4 & 0.3 & 0 & 0 & 0.4 & 0.3 & 0.4 & 0 \\ 0.2 & 0 & -0.2 & -0.3 & -0.3 & 0 & 0 & 0 & 0.2 & 0.2 \\ 0 & 0.1 & 0 & -0.1 & 0.4 & 0.3 & -0.2 & 0.3 & 0.4 & 0.3 \end{bmatrix}.$$

The origin of this system is GES as the spectral radius of A is 0.9544, and hence, less than 1. So we can compute the value $M \in \mathbb{N}$ by Procedure 17, and obtain for the 1-norm case a value of $M_1 = 17$, and for the infinity-norm case a value of $M_\infty = 20$. Thus, by Theorem 14, we obtain the global polyhedral Lyapunov function $W_1(\xi) = \|P_1\xi\|_1$, where the matrix $P_1 \in \mathbb{R}^{170 \times 10}$ is given by Theorem 14. Following Remark 15, we obtain the global polyhedral Lyapunov function $W_\infty(\xi) = \|P_\infty\xi\|_\infty$ with matrix $P_\infty \in \mathbb{R}^{200 \times 10}$.

V. CONCLUSIONS

In this paper we made use of the novel converse Lyapunov theorem for discrete-time systems presented in [1] to establish non-conservatism of the existence of conewise linear

Lyapunov functions for GES conewise linear discrete-time systems. Moreover, as a by-product, a tractable construction of polyhedral Lyapunov functions for linear systems was attained.

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