21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

A max-plus primal space fundamental solution for a class of difference Riccati equations

Huan Zhang[†] Peter M. Dower[†]

***Abstract*— Recently, a max-plus dual space fundamental solution for a class of difference Riccati equations (DREs) has been developed. This fundamental solution is represented in terms of the kernel of a specific max-plus linear operator that plays the role of the dynamic programming evolution operator in a max-plus dual space. In order to fully understand connections between this dual space fundamental solution and evolution of the value function of the underlying optimal control problem, a new max-plus primal space fundamental solution for the same class of difference Riccati equations is presented. Connections and commutation results between this new primal space fundamental solution and the recently developed dual space fundamental solution are developed.**

I. Introduction

The difference Riccati equation (DRE) arises naturally in optimal control and filtering theory [1], [2], [11]. In general terms, the solution of the DRE characterises the optimal performance and controller/estimator of the associated control/filtering problem. One of the important topics in the investigation of DREs is the characterisation and representation of all solutions by using some form of *fundamental solution* [3], [4], [10]. For example, in the continuous time case, the well-known Davison-Maki fundamental solution [4] exploits the solution of the corresponding Hamiltonian differential equation via a Bernoulli substitution technique [9]. Alternatively, a max-plus fundamental solution to a class of differential Riccati equations was developed in [13] by exploiting the linearity of the dynamic programming evolution operator associated with the attended optimal control problem, with respect to the max-plus algebra. It has been demonstrated that this max-plus fundamental solution facilitates efficient computation of particular solutions to the DRE [13], [5], [8], [6], [16]. This max-plus fundamental solution has also been applied in investigating properties of DRE solutions [15].

In developing the max-plus fundamental solution of interest, a specific bijection is identified that uniquely maps the value function to an element of a max-plus dual space. In this dual space, a family of time horizon indexed evolution operators that govern propagation of the dual of the value function define a semigroup of max-plus integral operators, analogously to the semigroup defined by the dynamic programming evolution operators in the primal space. The kernels of these max-plus integral operators also inherently define a semigroup (of horizon indexed quadratic functions). As all solutions of the DRE can be propagated from any initial data in a specific class via this semigroup, it can be regarded as a fundamental solution semigroup. It is referred to as a *dual space* fundamental solution (semigroup) due to its propagation in the aforementioned dual space.

The purpose of this paper is to introduce and explore the properties of a *max-plus primal space fundamental solution*. In principle, the construction of such a primal space fundamental solution involves the representation of the dynamic programming evolution operators in terms of a specific class of max-plus integral operators. This class of operators takes the same form as in the dual space case, but is defined entirely with respect to the primal space, with the corresponding operator kernels defining a semigroup in the primal space. Indeed Indeed, these kernels can be regarded as max-plus Green's functions [7]. It is shown that the primal and dual space fundamental solutions are in fact isomorphic.

In terms of organisation, Section II introduces the class of DREs and associated linear quadratic optimal control problem of interest. Section III establishes the notion of max-plus primal and dual spaces. Section IV summarises the existing max-plus dual space fundamental solution, followed by an analogous development of the primal space fundamental solution in Section V. This development includes a detailed analysis of the connection between the primal and dual space fundamental solutions. Section VI concludes the paper.

In terms of notation, $\mathbb{R}, \mathbb{N}, \mathbb{Z}_{\geq 0}$ denote the sets of reals, natural numbers and non-negative integers. The sets of extended reals are denoted by $\mathbb{R}^- \doteq \mathbb{R} \cup \{-\infty\}$ and $\mathbb{R}^+ \doteq \mathbb{R} \cup \{\infty\}$. The set of $n \times n$ real, symmetric matrices is denoted by $\mathbb{M}^{n\times n} \doteq \{P \in \mathbb{R}^{n\times n} | P = P^T\}$. The max-plus algebra is a semiring $(\mathbb{R}^-, \oplus, \otimes)$, where the max-plus addition and multiplication operations are defined by $a \oplus b \doteq \max\{a, b\}$ and $a \otimes b \doteq a + b$, respectively. The max-plus integral of a function $f : \mathbb{R}^n \to \mathbb{R}^-$ is defined as $\int_{\mathbb{R}^n}^{\oplus} f(x)\, dx \doteq \sup_{x\in\mathbb{R}^n}\{f(x)\}$. The identity operator is denoted by $\mathcal{I}$.

II. Linear quadratic optimal control problems and the difference Riccati equation

Attention is restricted to discrete-time linear systems of the form

$$x_{k+1} = A\, x_k + B\, w_k\,, \quad x_0 = x, \tag{1}$$

where $x_k \in \mathbb{R}^n, w_k \in \mathbb{R}^m, k \in \mathbb{Z}_{\geq 0}$ denote the state and input variables at time k, respectively. $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the system and input matrices. An input sequence

[†] Zhang and Dower are with the Department of Electrical and Electronic Engineering, University of Melbourne, Melbourne, Victoria 3010, Australia. Email: {hzhang5,pdower}@unimelb.edu.au. This research is supported by grants FA2386-12-1-4084 and DP120101549 from AFOSR and the Australian Research Council.

ISBN: 978-90-367-6321-9

on interval $[0, K-1]$ for any $K \in \mathbb{N}$ is denoted by $w_{0,K-1} \in \mathscr{W}[0, K-1] \doteq (\mathbb{R}^m)^K$. The optimal control problem of interest is defined via the value function $W_k : \mathbb{R}^n \to \mathbb{R}, k \in \mathbb{Z}_{\geq 0}$, where

$$W_K(x) \doteq \sup_{w_{0,K-1} \in \mathscr{W}[0,K-1]} J_K(x;\, w_{0,K-1}) \tag{2}$$

in which the total payoff $J_K : \mathbb{R}^n \times \mathscr{W}_{0,K-1} \to \mathbb{R}$ is

$$J_K(x; w_{0,K-1}) \doteq \sum_{k=0}^{K-1} \left(\frac{1}{2} x_k^T \, \Phi \, x_k - \frac{\gamma^2}{2} \, |w_k|^2 \right) + \frac{1}{2} x_K^T \Lambda x_K \,. \tag{3}$$

Here, $\Phi \in \mathbb{M}^{n \times n}, \Lambda \in \mathbb{M}^{n \times n}$ are positive semi-definite matrices, where the gain $\gamma \in \mathbb{R}_{>0}$ specifies the level of penalty on the input.

Applying dynamic programming to (2) yields

$$W_{k+1} = \mathcal{S}_1 \, W_k \tag{4}$$

subject to the initial condition $W_0(x) = \Psi(x) \doteq \frac{1}{2} x^T \Lambda x, x \in \mathbb{R}^n$. Here, the operator $\mathcal{S}_1$ denotes the (one-step) dynamic programming evolution operator defined by

$$(\mathcal{S}_1 \, \phi)(x) \doteq \sup_{w \in \mathbb{R}^m} \left\{ \frac{1}{2} x^T \, \Phi \, x - \frac{\gamma^2}{2} \, |w|^2 + \phi(A \, x + B \, w) \right\}. \tag{5}$$

Using (5), define the k-step dynamic programming evolution operator $\mathcal{S}_k, k \in \mathbb{N}$, iteratively

$$\mathcal{S}_{k+1} \, \phi \doteq \mathcal{S}_1 \left(\mathcal{S}_k \, \phi \right) , \tag{6}$$

and (as a matter of convention) define $\mathcal{S}_0 \doteq \mathcal{I}$. The operator $\mathcal{S}_k$ propagates the terminal payoff Ψ (equivalently, W_0) to the value function $W_k = \mathcal{S}_k \Psi$. A fundamental property of the operator $\mathcal{S}_k$ is that it is linear over max-plus algebra ([12], [16]), that is, $\mathcal{S}_k(a \otimes \phi_1 \oplus \phi_2) = a \otimes \mathcal{S}_k \phi_1 \oplus \mathcal{S}_k \phi_2$ for any $a \in \mathbb{R}^-$ and ϕ_1, ϕ_2 in a max-plus vector space (e.g., $\mathcal{S}_+^K(\mathbb{R}^n)$ of (10)). Consequently, the set $\{\mathcal{S}_k, k \in \mathbb{Z}_{\geq 0}\}$ defines a semigroup of max-plus linear operators. It is well-known [1] that the linear dynamics (1) and quadratic payoff (3) imply that the value function W_k of (2) is quadratic, with $W_k(x) = \frac{1}{2} x^T P_k x, x \in \mathbb{R}^n$, in which the Hessian P_k satisfies the difference Riccati equation (DRE)

$$P_{k+1} = \mathcal{R}(P_k), \quad P_0 \in \mathbb{M}^{n \times n}, \tag{7}$$

with initial condition $P_0 = \Lambda$ and $\mathcal{R} : \mathbb{M}^{n \times n} \to \mathbb{M}^{n \times n}$ defined by

$$\mathcal{R}(P) \doteq A^T P A + A^T P B (\gamma^2 \, I - B^T P B)^{-1} B^T P A + \Phi. \tag{8}$$

With operators $\mathcal{R}_k : \mathbb{M}^{n \times n} \to \mathbb{M}^{n \times n}, k \in \mathbb{Z}_{\geq 0}$ defined iteratively by

$$\mathcal{R}_{k+1} = \mathcal{R} \circ \mathcal{R}_k, \quad \mathcal{R}_0 = \mathcal{I}, \tag{9}$$

the solution P_k to the DRE (7) at time k is

$$P_k = \mathcal{R}_k(P_0).$$

III. MAX-PLUS PRIMAL AND DUAL SPACE

In order to introduce the max-plus dual space fundamental solution in [13], [16] and subsequently develop a max-plus primal space fundamental solution, the max-plus dual and primal spaces, and the duality pairing, are defined first.

Definition 3.1: ([12], [13]) Given $K \in \mathbb{M}^{n \times n}$, a function $\phi : \mathbb{R}^n \to \mathbb{R}^-$ is uniformly semiconvex with respect to K if the function $x \mapsto \phi(x) + \frac{1}{2} x^T K x : \mathbb{R}^n \to \mathbb{R}^-$ is convex on $\mathbb{R}^n$. Analogously, a function $\phi : \mathbb{R}^n \to \mathbb{R}^-$ is uniformly semiconcave with respect to K if the function $x \mapsto \phi(x) - \frac{1}{2} x^T K x : \mathbb{R}^n \to \mathbb{R}^-$ is concave on $\mathbb{R}^n$.

The spaces of uniformly semiconvex and semiconcave functions with respect to $K \in \mathbb{M}^{n \times n}$ are denoted by

$$\mathcal{S}_+^K(\mathbb{R}^n) \doteq \left\{ \phi : \mathbb{R}^n \to \mathbb{R}^- \middle| \begin{array}{c} \phi \text{ is uniformly} \\ \text{semiconvex w.r.t. } K \end{array} \right\}, \tag{10}$$

$$\mathcal{S}_-^K(\mathbb{R}^n) \doteq \left\{ \phi : \mathbb{R}^n \to \mathbb{R}^- \middle| \begin{array}{c} \phi \text{ is uniformly} \\ \text{semiconcave w.r.t. } K \end{array} \right\}. \tag{11}$$

Define a pair of operators $\mathcal{D}_\psi$ and $\mathcal{D}_\psi^{-1}$ for a given $M \in \mathbb{M}^{n \times n}$ by (see [12])

$$\mathcal{D}_\psi \, \phi = \left(\mathcal{D}_\psi \, \phi \right)(\cdot) \doteq - \int_{\mathbb{R}^n}^{\oplus} \psi(x, \cdot) \otimes (-\phi(x)) \, dx \,, \tag{12}$$

$$\mathcal{D}_\psi^{-1} \, a = \left(\mathcal{D}_\psi^{-1} \, a \right)(\cdot) \doteq \int_{\mathbb{R}^n}^{\oplus} \psi(\cdot, z) \otimes a(z) \, dz \,, \tag{13}$$

where the function $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is quadratic

$$\psi(x, z) \doteq \frac{1}{2} (x - z)^T \, M \, (x - z). \tag{14}$$

The following restrictions on $M \in \mathbb{M}^{n \times n}$ are assumed throughout.

Assumption 3.2: Given $B \in \mathbb{R}^{n \times m}$ as per (1), the matrix $M \in \mathbb{M}^{n \times n}$ in (14) satisfies the following inequalities.

$$\gamma^2 \, I - B^T \mathcal{R}_k(M) B > 0, \quad \forall \; k \in \mathbb{Z}_{\geq 0}, \tag{15}$$

$$\mathcal{R}(M) - M > 0, \tag{16}$$

$$M B (\gamma^2 \, I - B^T M B)^{-1} B^T M > 0. \tag{17}$$

The following result shows that the operators $\mathcal{D}_\psi$ and $\mathcal{D}_\psi^{-1}$ can be used to define a duality between the spaces $\mathcal{S}_+^{-M}(\mathbb{R}^n)$ and $\mathcal{S}_-^{-M}(\mathbb{R}^n)$ where $M \in \mathbb{M}^{n \times n}$ is as per (14). The proofs are provided in [17]. They are omitted here for brevity.

Theorem 3.3: The operator $\mathcal{D}_\psi$ of (12) is a bijection from space $\mathcal{S}_+^{-M}(\mathbb{R}^n)$ to space $\mathcal{S}_-^{-M}(\mathbb{R}^n)$ with inverse operator $\mathcal{D}_\psi^{-1}$ given by (13).

In view of Theorem 3.3, $D_\psi \phi$ is referred to as the *semiconvex dual* of $\phi \in \mathcal{S}_+^{-M}(\mathbb{R}^n)$. For the purpose of studying the solutions of the DRE (7), the domain of the operator $\mathcal{D}_\psi$ can be restricted to a space of quadratic functions

$$\mathcal{Q}_+^{-M}(\mathbb{R}^n) \doteq \left\{ \phi : \mathbb{R}^n \to \mathbb{R} \middle| \begin{array}{c} \phi(x) = \frac{1}{2} x^T \Omega x, \\ \Omega \in \mathbb{M}^{n \times n}, \Omega > M \end{array} \right\}. \tag{18}$$

For any function $\phi \in \mathcal{Q}_+^{-M}(\mathbb{R}^n)$, there exists $\Omega > M$ such that $\phi(x) = \frac{1}{2} x^T \Omega x$. Hence, the function $\breve{\phi} : \mathbb{R}^n \to \mathbb{R}$

defined by $\breve{\phi}(x) \doteq \phi(x) + \frac{1}{2}x^T(-M)x = \frac{1}{2}x^T(\Omega - M)x$ is convex on $\mathbb{R}^n$. This shows that $\mathcal{Q}_+^{-M}(\mathbb{R}^n) \subset \mathcal{S}_+^{-M}(\mathbb{R}^n)$. Define the range of the operator $\mathcal{D}_\psi$ over the space $\mathcal{Q}_+^{-M}(\mathbb{R}^n)$ by

$$\mathsf{ran}(\mathcal{D}_\psi) \doteq \left\{ \mathcal{D}_\psi \phi \,|\, \phi \in \mathcal{Q}_+^{-M}(\mathbb{R}^n) \right\}. \tag{19}$$

Then, $\mathsf{ran}(\mathcal{D}_\psi)$ can be expressed explicitly from the definition of the operator $\mathcal{D}_\psi$ of (12). To this end, define a matrix operation $\Upsilon : \mathbb{M}^{n\times n} \to \mathbb{M}^{n\times n}$ by

$$\Upsilon(\Omega) \doteq M(M - \Omega)^{-1}M - M \tag{20}$$

for $\Omega \in \mathbb{M}^{n\times n}$ such that $\Omega > M$. It can be verified directly that the inverse of Υ is

$$\begin{aligned} \Upsilon^{-1}(\Omega) &\doteq -M(M + \Omega)^{-1}M + M \\ &= -\Upsilon(-\Omega) \end{aligned} \tag{21}$$

for all $\Omega \in \mathbb{M}^{n\times n}$ such that $\Omega < -M$. Define

$$\mathcal{Q}_-^{-M}(\mathbb{R}^n) \doteq \left\{ \phi : \mathbb{R}^n \to \mathbb{R} \,\middle|\, \begin{array}{c} \phi(x) = \frac{1}{2}x^T\Upsilon(\Omega)x \\ \Omega \in \mathbb{M}^{n\times n}, \Omega > M \end{array} \right\}. \tag{22}$$

Theorem 3.4: The set $\mathcal{Q}_-^{-M}(\mathbb{R}^n)$ is the range of the operator $\mathcal{D}_\psi$ over the space $\mathcal{Q}_+^{-M}(\mathbb{R}^n)$. That is, $\mathcal{Q}_-^{-M}(\mathbb{R}^n) = \mathsf{ran}(\mathcal{D}_\psi)$.

Proof: For any $\phi \in \mathcal{Q}_+^{-M}(\mathbb{R}^n)$, let $\Omega \in \mathbb{M}^{n\times n}, \Omega > M$ be such that $\phi(x) = \frac{1}{2}x^T\Omega x$ for all $x \in \mathbb{R}^n$. Then from (12) and (20),

$$\begin{aligned} (\mathcal{D}_\psi\phi)(z) &= -\int_{\mathbb{R}^n}^{\oplus} \psi(x, z) \otimes (-\phi(x))\, dx \\ &= -\max_{x\in\mathbb{R}^n} \left\{ \tfrac{1}{2}(x - z)^T M(x - z) - \tfrac{1}{2}x^T\Omega x \right\} \\ &= \tfrac{1}{2}z^T\Upsilon(\Omega)z. \end{aligned}$$

Thus, every element in $\mathsf{ran}(\mathcal{D}_\psi)$ corresponds to an element in $\mathcal{Q}_-^{-M}(\mathbb{R}^n)$, and vice versa. Hence $\mathcal{Q}_-^{-M}(\mathbb{R}^n) = \mathsf{ran}(\mathcal{D}_\psi)$. ■

The following result follows immediately by combining Theorem 3.3 and Theorem 3.4.

Corollary 3.5: The operator $\mathcal{D}_\psi$ of (12) is a bijection from $\mathcal{Q}_+^{-M}(\mathbb{R}^n)$ to $\mathcal{Q}_-^{-M}(\mathbb{R}^n)$ with inverse $\mathcal{D}_\psi^{-1}$ given by (13).

Throughout, $\mathcal{Q}_+^{-M}(\mathbb{R}^n)$ is referred to as the *max-plus primal space* and $\mathcal{Q}_-^{-M}(\mathbb{R}^n)$ is referred to as the *max-plus dual space*. The dynamic programming evolution operator $\mathcal{S}_k$ of (6) propagates the value function W_k of (2) in the max-plus primal space $\mathcal{Q}_+^{-M}(\mathbb{R}^n)$. The domain of $\mathcal{S}_k$ is defined by

$$\mathsf{dom}(\mathcal{S}_k) \doteq \left\{ \phi \in \mathcal{Q}_+^{-M}(\mathbb{R}^n) \,\middle|\, \begin{array}{c} \mathcal{R}_k(\Omega) \text{ exists for} \\ \Omega \in \mathbb{M}^{n\times n} \text{ such that} \\ \phi(x) = \frac{1}{2}x^T\Omega x, x \in \mathbb{R}^n \end{array} \right\}. \tag{23}$$

It may be shown that the value function W_k stays in $\mathcal{Q}_+^{-M}(\mathbb{R}^n)$ when the initial condition $W_0 \in \mathcal{Q}_+^{-M}(\mathbb{R}^n)$. The proofs are referred to [17] and are omitted here for brevity.

Theorem 3.6: Suppose that Assumption 3.2 holds, and for $k \in \mathbb{Z}_{\geq 0}$, an initial value function $W_0 \in \mathsf{dom}(\mathcal{S}_k)$ is given by $W_0(x) = \frac{1}{2}x^T P_0 x, x \in \mathbb{R}^n$, with $P_0 > M$. Then, the value function $W_k = \mathcal{S}_k W_0 \in \mathcal{Q}_+^{-M}(\mathbb{R}^n)$.

Remark 3.7: From Theorem 3.6, the value function $W_k \in \mathcal{Q}_+^{-M}(\mathbb{R}^n)$ at time $k \in \mathbb{Z}_{\geq 0}$ if $W_0 \in \mathsf{dom}(\mathcal{S}_k) \subset \mathcal{Q}_+^{-M}(\mathbb{R}^n)$. Thus, the max-plus dual $\widehat{W}_k \doteq \mathcal{D}_\psi W_k$ exists. It follows from Theorem 3.4 that $\widehat{W}_k \in \mathcal{Q}_-^{-M}(\mathbb{R}^n)$ with a representation $\widehat{W}_k(z) = \frac{1}{2}z^T\Upsilon(\mathcal{R}_k(P_0))z, z \in \mathbb{R}^n$.

IV. Max-plus dual space fundamental solution

In [13], a max-plus dual space fundamental solution for the continuous time counterpart of DRE (7) has been developed. This analogous max-plus dual space fundamental solution has also been developed for infinite dimensional problems [5], [6], [8]. This max-plus dual space fundamental solution is obtained by exploring the propagation of the semi-convex dual of the value function in the dual space $\mathcal{Q}_-^{-M}(\mathbb{R}^n)$.

Define an auxiliary value function $\widehat{\mathrm{S}}_k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by applying the operator $\mathcal{S}_k$ to the functions $\psi(\cdot, z), z \in \mathbb{R}^n$,

$$\widehat{\mathrm{S}}_k(x, z) \doteq (\mathcal{S}_k\psi(\cdot, z))\,(x). \tag{24}$$

Applying the operator $\mathcal{D}_\psi$ of (12) to $\widehat{\mathrm{S}}_k(\cdot, z)$ of (24) yields a function $\mathrm{B}_k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$\begin{aligned} \mathrm{B}_k(y, z) &\doteq (\mathcal{D}_\psi\widehat{\mathrm{S}}_k(\cdot, z))(y) \\ &= -\int_{\mathbb{R}^n}^{\oplus} \psi(x, y) \otimes (-\widehat{\mathrm{S}}_k(x, z))\, dx. \end{aligned} \tag{25}$$

The function B_k can be used to define a max-plus integral operator $\mathcal{B}_k : \mathcal{Q}_-^{-M}(\mathbb{R}^n) \to \mathcal{Q}_-^{-M}(\mathbb{R}^n)$ by

$$(\mathcal{B}_k\, a)(y) = \int_{\mathbb{R}^n}^{\oplus} \mathrm{B}_k(y, z) \otimes a(z)\, dz. \tag{26}$$

$\mathcal{B}_k$ is related with $\mathcal{S}_k$ of (6) by [13]

$$\mathcal{B}_k = \mathcal{D}_\psi\, \mathcal{S}_k \mathcal{D}_\psi^{-1}. \tag{27}$$

The set of operators $\{\mathcal{B}_k, k \in \mathbb{Z}_{\geq 0}\}$ of (26) defines a semigroup that propagates the semiconvex dual $\widehat{W}_k \doteq \mathcal{D}_\psi W_k = \mathcal{B}_k(\mathcal{D}_\psi W_0)$ of the value function W_k. This provides one means of propagating the initial value functions W_0 to the final value function W_k. In particular, the value function can be propagated via three main steps: 1) the initial condition $W_0(x) = \Psi(x) = \frac{1}{2}x^T\Lambda x$ is mapped into the dual space by the operator $\mathcal{D}_\psi$ of (12); 2) this dual $\widehat{W}_0$ is propagated via the operators $\mathcal{B}_k, k \in \mathbb{N}$ to obtain the dual $\widehat{W}_k = \mathcal{B}_k\widehat{W}_0$ of the final value function W_k; 3) applying the inverse dual $\mathcal{D}_\psi^{-1}$ of (13) to $\widehat{W}_k$ then finally yields the value function $W_k = \mathcal{D}_\psi^{-1}\widehat{W}_k$. These steps are equivalent to application of the dynamic programming evolution operator $\mathcal{S}_k$ in the primal space, as summarized in Figure 1.

It is a consequence of linear quadratic nature of the optimal control problem (2) that the function B_k of (25) is quadratic

$$\begin{aligned} \mathrm{B}_k(y, z) &= \frac{1}{2}\begin{bmatrix} y \\ z \end{bmatrix}^T \Theta_k \begin{bmatrix} y \\ z \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} y \\ z \end{bmatrix}^T \begin{bmatrix} \Theta_k^{11} & \Theta_k^{12} \\ \Theta_k^{21} & \Theta_k^{22} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \end{aligned} \tag{28}$$

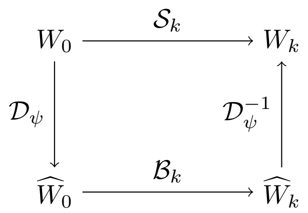

Fig. 1. Propagation of W_k by $\mathcal{S}_k$ of (6) or via $\mathcal{B}_k$ of (26).

with Hessian $\Theta_k \in \mathbb{M}^{2n\times 2n}$. Thus, a representation of the value function W_k in terms of B_k is given by

$$\begin{aligned} W_k(x) &= \tfrac{1}{2}x^T P_k x \qquad (29)\\ &= (\mathcal{S}_k\Psi)(x) = (\mathcal{D}_\psi^{-1}\mathcal{B}_k\mathcal{D}_\psi\Psi)(x)\\ &= \mathcal{D}_\psi^{-1}\left(\int_{\mathbb{R}^n}^{\oplus} \mathrm{B}_k(\cdot,z)\otimes(\mathcal{D}_\psi\Psi)(z)\,dz\right)(x). \end{aligned}$$

Using the definitions of the operators Υ of (20), and Υ^{-1} of (21), equation (29) can be represented in terms of Hessians P_k and Θ_k. This yields a representation of the solutions P_k to the DRE of (7) with initial condition $P_0 = \Lambda$ via Θ_k by

$$\begin{aligned} O_0 &= \Upsilon(\Lambda),\\ O_k &= \Theta_k^{11} - \Theta_k^{12}(O_0 + \Theta_k^{22})^{-1}\Theta_k^{21}, \qquad (30)\\ P_k &= \Upsilon^{-1}(O_k). \end{aligned}$$

The sequence $\{\Theta_k, k\in\mathbb{N}\}$ is a fundamental solution to the DRE (7) since this particular sequence does not depend on the initial condition $P_0 = \Lambda$, and it can be used to represent the DRE solutions P_k with $P_0 = \Lambda$ via (30). It is regarded as the max-plus dual space fundamental solution since it is the Hessian of the kernel of max-plus integral operators $\mathcal{B}_k$ of (26), which propagates the semiconvex dual in the dual space $\mathcal{Q}_-^{-M}(\mathbb{R}^n)$.

V. The max-plus primal space fundamental solution

A. Max-plus representation of the operator $\mathcal{S}_k$

Define a new auxiliary function $\mathrm{S}_k : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ by applying the operator $\mathcal{D}_\psi$ to the function $\widehat{\mathrm{S}}_k(x,\cdot)$ of (24),

$$\begin{aligned} \mathrm{S}_k(x,y) &\doteq \left(\mathcal{D}_\psi \widehat{\mathrm{S}}_k(x,\cdot)\right)(y) \qquad (31)\\ &= -\int_{\mathbb{R}^n}^{\oplus} \psi(z,y)\otimes(-\widehat{\mathrm{S}}_k(x,z))\,dz. \end{aligned}$$

This allows a representation of $\widehat{\mathrm{S}}_k$ in terms of S_k by applying the inverse dual operator $\mathcal{D}_\psi^{-1}$ of (13) to $\mathrm{S}_k(x,\cdot)$,

$$\begin{aligned} \widehat{\mathrm{S}}_k(x,z) &= \left(\mathcal{D}_\psi^{-1}\mathrm{S}_k(x,\cdot)\right)(z) \qquad (32)\\ &= \int_{\mathbb{R}^n}^{\oplus} \psi(z,y)\otimes \mathrm{S}_k(x,y)\,dy. \end{aligned}$$

Define a max-plus integral operator $\widetilde{\mathcal{S}}_k$ by

$$(\widetilde{\mathcal{S}}_k\phi)(x) \doteq \int_{\mathbb{R}^n}^{\oplus} \mathrm{S}_k(x,y)\otimes\,\phi(y)\,dy \qquad (33)$$

for all $\phi\in\mathcal{Q}_+^{-M}(\mathbb{R}^n)$ such that the max-plus integral in (33) is finite. It may be shown that $\widetilde{\mathcal{S}}_k$ is actually the dynamic programming evolution operator $\mathcal{S}_k$ of (6).

Theorem 5.1: For any $\phi\in\mathsf{dom}(\mathcal{S}_k)$,

$$\mathcal{S}_k\phi = \widetilde{\mathcal{S}}_k\phi. \qquad (34)$$

Proof: Define $\widehat{\mathcal{S}}_k \doteq \mathcal{S}_k\mathcal{D}_\psi^{-1}$ for $k\in\mathbb{Z}_{\geq 0}$. It is shown first that

$$\left(\widehat{\mathcal{S}}_k a\right)(x) = \int_{\mathbb{R}^n}^{\oplus} \widehat{\mathrm{S}}_k(x,z)\otimes\, a(z)\,dz \qquad (35)$$

for any $a\in\mathcal{Q}_-^{-M}(\mathbb{R}^n)$ such that $\mathcal{D}_\psi^{-1}a\in\mathsf{dom}(\mathcal{S}_k)$. By the max-plus linearity of the operator $\mathcal{S}_k$

$$\begin{aligned} \left(\widehat{\mathcal{S}}_k a\right)(x) &= \left(\mathcal{S}_k\mathcal{D}_\psi^{-1}\,a\right)(x)\\ &= \left(\mathcal{S}_k\left(\int_{\mathbb{R}^n}^{\oplus}\psi(\cdot,z)\otimes\, a(z)\,dz\right)\right)(x)\\ &= \int_{\mathbb{R}^n}^{\oplus}\left(\mathcal{S}_k\psi(\cdot,z)\right)(x)\otimes\, a(z)\,dz\\ &= \int_{\mathbb{R}^n}^{\oplus}\widehat{\mathrm{S}}_k(x,z)\otimes\, a(z)\,dz. \end{aligned}$$

Then, for any $x\in\mathbb{R}^n$, $\phi\in\mathsf{dom}(\mathcal{S}_k)\subset\mathcal{Q}_+^{-M}(\mathbb{R}^n)$, from(35), (31) and (32),

$$\begin{aligned} &(\mathcal{S}_k\phi)(x) = (\mathcal{S}_k\mathcal{D}_\psi^{-1})(\mathcal{D}_\psi\phi)(x)\\ &= \int_{\mathbb{R}^n}^{\oplus}\widehat{\mathrm{S}}_k(x,z)\otimes(\mathcal{D}_\psi\phi)(z)\,dz\\ &= \int_{\mathbb{R}^n}^{\oplus}\left(\int_{\mathbb{R}^n}^{\oplus}\mathrm{S}_k(x,y)\otimes\psi(z,y)\,dy\right)\otimes(\mathcal{D}_\psi\phi)(z)\,dz\\ &= \int_{\mathbb{R}^n}^{\oplus}\mathrm{S}_k(x,y)\otimes\left(\int_{\mathbb{R}^n}^{\oplus}\psi(y,z)\otimes(\mathcal{D}_\psi\phi)(z)\,dz\right)dy\\ &= \int_{\mathbb{R}^n}^{\oplus}\mathrm{S}_k(x,y)\otimes\phi(y)\,dy\\ &= (\widetilde{\mathcal{S}}_k\phi)(x), \end{aligned}$$

where the third equality uses the fact that $\psi(x,z)=\psi(z,x)$ for all $x,z\in\mathbb{R}^n$. ■

Theorem 5.1 and (33) show that the dynamic programming evolution operator $\mathcal{S}_k$ has a max-plus integral operator representation with kernel S_k of (31). That is,

$$(\mathcal{S}_k\phi)(x) = \int_{\mathbb{R}^n}^{\oplus}\mathrm{S}_k(x,y)\otimes\,\phi(y)\,dy \qquad (36)$$

for any $\phi\in\mathcal{S}_+^{-M}(\mathbb{R}^n)$. Using this representation, the value function W_k of (2) can be expressed as

$$W_k(x) = (\mathcal{S}_k\Psi)(x) = \int_{\mathbb{R}^n}^{\oplus}\mathrm{S}_k(x,y)\otimes\Psi(y)\,dy. \qquad (37)$$

Compared with (29), which represents the value function via the kernel B_k of the operator $\mathcal{B}_k$ in dual space, the representation of (37) is simpler since it does not involve the mapping to and back from the dual space by $\mathcal{D}_\psi$ and $\mathcal{D}_\psi^{-1}$. This representation of the value function in the primal space $\mathcal{Q}_+^{-M}(\mathbb{R}^n)$ yields a simpler representation of the solutions to DRE (7) via the max-plus primal space fundamental solution.

B. The max-plus primal space fundamental solution

The function S_k of (31) is related to $\widehat{\mathrm{S}}_k$ via (31). Similarly to B_k of (28), S_k is also quadratic. Let $\Lambda_k \in \mathbb{M}^{2n\times 2n}$ be the Hessian of S_k, that is,

$$\mathrm{S}_k(x,y) = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^T \Lambda_k \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} \Lambda_k^{11} & \Lambda_k^{12} \\ \Lambda_k^{21} & \Lambda_k^{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \tag{38}$$

Then, the representation of the value function W_k of (2) in terms of S_k via (37) yields a representation of the solution to the DRE of (7) via the sequence $\{\Lambda_k, k\in\mathbb{N}\}$ of (38).

Theorem 5.2: For any $\Lambda \in \mathbb{M}^{n\times n}$ such that $\Lambda_k^{22} + \Lambda < 0$, the solution P_k to the DRE of (7) with $P_0 = \Lambda$ has a representation

$$P_k = \Lambda_k^{11} - \Lambda_k^{12}(\Lambda + \Lambda_k^{22})^{-1}\Lambda_k^{21}. \tag{39}$$

Proof: From the fact that the Hessian P_k of the value function W_k of (2) is the solution to the DRE of (7),

$$\begin{aligned} W_k(x) &= \tfrac{1}{2}x^T P_k x \\ &= (\mathcal{S}_k\Psi)(x) = \int_{\mathbb{R}^n}^{\oplus} \mathrm{S}_k(x,y)\otimes \tfrac{1}{2}y^T\Lambda y\, dy \\ &= \max_{y\in\mathbb{R}^n}\left\{\frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} \Lambda_k^{11} & \Lambda_k^{12} \\ \Lambda_k^{21} & \Lambda_k^{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}y^T\Lambda y\right\} \\ &= \tfrac{1}{2}x^T(\Lambda_k^{11} - \Lambda_k^{12}(\Lambda+\Lambda_k^{22})^{-1}\Lambda_k^{21})x. \end{aligned}$$

Since this holds for all $x\in\mathbb{R}^n$, (39) follows. ∎

Analogous to the max-plus dual space fundamental solution $\{\Theta_k, k\in\mathbb{N}\}$, the sequence $\{\Lambda_k, k\in\mathbb{N}\}$ is referred to as the *max-plus primal space fundamental solution* to the DRE (7). It can be seen that the representation of the DRE solution P_k in terms of the max-plus primal space fundamental solution $\{\Lambda_k, k\in\mathbb{N}\}$ via (39) has a simpler form that the representation of (30) in terms of the max-plus dual space fundamental solution $\{\Theta_k, k\in\mathbb{N}\}$, insofar as the Υ operations are not required.

C. Connection with the max-plus dual space fundamental solution

The max-plus primal and dual space fundamental solutions Λ_k of (38) and Θ_k of (28) are the Hessian of S_k and B_k, respectively, which are kernels of the operators $\mathcal{S}_k$ of (6) and $\mathcal{B}_k$ of (26). From (26), the operators $\mathcal{S}_k$ and $\mathcal{B}_k$ are related via $\mathcal{S}_k = \mathcal{D}_\psi^{-1}\mathcal{B}_k\mathcal{D}_\psi$. Inherited from this relationship, there exists a one-to-one correspondence between Λ_k and Θ_k. To obtain this correspondence, first note that both functions B_k and S_k are derived from the auxiliary function $\widehat{\mathrm{S}}_k$ of (24), which has been shown in [16] to be a quadratic of the form

$$\widehat{\mathrm{S}}_k(x,z) = \frac{1}{2}\begin{bmatrix} x \\ z \end{bmatrix}^T Q_k \begin{bmatrix} x \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} Q_k^{11} & Q_k^{12} \\ Q_k^{21} & Q_k^{22} \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}. \tag{40}$$

Thus, the correspondence between Λ_k and Θ_k can be derived from the respective mapping between Λ_k and Q_k, and the mapping between Θ_k and Q_k. It has been proved in [16] that Q_k and Θ_k of (28) are related by

$$Q_k = \Gamma(\Theta_k), \quad \Theta_k = \Gamma^{-1}(Q_k). \tag{41}$$

Here, the operators $\Gamma : \mathbb{M}^{2n\times 2n} \to \mathbb{M}^{2n\times 2n}$ is defined by

$$\Gamma(\Omega) \doteq \Gamma\left(\begin{bmatrix} \Omega^{11} & \Omega^{12} \\ \Omega^{21} & \Omega^{22} \end{bmatrix}\right) = \begin{bmatrix} M - M\Pi_1^{-1}M & M\Pi_1^{-1}\Omega^{12} \\ \Omega^{21}\Pi_1^{-1}M & \Omega^{22} - \Omega^{21}\Pi_1^{-1}\Omega^{12} \end{bmatrix} \tag{42}$$

with $\Pi_1 = M + \Omega^{11}$, and the Γ^{-1} is given by

$$\Gamma^{-1}(\Omega) = -\Gamma(-\Omega) = \begin{bmatrix} M\Pi_2^{-1}M - M & M\Pi_2^{-1}\Omega^{12} \\ \Omega^{21}\Pi_2^{-1}M & \Omega^{21}\Pi_2^{-1}\Omega^{12} + \Omega^{22} \end{bmatrix} \tag{43}$$

with $\Pi_2 = M - \Omega^{11}$. To derive the mapping between Q_k and Λ_k of (38), define a matrix operation $\Delta : \mathbb{M}^{2n\times 2n} \to \mathbb{M}^{2n\times 2n}$ by

$$\Delta(\Omega) = \Delta\left(\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}\right) \doteq \begin{bmatrix} \Omega_{22} & \Omega_{21} \\ \Omega_{12} & \Omega_{11} \end{bmatrix}. \tag{44}$$

By inspection of (44), $\Delta^{-1} = \Delta$. Define a second matrix operation $\Pi : \mathbb{M}^{2n\times 2n} \to \mathbb{M}^{2n\times 2n}$ via the composition

$$\Pi \doteq \Delta\Gamma\Delta, \tag{45}$$

where $\Gamma, \Delta : \mathbb{M}^{2n\times 2n}$ are as per (42) and (44). From (43), $\Gamma^{-1}(\Omega) = -\Gamma(-\Omega)$, so that

$$\begin{aligned} \Pi^{-1}(\Omega) &= (\Delta\Gamma\Delta)^{-1}(\Omega) = (\Delta^{-1}\Gamma^{-1}\Delta^{-1})(\Omega) \\ &= \Delta(-\Gamma(-\Delta(\Omega))) = -(\Delta\Gamma\Delta)(-\Omega) \\ &= -\Pi(-\Omega). \end{aligned} \tag{46}$$

The matrix operations Π and Π^{-1} characterise the correspondence between Q_k of (40) and Λ_k of (38). They correspond to equations (31) and (32).

Theorem 5.3: The matrices Q_k of (40) and Λ_k of (38) satisfy

$$Q_k = \Pi(\Lambda_k), \quad \Lambda_k = \Pi^{-1}(Q_k). \tag{47}$$

Proof: From the definition (45) of Π,

$$\begin{aligned} \mathrm{S}_k(x,y) &= \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^T \Lambda_k \begin{bmatrix} x \\ y \end{bmatrix} = \left(\mathcal{D}_\psi \widehat{S}_k(x,\cdot)\right)(y) \\ &= -\int_{\mathbb{R}^n}^{\oplus} \psi(z,y)\otimes(-\widehat{\mathrm{S}}_k(x,z))\, dz \\ &= -\max_{z\in\mathbb{R}^n}\left\{\psi(z,y) - \frac{1}{2}\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} Q_k^{11} & Q_k^{12} \\ Q_k^{21} & Q_k^{22} \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}\right\} \\ &= -\max_{z\in\mathbb{R}^n}\left\{\psi(z,y) + \frac{1}{2}\begin{bmatrix} z \\ x \end{bmatrix}^T \Delta(-Q_k)\begin{bmatrix} z \\ x \end{bmatrix}\right\} \\ &= -\frac{1}{2}\begin{bmatrix} y \\ x \end{bmatrix}^T \Gamma(\Delta(-Q_k))\begin{bmatrix} y \\ x \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^T (-\Delta\Gamma\Delta)(-Q_k)\begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

$$= \frac{1}{2} \begin{bmatrix} x \\ z \end{bmatrix}^T \Pi^{-1}(Q_k) \begin{bmatrix} x \\ z \end{bmatrix}.$$

That is $\Lambda_k = \Pi^{-1}(Q_k)$. Since Π is an invertible operator, it follows that $Q_k = \Pi(\Lambda_k)$.

■

To obtain the correspondence between Λ_k and Θ_k, define a matrix operation $\Xi \doteq \Pi^{-1}\Gamma$. The inverse Ξ^{-1} is given by

$$\Xi^{-1}(\Omega) = (\Pi^{-1}\Gamma)^{-1}(\Omega) = (\Gamma^{-1}\Pi)(\Omega) = -\Gamma(-\Pi(\Omega)).$$

Theorem 5.4: The matrices Λ_k of (38) and Θ_k of (28) satisfy

$$\Lambda_k = \Xi(\Theta_k), \quad \Theta_k = \Xi^{-1}(\Lambda_k). \tag{48}$$

Proof: From (41) and (47),

$$\begin{aligned} \Lambda_k &= \Pi^{-1}(Q_k) = \Pi^{-1}(\Gamma(\Theta_k)) \\ &= (\Pi^{-1}\Gamma)(\Theta_k) = \Xi(\Theta_k). \end{aligned}$$

■

The correspondence among matrices Λ_k, Q_k, Θ_k follows (41), (47) and (48), which is summarised in Figure 2.

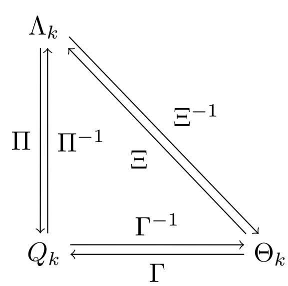

Fig. 2. Correspondence among matrices Q_k, Θ_k, and Λ_k, of (40), (28), and (38).

D. Propagation of the max-plus primal space fundamental solution

It has been shown in [13], [16] that the propagation of the max-plus dual space fundamental solution $\{\Theta_k, k \in \mathbb{N}\}$ of (28) is specified by

$$\Theta_{k_1+k_2} = \Theta_{k_1} \circledast \Theta_{k_2}, \quad \forall\ k_1, k_2 \in \mathbb{N}.$$

Here, the $\circledast$ operation on two matrices $\Omega_1, \Omega_2 \in \mathbb{M}^{2n\times 2n}$ is defined by

$$\Omega_1 \circledast \Omega_2 \doteq \begin{bmatrix} \Omega_1^{11} - \Omega_1^{12}\Pi_3^{-1}\Omega_1^{21} & -\Omega_1^{12}\Pi_3^{-1}\Omega_2^{12} \\ -\Omega_2^{21}\Pi_3^{-1}\Omega_1^{21} & \Omega_2^{22} - \Omega_2^{21}\Pi_3^{-1}\Omega_2^{12} \end{bmatrix}. \tag{49}$$

with $\Pi_3 = \Omega_1^{22} + \Omega_2^{11}$, where the partitioning of Ω_1 and Ω_2 is as per (42). In a similar way to the dual space fundamental solution, it can be shown that the propagation of the primal space fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$ follows the same rule.

Theorem 5.5: Suppose that the sequence $\{\Lambda_k, k \in \mathbb{N}\}$ of (38) exists. Then, for any $k_1, k_2 \in \mathbb{N}$,

$$\Lambda_{k_1+k_2} = \Lambda_{k_1} \circledast \Lambda_{k_2}. \tag{50}$$

Proof: For any $k_1, k_2 \in \mathbb{N}$ and $\phi \in \mathsf{dom}(\mathcal{S}_{k_1+k_2})$, from (34),

$$\begin{aligned} (\mathcal{S}_{k_1+k_2}\phi)(x) &= \int_{\mathbb{R}^n}^{\oplus} \mathrm{S}_{k_1+k_2}(x,y) \otimes \phi(y)\, dy \\ &= (\mathcal{S}_{k_1}(\mathcal{S}_{k_2}\phi))(x) \\ &= \int_{\mathbb{R}^n}^{\oplus} \mathrm{S}_{k_1}(x,\rho) \otimes (\mathcal{S}_{k_2}\phi)(\rho)\, d\rho \\ &= \int_{\mathbb{R}^n}^{\oplus} \mathrm{S}_{k_1}(x,\rho) \otimes \left(\int_{\mathbb{R}^n}^{\oplus} \mathrm{S}_{k_2}(\rho,y) \otimes \phi(y)\, dy \right) d\rho \\ &= \int_{\mathbb{R}^n}^{\oplus} \left(\int_{\mathbb{R}^n}^{\oplus} \mathrm{S}_{k_1}(x,\rho) \otimes \mathrm{S}_{k_2}(\rho,y)\, d\rho \right) \otimes \phi(y)\, dy. \end{aligned}$$

Since $\phi \in \mathsf{dom}(\mathcal{S}_{k_1+k_2})$ is arbitrary, it follows that

$$\mathrm{S}_{k_1+k_2}(x,y) = \int_{\mathbb{R}^n}^{\oplus} \mathrm{S}_{k_1}(x,\rho) \otimes \mathrm{S}_{k_2}(\rho,y)\, d\rho. \tag{51}$$

Applying the quadratic form of (38) for $\mathrm{S}_{k_1+k_2}, \mathrm{S}_{k_1}$ and S_{k_2} and evaluating the quadratic maximisation with respect to $\rho \in \mathbb{R}^n$ in (51) explicitly yields (50). ■

The propagations of the max-plus fundamental solutions $\{\Lambda_k, k \in \mathbb{N}\}$ and $\{\Theta_k, k \in \mathbb{N}\}$ are summarized in Figure 3.

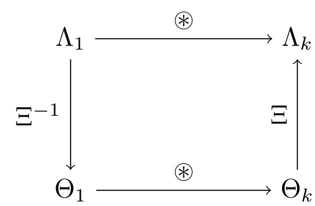

Fig. 3. Propagation of $\{\Lambda_k, k \in \mathbb{N}\}$ and $\{\Theta_k, k \in \mathbb{N}\}$.

E. The primal space fundamental solution interpreted as a Max-plus Green's function

In [7], the concept of max-plus green's function was proposed in the context of two point boundary value problems for infinite dimensional systems. Similarly, the kernel S_k of (31) can be interpreted as a max-plus green's function. In particular, define the max-plus delta function $\delta_y : \mathbb{R}^n \to \mathbb{R}^-$ centred at $y \in \mathbb{R}^n$, by

$$\delta_y(x) \doteq \begin{cases} 0, & x = y, \\ -\infty, & x \neq y. \end{cases} \tag{52}$$

Note that 0 is the multiplication identity and $-\infty$ is the additive identity in the max-plus algebra $(\mathbb{R}^-, \oplus, \otimes)$. That is, $0 \otimes a = a$ and $-\infty \oplus a = a$ for any $a \in \mathbb{R}^-$. Consequently, the function δ_y of (52) can be interpreted as a delta function defined with respect to the max-plus algebra, with $\int_{\mathbb{R}^n}^{\oplus} \delta_y(x)\, dx = 0$. Futhermore, applying the dynamic programming evolution operator $\mathcal{S}_k$ of (6) to δ_y yields

$$(\mathcal{S}_k \delta_y)(x) = \int_{\mathbb{R}^n}^{\oplus} \mathrm{S}_k(x,\rho) \otimes \delta_y(\rho)\, d\rho = \mathrm{S}_k(x,y). \tag{53}$$

Thus, the kernel function S_k (31) has an interpretation of a particular value function $(\mathcal{S}_k \delta_y)(x)$. That is, the function

S_k is a Green's function for the integral operator $\mathcal{S}_k$ over max-plus algebra $(\mathbb{R}^-, \oplus, \otimes)$.

It is also interesting to note that the max-plus delta function δ_y of (52) can be regarded as the limit of the quadratic function ψ of (14), where the Hessian $M \rightarrow -\infty\, I$. When $\psi(x,y) = \delta_y(x)$, the max-plus duality operators $\mathcal{D}_\psi$ and $\mathcal{D}_\psi^{-1}$ of (12) and (13) simplify to the special case in which $\mathcal{D}_\psi(\phi) = \phi = \mathcal{D}_\psi^{-1}\phi$. Thus, in this limit, the primal space fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$ and the dual space fundamental solution $\{\Theta_k, \mathbb{N}\}$ are the same. The discovery of S_k of (31) via the limit of B_k as $M \rightarrow -\infty\, I$ was discussed in [14].

VI. Conclusions

A new max-plus fundamental solution to a class of difference Riccati equations (DREs) is developed. This max-plus fundamental solution admits computation of all solutions of a DRE in a specified class, without invocation of duality via the Legendre-Fenchel transform. Connections between this new primal space fundamental solution, and the previously known dual space fundamental solution, are established.

References

[1] B.D.O Anderson and J.B. Moore. *Optimal Control: Linear Quadratic Methods*. Prentice-Hall, Englewood Cliffs, N.J., 1989.

[2] D.P. Bertsekas. *Dynamic Programming and Optimal Control*. Athena Scientific, 3rd edition, 2005.

[3] F.M. Callier and J.J. Winkin. Asymptotic behaviours of the solution of the projection Riccati differential equation. *IEEE Transactions on Automatic Control*, 41(5):646–659, 1996.

[4] E.J. Davison and M.C. Maki. The numerical solution of the matrix Riccati differential equation. *IEEE Transactions on Automatic Control*, 18:71–73, 1973.

[5] P.M. Dower and W.M. McEneaney. A max-plus based fundamental solution for a class of infinite dimensional Riccati equations. In *Proc.* 50^{th} *IEEE Conference on Decision and Control / European Control Conference*, pages 615–620, 2011.

[6] P.M. Dower and W.M. McEneaney. A max-plus method for optimal control of a diffusion equation. In *Proc.* 51^{st} *IEEE Conference on Decision & Control (Maui HI, USA)*, pages 618–623, 2012.

[7] P.M. Dower and W.M. McEneaney. A fundamental solution for an infinite dimensional two-point boundary value problem via the principle of stationary action. In *Proc. of the* 3^{rd} *Australian Control Conference (Perth, Australia)*, 2013.

[8] P.M. Dower and W.M. McEneaney. A max-plus dual space fundamental solution for a class of operator differential Riccati equations. *In review, SIAM J. Control & Optimization, preprint arXiv:1404.7209*, 2014.

[9] C.S. Kenney and R. Leipnik. Numerical integration of the differential matrix Riccati equation. *IEEE Transactions on Automatic Control*, 30:962–970, 1985.

[10] C. Kojima, K. Takaba, O. Kaneko, and P. Rapisarda. A characterization of solutions of the discrete-time algebraic Riccati equation based on quadratic difference forms. *Linear Algebra and its Applications*, 416:1060–1082, 2006.

[11] F.L. Lewis and V. Syrmos. *Optimal Control*. John Wiley & Sons, 1995.

[12] W.M. McEneaney. *Max-plus methods for nonlinear control and estimation*. Systems & Control: Foundations & Applications. Birkhauser, 2006.

[13] W.M. McEneaney. A new fundamental solution for differential Riccati equations arising in control. *Automatica*, 44(4):920–936, 2008.

[14] W.M. McEneaney and P.M. Dower. The principle of least action and fundamental solutions of mass-spring and N-body two-point boundary value problems. *In review for SIAM Journal on Control and Optimization*, 2013.

[15] H. Zhang and P.M. Dower. Analysis of difference Riccati equations via a new max-plus based fundamental solution. In *MTNS2014*, 2014.

[16] H. Zhang and P.M. Dower. A max-plus based fundamental solution for a class of discrete time linear regulator problems. *Revised for Linear Algebra and its Applications, preprint ArXiv:1306.5060*, 2014.

[17] H. Zhang and P.M. Dower. Max-plus fundamental solution semigroups for a class of difference Riccati equations. *In review, Automatica, preprint arXiv:1404.7593*, 2014.