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Analysis of difference Riccati equations via a new max-plus based fundamental solution

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Abstract— **The recently developed max-plus primal space fundamental solution provides a new explicit representation of solutions to the difference Riccati equation (DRE) that is ubiquitous in system and optimal control / filtering theory. This representation provides a new tool to explore various key properties of the DRE. This paper presents new results on these properties for a class of DREs via the associated max-plus primal space fundamental solution.**

I. INTRODUCTION

The difference Riccati equation (DRE) (differential Riccati equation in continuous time) plays a central role in optimal control theory. In particular, the solution of a DRE corresponds to the Hessian of the quadratic optimal cost / payoff of an associated linear quadratic regulator (LQR) problem; see e.g. [1], [2], [10] and the reference therein. As a consequence of its fundamental importance, the DRE has been one of the most intensively studied equations in the field of systems and control [4], [5], [6], [14], [15], [16]. Indeed, after more than 40 years of intensive research, the study of DREs still remains an active research topic, with many interesting new results [3], [9], [11], [12], [13]. Of these recent research, particular attention is paid to the study of the fundamental properties of existence [3], convergence [9], [12] and rate of convergence [13] of the solutions.

This paper presents a new perspective on these properties by using a new tool that was created in applying the max-plus based methods to optimal control problems [7], [8], [11], [18], [19]. In particular, a new closed-form representation of the DRE solution provided by a max-plus primal space fundamental solution [18], [19] is exploited in the derivation of new results on the aforementioned DRE properties. It is known that the closed-form representation of DRE solution via the Bernoulli substitution, e.g. Davison-Maki solution [6], is inconvenient for studying the important properties of the DRE such as finite escape, convergence and mechanism for attraction [12]. In this paper, the max-plus primal space fundamental solution [18], [19] is shown to be particularly useful in the study of these properties.

In terms of organization, Section II introduces the class of DREs and the max-plus primal space fundamental solution. In Section III, the connection between the DREs and the max-plus primal space fundamental solution is explored which presents results on various properties of the DREs via the fundamental solution. Two examples demonstrating various results in Section III are given in Section IV followed by conclusions in Section V.

In terms of notation, $\mathbb{R}, \mathbb{N}, \mathbb{Z}_{\geq 0}$ denote the set of reals, natural numbers and non-negative integers. The set of extended reals is denoted by $\mathbb{R}^- \doteq \mathbb{R} \cup \{-\infty\}$ and $\mathbb{R}^+ \doteq \mathbb{R} \cup \{\infty\}$. $\mathbb{R}^n$ denotes the standard n-dimensional real Euclidean space, and $\mathbb{R}^n_{>0} \doteq \{[x_1, x_2, \cdots, x_n]^T \in \mathbb{R}^n \,|\, x_i > 0, i = 1, 2, \cdots, n\}$. The set of $n \times n$ real, symmetric matrices are denoted by $\mathbb{M}^{n\times n} \doteq \{P \in \mathbb{R}^{n\times n} | P = P^T\}$. The triple $(\mathbb{R}^-, \oplus, \otimes)$ denotes the max-plus algebra (a semiring), where $\oplus$ and $\otimes$ denote the max-plus addition and multiplication operations defined by $a \oplus b \doteq \max\{a, b\}$ and $a \otimes b \doteq a + b$ for $a, b \in \mathbb{R}^-$. The max-plus integral of a function $f : \mathbb{R}^n \to \mathbb{R}^-$ is $\int^{\oplus}_{\mathbb{R}^n} f(x)\, dx \doteq \sup_{x\in\mathbb{R}^n}\{f(x)\}$. The spectrum of a matrix $\Omega \in \mathbb{R}^{n\times n}$ is denoted by $\sigma(\Omega)$, and its spectral radius is denoted by $\rho(\Omega) \doteq \max\{|\lambda| \,|\, \lambda \in \sigma(\Omega)\}$. $\|\Omega\|_2 \doteq \sqrt{\rho(\Omega^T\Omega)}$ denotes the spectra norm of $\Omega \in \mathbb{R}^{n\times n}$. For a transfer function $T(z)$ of a discrete time linear time invariant system, $\|T(\cdot)\|_\infty$ denotes to its $\mathcal{H}_\infty$-norm. The identity operator is denoted by $\mathcal{I}$.

When the required inverses exist, the identity (1) known as Woodbury's lemma holds [20]:

II. CLASS OF DREs AND THEIR MAX-PLUS PRIMAL SPACE FUNDAMENTAL SOLUTION

A. A class of DREs

Consider the following difference Riccati equation (DRE)

$$P_{k+1} = \mathcal{R}(P_k), \quad P_0 \in \mathbb{M}^{n\times n}, \tag{2}$$

with the operator $\mathcal{R} : \mathbb{M}^{n\times n} \to \mathbb{M}^{n\times n}$ given by

$$\mathcal{R}(P) \doteq A^T PA + A^T PB(\gamma^2 I - B^T PB)^{-1} B^T PA + \Phi. \tag{3}$$

Here, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}, n \geq m$, $\Phi \in \mathbb{M}^{n\times n}, \Phi > 0$ are real matrices, and the gain $\gamma > 0$ is a real number. This paper considers the special case where $B \in \mathbb{R}^{n\times n}$ is invertible. The general case is considered in a full version. The DRE (2) is an example of the indefinite DRE [13], [14], [15], [20] as the so-called Popov matrix $\Pi = \begin{bmatrix} \Phi & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$ is indefinite. When $k \to \infty$, the DRE (2) turns into the corresponding algebraic Riccati equation (ARE)

$$P = \mathcal{R}(P). \tag{4}$$

Let $C \in \mathbb{R}^{n\times n}$ be such that $C^T C = \Phi$, and $T(z) \doteq C(z\,I - A)^{-1}B$ be the transfer function of a discrete-time

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$$(\Omega^{11} - \Omega^{12}(\Omega^{22})^{-1}\Omega^{21})^{-1} = (\Omega^{11})^{-1} + (\Omega^{11})^{-1}\Omega^{12}(\Omega^{22} - \Omega^{21}(\Omega^{11})^{-1}\Omega^{12})^{-1}\Omega^{21}(\Omega^{11})^{-1}. \tag{1}$$

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linear system

$$\left\{ \begin{array}{rl} x_{k+1} & = Ax_k + Bw_k, \quad x_0 = x \in \mathbb{R}^n, \\ y_k & = Cx_k. \end{array} \right. \tag{5}$$

It is known [20] that there exist two extreme solutions $P_-, P_+ \in \mathbb{M}^{n\times n}, P_- < P_+$ to the ARE (4) if $\|T(\cdot)\|_\infty < \gamma$. Any other solution $P \in \mathbb{M}^{n\times n}$ satisfies $P_- \leq P \leq P_+$. Furthermore, the minimum solution P_- is stabilising and the maximum solution P_+ is anti-stabilising.

The DRE (2) arises from the study of the optimal control of the linear system (5), in which the value function $W_k : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$W_k(x) \doteq \sup_{w_{0,k-1}\in(\mathbb{R}^m)^k} J_k(x; w_{0,k-1}) \tag{6}$$

with

$$J_k(x; w_{0,k-1}) \doteq \sum_{i=0}^{k-1} \left(\frac{1}{2} x_i^T \Phi x_i - \frac{1}{2}\gamma^2 w_i^T w_i \right) + \frac{1}{2} x_k^T P_0 x_k. \tag{7}$$

It is known from the linear quadratic regulator (LQR) theory [1] that W_k is quadratic of the form $W_k(x) = \frac{1}{2} x^T P_k\, x, x \in \mathbb{R}^n$, in which $P_k \in \mathbb{M}^{n\times n}$ is the solution of the DRE (2) with initial condition P_0.

If define operators $\mathcal{R}_k : \mathbb{M}^{n\times n} \to \mathbb{M}^{n\times n}, k \in \mathbb{Z}_{\geq 0}$ iteratively by

$$\mathcal{R}_{k+1} = \mathcal{R} \circ \mathcal{R}_k, \quad \mathcal{R}_0 = \mathcal{I}, \tag{8}$$

then, the solution P_k to the DRE (2) at time k is $P_k = \mathcal{R}_k(P_0)$. That is, $\mathcal{R}_k$ is the operator that propagates the solution of the DRE (2) from initial condition $P_0 \in \mathbb{M}^{n\times n}$. To study the existence of a solution P_k to the DRE (2), define the domain of the operator $\mathcal{R}_k$ to be

$$\mathsf{dom}(\mathcal{R}_k) \doteq \left\{ P \in \mathbb{M}^{n\times n} \middle| \begin{array}{c} \gamma^2\, I - B^T \mathcal{R}_i(P) B > 0, \\ i = 1, 2, \cdots, k-1 \end{array} \right\}. \tag{9}$$

By inspection, $\mathsf{dom}(\mathcal{R}_{k+1}) \subset \mathsf{dom}(\mathcal{R}_k)$ for $k \in \mathbb{N}$.

B. Max-plus primal space fundamental solution

This paper employs a new representation of the solution $P_k = \mathcal{R}_k(P)$ via a max-plus primal space fundamental solution in [18] and [19] to study the various properties of the DRE (2). In particular, it is shown that the DRE (2) is closely related with a particular sequence of matrices $\{\Lambda_k \in \mathbb{R}^{2n\times 2n}, k \in \mathbb{N}\}$ [18], [19] that is generated iteratively by

$$\Lambda_{k+1} = \Lambda_1 \circledast \Lambda_k, \quad \Lambda_1 = \Lambda, \tag{10}$$

with

$$\Lambda = \left[\begin{array}{cc} \Phi - \gamma^2 A^T (BB^T)^{-1} A & \gamma^2 A^T (BB^T)^{-1} \\ \gamma^2 (BB^T)^{-1} A & -\gamma^2 (BB^T)^{-1} \end{array} \right]. \tag{11}$$

Here, the operation $\circledast$ on two matrices $\Omega_i \in \mathbb{R}^{2n\times 2n}, i = 1, 2$ of the form $\Omega_i = \left[\begin{array}{cc} \Omega_i^{11} & \Omega_i^{12} \\ \Omega_i^{21} & \Omega_i^{22} \end{array} \right]$ such that $\Omega_i = \Omega_i^T$ and $\Omega_1^{22} + \Omega_2^{11} < 0$ is defined by [11], [18]

$$\Omega_1 \circledast \Omega_2 \doteq \left[\begin{array}{cc} \Omega_1^{11} - \Omega_1^{12}(\Omega_d)^{-1}\Omega_1^{21} & -\Omega_1^{12}(\Omega_d)^{-1}\Omega_2^{12} \\ -\Omega_2^{21}(\Omega_d)^{-1}\Omega_1^{21} & \Omega_2^{22} - \Omega_2^{21}(\Omega_d)^{-1}\Omega_2^{12} \end{array} \right] \tag{12}$$

with $\Omega_d = \Omega_1^{22} + \Omega_2^{11}$. Note that, in general, the $\circledast$ operation is non-commutative. That is, $\Omega_1 \circledast \Omega_2 \neq \Omega_2 \circledast \Omega_1$, however, it can be verified that $\Lambda_{k_1} \circledast \Lambda_{k_2} = \Lambda_{k_2} \circledast \Lambda_{k_1}, k_1, k_2 \in \mathbb{N}$ for the sequence $\{\Lambda_k, k \in \mathbb{N}\}$ of (10). Thus, $\{\Lambda_k, k \in \mathbb{N}\}$ of (10) maybe regarded as exponentiation

$$\Lambda_k = \Lambda^{\circledast k} \doteq \underbrace{\Lambda \circledast \Lambda \circledast \cdots \circledast \Lambda}_{k}.$$

This particular sequence $\{\Lambda_k, k \in \mathbb{N}\}$ is the max-plus primal space fundamental solution [18], [19] in the sense that it can be used to represent all solutions $\mathcal{R}_k(P_0)$ when it exists and $P_0 + \Lambda_k^{22} < 0$,

$$\mathcal{R}_k(P_0) = \Lambda_k^{11} - \Lambda_k^{12}(P_0 + \Lambda_k^{22})^{-1}\Lambda_k^{21}. \tag{13}$$

C. Convergence of fundamental solution

The convergence of the max-plus primal space fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$ of (10) will be useful in the investigation of the infinite horizon limit of the solutions $\mathcal{R}_k$ to the DRE (2) in Section III-B. It may be noted that the sequence $\{\Lambda_k, k \in \mathbb{N}\}$ is completely determined by the initial matrix Λ of (11). A sufficient condition for the convergence of $\{\Lambda_k, k \in \mathbb{N}\}$ has been developed in [17], [18]. Here, a similar but less conservative condition is proposed (see Theorem 2.3). To this end, the convergence of a sequence $\{\lambda_k, k \in \mathbb{N}\}$ that is generated by

$$\lambda_{k+1} = \lambda - \sigma\lambda_k^{-1}, \quad \lambda_1 = \lambda, \tag{14}$$

with a given pair $(\lambda, \sigma) \in \mathbb{R}^2_{>0}$, is presented first.

Lemma 2.1: Suppose that $(\lambda, \sigma) \in \mathbb{R}^2_{>0}$ satisfies $\lambda > 2\sqrt{\sigma}$. Then, the sequence $\{\lambda_k, k \in \mathbb{N}\}$ of (14) satisfies $\lambda_{k+1} \leq \lambda_k, k \in \mathbb{N}$ and $\lambda_k \to \lambda_\infty, k \to \infty$ with

$$\lambda_\infty \doteq \frac{\lambda + \sqrt{\lambda^2 - 4\sigma}}{2}. \tag{15}$$

Proof: It can be verified directly that λ_∞ of (15) is an equilibrium of (14), that is, $\lambda_\infty = \lambda - \sigma{\lambda_\infty}^{-1}$. An inductive argument is used to show the monotonicity and convergence of the sequence $\{\lambda_k, k \in \mathbb{N}\}$. By inspection of the definition of λ_∞ of (15), it is clear that $\lambda > \lambda_\infty > 0$ by the assumption $(\lambda, \sigma) \in \mathbb{R}^2_{>0}$ and $\lambda > 2\sqrt{\sigma}$. Then, $\lambda_1 > \lambda_1 - \sigma\lambda_1^{-1} = \lambda - \sigma\lambda_1^{-1} = \lambda_2$. and $\lambda_2 = \lambda - \sigma\lambda_1^{-1} > \lambda - \sigma\lambda_\infty^{-1} = \lambda_\infty$. To apply the mathematical inductive argument, assume that for $k \in \mathbb{N}$, $\lambda_k > \lambda_{k+1} > \lambda_\infty$. Then, for $k+1$, $\lambda_{k+1} = \lambda - \sigma\lambda_k^{-1} > \lambda - \sigma\lambda_{k+1}^{-1} = \lambda_{k+2}$ and $\lambda_{k+2} = \lambda - \sigma\lambda_{k+1}^{-1} > \lambda - \sigma\lambda_\infty^{-1} = \lambda_\infty$. Thus, the sequence $\{\lambda_k, k \in \mathbb{N}\}$ strictly decreases and converges to a number $\tilde{\lambda} \geq \lambda_\infty$, and $\tilde{\lambda}$ must be an equilibrium point of the dynamics $\eta_{k+1} = \lambda - \sigma \eta_k^{-1}$. But λ_∞ is the unique equilibrium point of this dynamics on set $\{\eta \in \mathbb{R} \,|\, \eta \geq \lambda_\infty\}$. So $\tilde{\lambda} = \lambda_\infty$. ■

Remark 2.2: Suppose that $(\lambda, \sigma, \lambda_\infty) \in \mathbb{R}^3_{>0}$ are given as per Lemma 2.1 . Then, the sequence $\{\sigma_k, k \in \mathbb{N}\}$ that is defined by

$$\sigma_{k+1} = (\sigma \lambda_\infty^{-2})\, \sigma_k, \quad \sigma_1 = \sigma \tag{16}$$

decreases and $\sigma_k \to 0$ if and only if $\sigma \lambda_\infty^{-2} < 1$. This is immediate from the fact that $\sigma_k = (\sigma \lambda_\infty^{-2})^{k-1} \sigma, k \in \mathbb{N}$.

The next theorem provides sufficient conditions for the monotonicity properties of the fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$.

Theorem 2.3: Suppose that there exists $(\lambda, \sigma) \in \mathbb{R}^2_{>0}$ such that $\lambda > 2\sqrt{\sigma}$ and $\sigma \lambda_\infty^{-2} < 1$ with λ_∞ given by (15), and that the matrix Λ of (11) satisfies

$$\left\{ \begin{array}{l} \Lambda^{11} + \Lambda^{22} \leq -\lambda\, I, \\ \Lambda^{12}\Lambda^{21} \leq \sigma\, I, \quad \Lambda^{21}\Lambda^{12} \leq \sigma\, I. \end{array} \right. \tag{17}$$

Then, the fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$ of (10) satisfies for all $k \in \mathbb{N}$

$$\left\{ \begin{array}{l} \Lambda^{11}_{k+1} \geq \Lambda^{11}_k, \quad \Lambda^{22}_{k+1} \geq \Lambda^{22}_k, \\ \Lambda^{22} + \Lambda^{11}_k \leq -\lambda_k\, I, \quad \Lambda^{11} + \Lambda^{22}_k \leq -\lambda_k\, I \end{array} \right. \tag{18}$$

and

$$\left\{ \begin{array}{l} \Lambda^{12}_{k+1}\Lambda^{21}_{k+1} \leq \Lambda^{12}_k\Lambda^{21}_k, \quad \Lambda^{21}_{k+1}\Lambda^{12}_{k+1} \leq \Lambda^{21}_k\Lambda^{12}_k, \\ \Lambda^{12}_k\Lambda^{21}_k \leq \sigma_k\, I, \quad \Lambda^{21}_k\Lambda^{12}_k \leq \sigma_k\, I. \end{array} \right. \tag{19}$$

Here, the sequence $\{\lambda_k, \sigma_k\}_{k=1}^{\infty}$ is generated from (14) and (16), respectively.

Proof: From (10) and the $\circledast$ operation of (12),

$$\begin{aligned} &\left[\begin{array}{cc} \Lambda^{11}_{k+1} & \Lambda^{12}_{k+1} \\ \Lambda^{21}_{k+1} & \Lambda^{22}_{k+1} \end{array} \right] \\ &= \left[\begin{array}{cc} \Lambda^{11} - \Lambda^{12}(\Lambda_{d,k})^{-1}\Lambda^{21} & -\Lambda^{12}(\Lambda_{d,k})^{-1}\Lambda^{12}_k \\ -\Lambda^{21}_k(\Lambda_{d,k})^{-1}\Lambda^{21} & \Lambda^{22}_k - \Lambda^{21}_k(\Lambda_{d,k})^{-1}\Lambda^{12}_k \end{array} \right] \\ &= \left[\begin{array}{cc} \Lambda^{11}_k - \Lambda^{12}_k(\Lambda^2_{d,k})^{-1}\Lambda^{21}_k & -\Lambda^{12}_k(\Lambda^2_{d,k})^{-1}\Lambda^{12} \\ -\Lambda^{21}(\Lambda^2_{d,k})^{-1}\Lambda^{21}_k & \Lambda^{22} - \Lambda^{21}(\Lambda^2_{d,k})^{-1}\Lambda^{12} \end{array} \right]. \end{aligned} \tag{20}$$

with $\Lambda^1_{d,k} \doteq \Lambda^{22} + \Lambda^{11}_k$ and $\Lambda^2_{d,k} \doteq \Lambda^{22}_k + \Lambda^{11}$.

An inductive argument is used to prove (18) and (19). In order to show (18), let $k = 1$. From (20), $\Lambda^{11}_2 = \Lambda^{11}_1 - \Lambda^{12}_1(\Lambda^{22}_1 + \Lambda^{11}_1)^{-1}\Lambda^{21}_1 \geq \Lambda^{11}_1$ since $\Lambda_1 = \Lambda$, $\Lambda^{22} + \Lambda^{11} < -\lambda\, I = -\lambda_1\, I < 0$, and $\Lambda^{12} = (\Lambda^{21})^T$. The inequality (18) for $\Lambda^{22}_2 \geq \Lambda^{22}_1$ can be shown similarly. Applying the argument of mathematical induction, suppose that (18) holds for k, then (18) needs to be shown for $k+1$. From (20),

$$\Lambda^{11}_{k+1} = \Lambda^{11}_k - \Lambda^{12}_k(\Lambda^{22}_k + \Lambda^{11})^{-1}\Lambda^{21}_k \geq \Lambda^{11}_k.$$

Also,

$$\begin{aligned} \Lambda^{22} + \Lambda^{11}_{k+1} &= \Lambda^{22} + \Lambda^{11} - \Lambda^{12}(\Lambda^{22} + \Lambda^{11}_k)^{-1}\Lambda^{21} \\ &\leq -\lambda\, I - \Lambda^{12}(-\lambda_k^{-1}\, I)\Lambda^{21} \\ &\leq -(\lambda - \lambda_k^{-1}\sigma)\, I \leq -\lambda_{k+1}\, I\, \Lambda. \end{aligned}$$

The inequalities of $\Lambda^{22}_{k+1} \geq \Lambda^{22}_k$ and $\Lambda^{11} + \Lambda^{22}_{k+1} \leq -\lambda_{k+1}\, I$ can be shown by a similar argument.

A similar inductive argument is used to prove (19). For $k = 1$, the inequality $\Lambda^{12}_1\Lambda^{21}_1 \leq \sigma_1\, I = \sigma\, I$ is immediate from (17). From (20), the inequality $\lambda > \lambda_\infty$, and the assumption $\sigma\lambda_\infty^{-2} < 1$,

$$\begin{aligned} \Lambda^{21}_2\Lambda^{12}_2 &= \Lambda^{21}(\Lambda^{22} + \Lambda^{11})^{-1}\Lambda^{21}\Lambda^{12}(\Lambda^{22} + \Lambda^{11})^{-1}\Lambda^{12} \\ &\leq \sigma\, \Lambda^{21}(\Lambda^{22} + \Lambda^{11})^{-1}(\Lambda^{22} + \Lambda^{11})^{-1}\Lambda^{12} \\ &\leq (\sigma\lambda^{-2})\Lambda^{21}\Lambda^{12} = (\sigma\lambda^{-2})\Lambda^{21}_1\Lambda^{12}_1 \\ &\leq (\sigma{\lambda_\infty}^{-2})\Lambda^{21}_1\Lambda^{12}_1 < \Lambda^{21}_1\Lambda^{12}_1. \end{aligned}$$

The proof of the inequality $\Lambda^{12}_2\Lambda^{21}_2 \leq \Lambda^{12}_1\Lambda^{21}_1$ follows similarly. In order to use a mathematical inductive argument, suppose that (19) holds for $k \in \mathbb{N}$, it needs to show (19) for $k+1$. From (20),

$$\begin{aligned} \Lambda^{21}_{k+1}\Lambda^{12}_{k+1} &= \Lambda^{21}_k(\Lambda^{22} + \Lambda^{11}_k)^{-1}\Lambda^{21}\Lambda^{12}(\Lambda^{22} + \Lambda^{11}_k)^{-1}\Lambda^{12}_k \\ &\leq \sigma\, \Lambda^{21}_k(\Lambda^{22} + \Lambda^{11}_k)^{-1}(\Lambda^{22} + \Lambda^{11}_k)^{-1}\Lambda^{12}_k \\ &\leq (\sigma\lambda_k^{-2})\Lambda^{21}_k\Lambda^{12}_k \leq (\sigma{\lambda_\infty}^{-2})\Lambda^{21}_k\Lambda^{12}_k. \end{aligned} \tag{21}$$

Also by the assumption $\Lambda^{21}_k\Lambda^{12}_k \leq \sigma_k\, I$, (21) implies $\Lambda^{21}_{k+1}\Lambda^{12}_{k+1} \leq (\sigma\lambda_k^{-2})\sigma_k\, I = \sigma_{k+1}\, I$. The inequalities in (19) for $\Lambda^{12}_{k+1}\Lambda^{21}_{k+1}$ follow similarly.

■

The monotonicity properties in Theorem 2.3 imply the convergence of the fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$ as shown in the next theorem.

Theorem 2.4: Suppose that the conditions as per Theorem 2.3 hold for the matrix Λ of (11). Then, there exists a matrix $\Lambda_\infty = \left[\begin{array}{cc} \Lambda^{11}_\infty & 0 \\ 0 & \Lambda^{22}_\infty \end{array} \right] \in \mathbb{M}^{2n \times 2n}$ such that $\Lambda_k \to \Lambda_\infty$ as $k \to \infty$.

Proof: From (19) in Theorem 2.3, $\Lambda^{21}_k\Lambda^{12}_k \leq \sigma_k\, I$, $\Lambda^{12}_k\Lambda^{21}_k \leq \sigma_k\, I$ for all $k \in \mathbb{N}$. That is, $\|\Lambda^{21}_k\|_2^2 \leq \sigma_k, \|\Lambda^{12}_k\|_2^2 \leq \sigma_k$ since $\Lambda^{12}_k = (\Lambda^{21}_k)^T$. Thus, $\Lambda^{21}_k \to 0, \Lambda^{12}_k \to 0$ since $\sigma_k \to 0$. From (18), $\Lambda^{22}_1 + \Lambda^{11}_k \leq -\lambda_k\, I \leq -\lambda_\infty\, I$ and $\Lambda^{11}_1 + \Lambda^{22}_k \leq -\lambda_k\, I \leq -\lambda_\infty\, I$ for any $k \in \mathbb{N}$. Thus, $(-(\Lambda^{22}_1 + \Lambda^{11}_k))^{-1} \leq \lambda_\infty^{-1}\, I$ and $(-(\Lambda^{11}_1 + \Lambda^{22}_k))^{-1} \leq \lambda_\infty^{-1}\, I$. From (20),

$$\begin{aligned} \Lambda^{22}_{k+1} - \Lambda^{22}_k &= -\Lambda^{21}_k(\Lambda^{22}_1 + \Lambda^{11}_k)^{-1}\Lambda^{12}_k \\ &= \Lambda^{21}_k(-(\Lambda^{22}_1 + \Lambda^{11}_k))^{-1}\Lambda^{12}_k \\ &\leq \lambda_\infty^{-2}\Lambda^{21}_k\Lambda^{12}_k \leq \lambda_\infty^{-2}\sigma_k\, I. \end{aligned}$$

Hence, $\|\Lambda^{22}_{k+1} - \Lambda^{22}_k\|_2^2 \leq \lambda_\infty^{-2}\sigma_k \to 0$. Similarly, it can be shown that $\|\Lambda^{11}_{k+1} - \Lambda^{11}_k\| \to 0$. (18) also implies that Λ^{11}_k and Λ^{22}_k are bounded from above by $\Lambda^{11}_k \leq -\lambda_\infty\, I - \Lambda^{22}_1$ and $\Lambda^{22}_k \leq -\lambda_\infty - \Lambda^{11}_1$. Consequently, there exist $\Lambda^{11}_\infty, \Lambda^{22}_\infty \in \mathbb{M}^{n \times n}$ such that $\Lambda^{11}_k \to \Lambda^{11}_\infty$ and $\Lambda^{22}_k \to \Lambda^{22}_\infty$ as $k \to \infty$.

■

III. PROPERTIES OF THE DREs

The representation (13) of all solutions P_k to the DRE (2) via of the max-plus primal space fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$ of (10) provides a new tool to study various properties of the DRE. This paper concerns with three specific properties.

❶ Existence: Characterise the sets $\mathsf{dom}(\mathcal{R}_k)$ for all $k \in \mathbb{N}$;

❷ Convergence: Find conditions that guarantee the convergence of the propagation of $P_k = \mathcal{R}_k(P_0) \to \mathcal{R}_\infty(P_0) \in \mathbb{M}^{n\times n}$ as $k \to \infty$;

❸ Rate of convergence: Find bounds for the propagation error

$$E_k(P_0^1, P_0^2) \doteq \mathcal{R}_k(P_0^1) - \mathcal{R}_k(P_0^2) \tag{22}$$

for admissible $P_0^1, P_0^2 \in \mathsf{dom}(\mathcal{R}_k)$.

The existence property is investigated for the standard DREs in [3], and the convergence property for standard DREs arising from the LQR problem is an old topic with numerous mature results [1], [4], [5], [9], [12], [20] and the references therein. The rate of convergence of a class of continuous-time DRE was investigated in [13].

A. Existence of solutions

The representation (13) for the solution $P_k = \mathcal{R}_k(P_0)$ via the max-plus primal space fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$ holds when $P_k = \mathcal{R}_k(P_0)$ exists and $P_0 + \Lambda_k^{22} < 0$. The next theorem shows that the existence of P_k is equivalent to the inequality $P_0 + \Lambda_k^{22} < 0$.

Theorem 3.1: Suppose that the conditions on Theorem 2.3 holds for the matrix Λ of (11). Then, for any $k \in \mathbb{N}$,

$$P_0 \in \mathsf{dom}(\mathcal{R}_k) \quad \Longleftrightarrow \quad P_0 + \Lambda_k^{22} < 0. \tag{23}$$

Proof: In order to use an inductive argument, (23) is demonstrated for $k = 1$ first. From (9) and (11), $\Lambda_1^{22} = -\gamma^2(BB^T)^{-1}$. Thus,

$$\begin{aligned} &P_0 \in \mathsf{dom}(\mathcal{R}_1) \Longleftrightarrow \gamma^2 I - B^T P_0 B > 0 \Longleftrightarrow \\ &\gamma^2 (BB^T)^{-1} - P_0 > 0 \Longleftrightarrow -\Lambda_1^{22} - P_0 > 0 \Longleftrightarrow \\ &P_0 + \Lambda_1^{22} < 0. \end{aligned}$$

Suppose that (23) holds for k. That is, the representation of

$$\mathcal{R}_k(P_0) = \Lambda_k^{11} - \Lambda_{k-1}^{12}(P_0 + \Lambda_k^{22})^{-1}\Lambda_k^{21} \tag{24}$$

holds for $P_0 \in \mathsf{dom}(\mathcal{R}_k)$. It is required to show inequality (23) for $k+1$,

$$\gamma^2 I - B^T\mathcal{R}_k(P_0)B > 0 \quad \Longleftrightarrow \quad P_0 + \Lambda_{k+1}^{22} < 0. \tag{25}$$

Suppose first that $\gamma^2 I - B^T\mathcal{R}_k(P_0)B > 0$. Using (24),

$$\begin{aligned} &\gamma^2 I - B^T\mathcal{R}_k(P_0)B \\ &= \gamma^2 I - B^T(\Lambda_k^{11} - \Lambda_k^{12}(P_0 + \Lambda_k^{22})^{-1}\Lambda_k^{21})B > 0. \end{aligned} \tag{26}$$

Multiplying $(B^T)^{-1}$ from the left and B^{-1} from the right to the righthand side of (26) yields

$$-\gamma^2 (BB^T)^{-1} + \Lambda_k^{11} - \Lambda_k^{12}(P_0 + \Lambda_k^{22})^{-1}\Lambda_k^{21} < 0,$$

which is, by (11),

$$(\Lambda_1^{22} + \Lambda_k^{11}) - \Lambda_k^{12}(P_0 + \Lambda_k^{22})^{-1}\Lambda_k^{21} < 0. \tag{27}$$

On the otherhand, by (20),

$$P_0 + \Lambda_{k+1}^{22} = (P_0 + \Lambda_k^{22}) - \Lambda_k^{21}(\Lambda_1^{22} + \Lambda_k^{11})^{-1}\Lambda_k^{12}. \tag{28}$$

By the assumption for step k, $P_0 + \Lambda_k^{22} < 0$. It also holds $\Lambda_1^{22} + \Lambda_k^{11} \leq -\lambda_k I < 0$ from Theorem 2.3. Using (27), the following matrix is well defined

$$\begin{aligned} &(P_0 + \Lambda_k^{22})^{-1} + (P_0 + \Lambda_k^{22})^{-1}\Lambda_k^{21} \\ &\quad\big((\Lambda_1^{22} + \Lambda_k^{11}) - \Lambda_k^{12}(P_0 + \Lambda_k^{22})^{-1}\Lambda_k^{21}\big)^{-1} \\ &\quad\quad\Lambda_k^{12}(P_0 + \Lambda_k^{22})^{-1} < 0, \end{aligned}$$

which is $(P_0 + \Lambda_{k+1}^{22})^{-1} < 0$ by (28) and the matrix inverse equality (1).

Conversely, suppose that $P_0 + \Lambda_{k+1}^{22} < 0$. That is, $(P_0 + \Lambda_k^{22}) - \Lambda_k^{21}(\Lambda_1^{22} + \Lambda_k^{11})^{-1}\Lambda_k^{12} < 0$ by (20). By the invertibility of the matrices $\Lambda_1^{22} + \Lambda_k^{11} \leq -\lambda_k I < 0$ and $P_0 + \Lambda_k^{22} < 0$, the following matrix is well defined and negative definite

$$\begin{aligned} &(\Lambda_1^{22} + \Lambda_k^{11})^{-1} + (\Lambda_1^{22} + \Lambda_k^{11})^{-1}\Lambda_k^{12} \\ &\quad\big((P_0 + \Lambda_k^{22}) - \Lambda_k^{21}(\Lambda_1^{22} + \Lambda_k^{11})^{-1}\Lambda_k^{12}\big)^{-1} \\ &\quad\quad\Lambda_k^{21}(\Lambda_1^{22} + \Lambda_k^{11})^{-1} < 0, \end{aligned}$$

which is $((\Lambda_1^{22} + \Lambda_k^{11}) - \Lambda_k^{12}(P_0 + \Lambda_k^{22})^{-1}\Lambda_k^{21})^{-1} < 0$ by the matrix inverse equality (1). This implies that $\gamma^2 I - B^T\mathcal{R}_k(P_0)B > 0$ from (26) and (27). ■

Remark 3.2: The sequence $\{\Lambda_k, k \in \mathbb{N}\}$ is independent of the initial condition P_0 since it is a fundamental solution. Theorem 3.1 provides a necessary and sufficient condition for the existence of solutions $P_k = \mathcal{R}_k(P_0)$, with respect to an initial condition P_0, in terms of an inequality $P_0 + \Lambda_k^{22} < 0$. This provides a characterisation of the domain of the operator $\mathsf{dom}(\mathcal{R}_k) = \{P \in \mathbb{M}^{n\times n} \,|\, P < -\Lambda_k^{22}\}$.

Using the sequence $\{\lambda_k, k \in \mathbb{N}\}$, a sufficient condition for the existence of a solution $P_k = \mathcal{R}_k(P_0)$ can be derived without computing the fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$.

Corollary 3.3: Suppose that the conditions on Theorem 2.3 hold for Λ of (11). Then, for $k \in \mathbb{N}, k \geq 2$,

$$P_0 < \gamma^2 (BB^T)^{-1} - \sigma\lambda_{k-1}^{-1} I \quad \Longrightarrow \quad P_0 \in \mathsf{dom}(\mathcal{R}_k). \tag{29}$$

Proof: From (20) and Theorem 2.3,

$$\begin{aligned} P_0 + \Lambda_k^{22} &= P_0 + \Lambda_1^{22} - \Lambda_1^{21}(\Lambda_{k-1}^{22} + \Lambda_1^{11})^{-1}\Lambda_1^{12} \\ &= P_0 - \gamma^2 (BB^T)^{-1} - \Lambda_1^{21}(\Lambda_{k-1}^{22} + \Lambda_1^{11})^{-1}\Lambda_1^{12} \\ &\leq P_0 - \gamma^2 (BB^T)^{-1} + \lambda_{k-1}^{-1}\Lambda_1^{21}\Lambda_1^{12} \\ &\leq P_0 - \gamma^2 (BB^T)^{-1} + \sigma\lambda_{k-1}^{-1} I. \end{aligned}$$

Thus, $P_0 < \gamma^2 (BB^T)^{-1} - \sigma\lambda_{k-1}^{-1} I \Longrightarrow P_0 + \Lambda_k^{22} < 0$, which, by Theorem 3.1, implies that $P_0 \in \mathsf{dom}(\mathcal{R}_k)$. ■

B. Infinite horizon limit

The investigation of the infinite horizon limit of the solutions $P_k = \mathcal{R}_k(P_0)$ to the DRE (2) requires to identify those matrices $P_0 \in \mathbb{M}^{n\times n}$ such that $\mathcal{R}_k(P_0)$ converges as $k \to \infty$. To this end, a monotonicity property of the operator $\mathcal{R}_k$ is presented first.

Lemma 3.4: For any $k \in \mathbb{N}, P_0^1, P_0^2 \in \mathsf{dom}(\mathcal{R}_k)$

$$P_0^1 \leq P_0^2 \Longrightarrow \mathcal{R}_k(P_0^1) \leq \mathcal{R}_k(P_0^2). \tag{30}$$

Proof: It is shown in Section II-A that $P_k^i = \mathcal{R}_k(P_0^i), i = 1,2$ are the Hessian of the value function $W_k^i, i = 1,2$ of (6), that is,

$$\begin{aligned} W_k^i(x) &\doteq \tfrac{1}{2}x^T P_k^i x \\ &= \sup_{w_{0,k-1}\in(\mathbb{R}^n)^k} \left\{ \begin{array}{c} \displaystyle\sum_{j=0}^{k-1}\left(\tfrac{1}{2}x_j^T \Phi x_j - \frac{1}{2}\gamma^2 w_j^T w_j\right) + \\ \tfrac{1}{2}x_k^T P_0^i x_k. \end{array} \right\} \end{aligned}$$

It is immediate that $W_k^1(x) \le W_k^2(x), x \in \mathbb{R}^n$ for $P_0^1 \le P_0^2$. Hence $P_k^1 = \mathcal{R}_k(P_0^1) \le P_k^2 = \mathcal{R}_k(P_0^2)$. ■

Theorem 2.4 proposes a convergence result for the fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$. It is shown that the limit is a block diagonal matrix $\Lambda_\infty = \begin{bmatrix} \Lambda_\infty^{11} & 0 \\ 0 & \Lambda_\infty^{22} \end{bmatrix}$. Next, it is shown that the sequence $\{\Lambda_k^{11}, k \in \mathbb{N}\}$ is the minimum solution to the DRE (2), and its limit Λ_∞^{11} is the minimum solution P_- of the the ARE (4).

Theorem 3.5: For any $k \in \mathbb{N}$, suppose that $\{\Lambda_k, k \in \mathbb{N}\}$ of (10) exists . Then, $\Lambda_k^{11}, k \in \mathbb{N}$ satisfies the DRE (2), and $\Lambda_k^{11} \le \mathcal{R}_k(P_0)$ for any $P_0 \in \mathsf{dom}(\mathcal{R}_k)$.

Proof: The fact that the sequence $\{\Lambda_k^{11}, k \in \mathbb{N}\}$ satisfies the DRE (2) $\Lambda_{k+1}^{11} = \mathcal{R}(\Lambda_k^{11})$ is proved in Theorem 3.1 of [18]. Thus, $\Lambda_k^{11} = \mathcal{R}_{k-1}(\Lambda_1^{11})$. $P_0 \in \mathsf{dom}(\mathcal{R}_1)$ for any $P_0 \in \mathsf{dom}(\mathcal{R}_k)$ since $\mathsf{dom}(\mathcal{R}_k) \subset \mathsf{dom}(\mathcal{R}_1)$. Thus, $P_1 = \mathcal{R}_1(P_0)$ exists. For any $x \in \mathbb{R}^n$, take $w^* = -B^{-1}Ax$. From the definition of the value function $W_1(x), x \in \mathbb{R}^n$ of (6)

$$\begin{aligned} \tfrac{1}{2}x^T\mathcal{R}_1(P_0)x &= W_1(x) \\ &= \max_{w\in\mathbb{R}^n}\left\{ \begin{array}{c} \tfrac{1}{2}x^T\Phi x - \tfrac{1}{2}\gamma^2 w^T w + \\ \tfrac{1}{2}(Ax+Bw)^T P_0(Ax+Bw) \end{array} \right\} \\ &\ge \tfrac{1}{2}x^T\Phi x - \tfrac{1}{2}\gamma^2(w^*)^T w^* \\ &= \tfrac{1}{2}x^T\Phi x - \tfrac{1}{2}\gamma^2 x^T(A^T(BB^T)^{-1}A)x \\ &= \tfrac{1}{2}x^T\Lambda_1^{11}x. \end{aligned}$$

Thus, $\mathcal{R}(P_0) \ge \Lambda_1^{11}$. From monotonicity property of the Riccati operator shown in Lemma 3.4, $\mathcal{R}_k(P_0) = \mathcal{R}_{k-1}(\mathcal{R}_1(P_0)) \ge \mathcal{R}_{k-1}(\Lambda_1^{11}) = \Lambda_k^{11}$. ■

Theorem 3.6: Suppose that the conditions of Theorem 2.3 hold for Λ of (11). Then, the matrix Λ_∞^{11} in Theorem 2.4 is the minimum solution P_- to the ARE (4).

Proof: According to Theorem 2.3, $\Lambda_k^{11} \to \Lambda_\infty^{11}, k \to \infty$ and $\Lambda_1^{22} + \Lambda_\infty^{11} \le -\lambda_\infty I < 0$. Thus,

$$\begin{aligned} \gamma^2 I - B^T\Lambda_\infty^{11}B &= -B^T(-\gamma^2(BB^T)^{-1} + \Lambda_\infty^{11})B \\ &= -B^T(\Lambda_1^{22} + \Lambda_\infty^{11})B > 0. \end{aligned}$$

Hence, the limit $k \to \infty$ can be taken from both sides of the DRE $\Lambda_{k+1}^{11} = \mathcal{R}(\Lambda_k^{11})$, which is $\Lambda_\infty^{11} = \mathcal{R}(\Lambda_\infty^{11})$. That is, Λ_∞^{11} is the solution to the ARE (4).

Suppose that $P \in \mathbb{M}^{n\times n}$ is a solution to the ARE (4). That is, $P = \mathcal{R}(P)$ and $\gamma^2 I - B^T PB > 0$. If take $P_0 = P$, then $P_k = \mathcal{R}_k(P) = P, k \in \mathbb{N}$. From Theorem 3.1, $P_k + \Lambda_k^{22} < 0$, and

$$P = P_k = \Lambda_k^{11} - \Lambda_k^{12}(P_k + \Lambda_k^{22})^{-1}\Lambda_k^{21} \ge \Lambda_k^{11}.$$

Thus, $P \ge \Lambda_\infty^{11}$ by sending $k \to \infty$. ■

When $\Lambda_k \to \Lambda_\infty, k \to \infty$, the matrix Λ_∞^{22}, which is the limit of the sequence $\{\Lambda_k^{22}, k \in \mathbb{N}\}$, specifies a condition on the initial matrix $P_0 \in \mathbb{M}^{n\times n}$ that $\mathcal{R}_k(P_0) \to \Lambda_\infty^{11}$.

Theorem 3.7: Suppose that the conditions of Theorem 2.3 hold on the matrix Λ of (11). Then,

$$\mathcal{R}_k(P_0) \text{ exists for } k \in \mathbb{N} \quad\Longrightarrow\quad P_0 + \Lambda_\infty^{22} \le 0; \tag{31}$$

Conversely,

$$P_0 + \Lambda_\infty^{22} < 0 \quad\Longrightarrow\quad \mathcal{R}_k(P_0) \to \Lambda_\infty^{11}. \tag{32}$$

Proof: If $\mathcal{R}_k(P_0)$ exists for $k \in \mathbb{N}$, according to Theorem 3.1, $P_0 + \Lambda_k^{22} < 0,\ k \in \mathbb{N}$. Taking the limit as $k \to \infty$ yields $P_0 + \Lambda_\infty^{22} \le 0$. Conversely, if $P_0 + \Lambda_\infty^{22} < 0$, then $P_0 + \Lambda_k^{22} \le P_0 + \Lambda_\infty^{22} < 0,\ k \in \mathbb{N}$ by $\Lambda_\infty^{22} \ge \Lambda_k^{22},\ k \in \mathbb{N}$ from Theorem 2.3. By Theorem 3.1, $\mathcal{R}_k(P_0) = \Lambda_k^{11} - \Lambda_k^{12}(P_0 + \Lambda_k^{22})^{-1}\Lambda_k^{21}$ and $\Lambda_k^{12} \to 0, \Lambda_k^{21} \to 0$ as $k \to \infty$. Consequently, $\mathcal{R}_k(P_0) \to \Lambda_\infty^{11},\ k \to \infty$. ■

C. Rate of convergence

The following result provides an explicit upper bound for the propagation error $E_k(P_0^1, P_0^2) = \mathcal{R}_k(P_0^1) - \mathcal{R}_k(P_0^2)$ (22) for given $P_0^1, P_0^2 \in \mathsf{dom}(\mathcal{R}_k)$.

Theorem 3.8: Suppose that the conditions of Theorem 2.3 hold on Λ of (11), and there exists $\varepsilon > 0$ such that two $P_0^1, P_0^2 \in \mathbb{M}^{n\times n}$ satisfy

$$0 \le P_0^1 - P_0^2 \le \varepsilon I,\ P_0^1 + \Lambda_\infty^{22} < 0,\ P_0^1 + \Lambda_\infty^{22} < 0. \tag{33}$$

Then, for any $k \in \mathbb{N}$,

$$0 \le \mathcal{R}_k(P_0^1) - \mathcal{R}_k(P_0^2) \le \bar\rho^{-2}(\sigma\lambda_\infty^{-2})^{k-1}\sigma\varepsilon\, I, \tag{34}$$

where $\bar\rho \doteq \rho(P_0^1 + \Lambda_\infty^{22}) < 0$, and σ, λ_∞ are as per (14) and (15),

Proof: The monotonicity of the Riccati operator $\mathcal{R}_k$ from Lemma 3.4 implies $\mathcal{R}_k(P_0^1) - \mathcal{R}_k(P_0^2) \ge 0$. It is left to show the upper bound of (34). Since $P_0^1 + \Lambda_k^{22} \le P_0^1 + \Lambda_\infty^{22} \le \bar\rho I < 0$, and $P_0^2 \le P_0^1$,

$$P_0^2 + \Lambda_k^{22} \le \bar\rho\, I < 0,\ k \in \mathbb{N}.$$

By Theorem 2.3, $\mathcal{R}_k(P_0^1) = \Lambda_k^{11} - \Lambda_k^{12}(P_0^1 + \Lambda_k^{22})^{-1}\Lambda_k^{21}$ and $\mathcal{R}_k(P_0^2) = \Lambda_k^{11} - Q_k^{12}(P_0^2 + \Lambda_k^{22})^{-1}\Lambda_k^{21}$. by Theorem 3.1. Then,

$$\begin{aligned} &E_k(P_0^1 - P_0^2) = \mathcal{R}_k(P_0^1) - \mathcal{R}_k(P_0^2) \\ &= \Lambda_k^{12}(\Lambda_k^{22} + P_0^2)^{-1}\Lambda_k^{21} - \Lambda_k^{12}(\Lambda_k^{22} + P_0^1)^{-1}\Lambda_k^{21} \\ &= \Lambda_k^{12}\left((\Lambda_k^{22} + P_0^2)^{-1} - (\Lambda_k^{22} + P_0^1)^{-1}\right)\Lambda_k^{21}. \end{aligned} \tag{35}$$

Using equality (1) for $\Omega^{11} = \Lambda_k^{22} + P_0^1,\ \Omega^{22} = (P_0^1 - P_0^2)^{-1}, \Omega^{12} = \Omega^{21} = I$,

$$\begin{aligned} &(\Lambda_k^{22} + P_0^2)^{-1} = ((\Lambda_k^{22} + P_0^1) - (P_0^1 - P_0^2))^{-1} \\ &= (\Lambda_k^{22} + P_0^1)^{-1} + (\Lambda_k^{22} + P_0^1)^{-1} \\ &\qquad ((P_0^1 - P_0^2)^{-1} - (\Lambda_k^{22} + P_0^1)^{-1})^{-1}(\Lambda_k^{22} + P_0^1)^{-1}. \end{aligned} \tag{36}$$

Thus,

$$\Lambda_k^{12}\left((\Lambda_k^{22} + P_0^2)^{-1} - (\Lambda_k^{22} + P_0^1)^{-1}\right)\Lambda_k^{21}$$

$$\begin{aligned} &= \Lambda_k^{12}(\Lambda_k^{22} + P_0^1)^{-1} \\ &\quad ((P_0^1 - P_0^2)^{-1} - (\Lambda_k^{22} + P_0^1)^{-1})^{-1}(\Lambda_k^{22} + P_0^1)^{-1}\Lambda_k^{21} \\ &\leq \Lambda_k^{12}(\Lambda_k^{22} + P_0^1)^{-1}(P_0^1 - P_0^2)(\Lambda_k^{22} + P_0^1)^{-1}\Lambda_k^{21} \\ &\leq \varepsilon\, \Lambda_k^{12}(\Lambda_k^{22} + P_0^1)^{-1}(\Lambda_k^{22} + P_0^1)^{-1}\Lambda_k^{21} \qquad (37) \\ &\leq \varepsilon\bar{\rho}^{-2}\Lambda_k^{12}\Lambda_k^{21} \leq \varepsilon\bar{\rho}^{-2}\sigma_k\, I \\ &= \varepsilon\bar{\rho}^{-2}(\sigma\lambda_\infty^{-2})^{k-1}\sigma\, I. \end{aligned}$$

Here, the last inequality follows from Theorem 2.3, and the first inequality follows from the assumption that $-(\Lambda_k^{22} + P_0^1)^{-1} > 0$. Then, the error bound (34) follows from equality (35) and inequality (37). ■

Remark 3.9: Theorem 3.8 implies contraction of the solution $\mathcal{R}_k(P_0)$ to the DRE (2) with initial condition P_0 since $\sigma\lambda_\infty^{-2} < 1$ from Theorem 3.8.

IV. EXAMPLES

Two examples are presented to demonstrate various results regarding the relationship between DRE solutions and the max-plus primal space fundamental solution. In particular, the examples are used to show the convergence of the sequence $\{\Lambda_k, k \in \mathbb{N}\}$ of (10) as per Theorem 2.4, and the rate of convergence from Theorem 3.8.

A. Example 1

Consider a difference Riccati Equation of (2) with data specified by

$$A = \begin{bmatrix} 0.1 & -0.1 \\ -0.05 & 0.1 \end{bmatrix}, B = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.5 \end{bmatrix},$$
$$\Phi = \begin{bmatrix} 0.3 & 0 \\ 0 & 1.1 \end{bmatrix}, \gamma = \sqrt{5}.$$

The stabilising solution of the corresponding ARE (4) can be obtained using MATLAB to be $P_- = \begin{bmatrix} 0.3095 & -0.0153 \\ -0.0153 & 1.1270 \end{bmatrix}$. From definition (11), the matrix Λ is explicitly given by

$$\Lambda = \left[\begin{array}{cc|cc} 0.2597 & 0.0458 & 0.3472 & -0.1111 \\ 0.0458 & 1.0431 & -0.3472 & 0.2222 \\ \hline 0.3472 & -0.3472 & -3.4722 & 0.000 \\ -0.1111 & 0.2222 & 0.000 & -2.2222 \end{array}\right].$$

Taking σ and λ to be

$$\sigma = \max\{\rho(\Lambda^{12}\Lambda^{21}), \rho(\Lambda^{21}\Lambda^{12})\} = 0.2979, \qquad (38)$$
$$\lambda = \min \sigma(-(\Lambda^{11} + \Lambda^{22})) = 1.1781,$$

then the condition in Theorem 2.3 are satisfied. According to (15), $\lambda_\infty = \frac{\lambda + \sqrt{\lambda^2 - 4\sigma}}{2} = 0.8107$. Figure 1 shows the sequence $\sigma_k \to 0$ and $\lambda_k \to \lambda_\infty$ as per Lemma 2.1. From Theorem 2.3, the sequences σ_k is an upper bound of $\rho(\Lambda_k^{12}\Lambda_k^{21})$ and $\rho(\Lambda_k^{21}\Lambda_k^{12})$, and the sequence λ_k is an upper bound for $\rho(\Lambda^{11} + \Lambda_k^{22})$ and $\rho(\Lambda_k^{11} + \Lambda^{22})$. These are verified as shown in Figure 1. The sequence $\{\Lambda_k, k \in \mathbb{N}\}$ of (10) converges to a matrix

$$\Lambda_\infty = \left[\begin{array}{cc|cc} 0.3095 & -0.0153 & 0.0000 & 0.0000 \\ -0.0153 & 1.1270 & 0.0000 & 0.0000 \\ \hline 0.0000 & 0.0000 & -3.3237 & -0.0821 \\ 0.0000 & 0.0000 & -0.0821 & -2.1740 \end{array}\right].$$

It can be observed that the upper left block $\Lambda_\infty^{11} = P_-$, which confirms Theorem 3.6. Figure 2 shows an example of propagation error as stated in Theorem 3.8. Here, $P_0^1 = \begin{bmatrix} -1.3 & -0.1 \\ -0.1 & -3.8 \end{bmatrix}$ and $P_0^2 = \begin{bmatrix} -6.8 & 0.3 \\ 0.3 & -2.0 \end{bmatrix}$. For these P_0^1 and P_0^2, $\varepsilon = 5.5219$ and $\bar{\rho} = -4.5996$, respectively. The propagated error $E_k(P_0^1, P_0^2)$ as per (22) and its upper bound of (34) in Theorem 3.8 are shown in Figure 2.

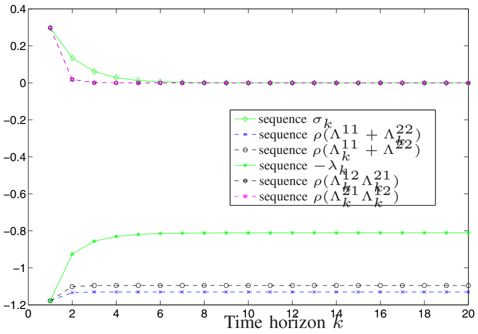

Fig. 1. Various sequences in Theorem 2.3 for example 1.

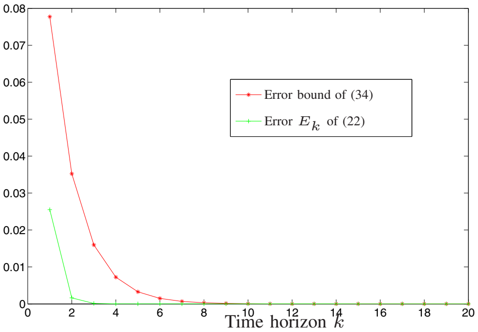

Fig. 2. The propagated error $E_k(P_0^1, P_0^2)$ of (22) and the upper bound of (34) for example 1.

B. Example 2

To further verify various developed results, a second example with different data is presented. Here, the data is given by

$$A = \begin{bmatrix} -0.1 & -0.1 \\ -0.15 & 0.3 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1.15 \end{bmatrix},$$
$$\Phi = \begin{bmatrix} 1.3 & 0 \\ 0 & 2.1 \end{bmatrix}, \gamma = \sqrt{7}.$$

The stabilising solution of the ARE (4) is $P_- = \begin{bmatrix} 1.4198 & -0.2087 \\ -0.2087 & 2.6183 \end{bmatrix}$. The matrix Λ in the fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$ is

$$\Lambda = \left[\begin{array}{cc|cc} 1.1109 & 0.1682 & -0.7000 & -0.7940 \\ 0.1682 & 1.5536 & -0.7000 & 1.5879 \\ \hline -0.7000 & -0.7000 & -7.0000 & 0.0000 \\ -0.7940 & 1.5879 & 0.0000 & -5.2930 \end{array}\right].$$

From Λ, λ and σ as defined in (38) can be similar computed to be $\lambda = 3.7263, \sigma = 3.2857$. Again, the convergence conditions in Theorem 2.3 are satisfied with these λ and σ. In particular, $\sigma_k \to 0$ and $\lambda_k \to \lambda_\infty = \frac{\lambda+\sqrt{\lambda^2-4\sigma}}{2} = 2.2939$. The convergence is of various sequences are as shown in in Figure 3. The limit of the primal space fundamental solution $\{\Lambda_k, k \in \mathbb{N}\}$ is given by

$$\Lambda_\infty = \left[\begin{array}{cc|cc} 1.4198 & -0.2087 & 0.0000 & 0.0000 \\ -0.2087 & 2.6183 & 0.0000 & 0.0000 \\ \hline 0.0000 & 0.0000 & -6.7372 & -0.3158 \\ 0.0000 & 0.0000 & -0.3158 & -4.2025 \end{array}\right].$$

Theorem 3.6 is verified again that the $\Lambda_\infty^{11} = P_-$. To verify Theorem 3.8, take $P_0^1 = \begin{bmatrix} 1.3 & 0.1 \\ 0.1 & 0.8 \end{bmatrix}$ and $P_0^2 = \begin{bmatrix} -0.8 & -0.3 \\ -0.3 & -2 \end{bmatrix}$, respectively. The value of ε in (33) is taken to be $\varepsilon \doteq \rho(P_0^1 - P_0^2)$. The propagated error $E_k(P_0^1, P_0^2)$ as per (22) and its upper bound of (34) in Theorem 3.8 are shown in Figure 4.

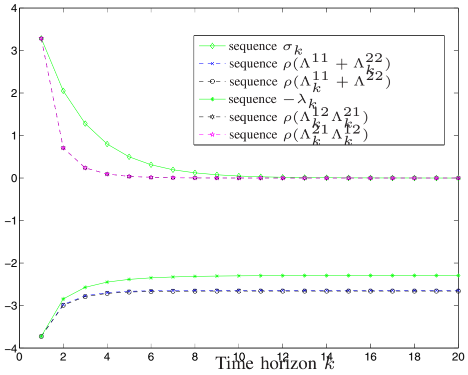

Fig. 3. Various sequences in Theorem 2.3 for example 2.

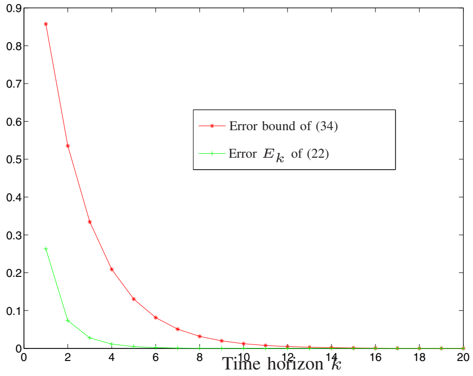

Fig. 4. The propagated error $E_k(P_0^1, P_0^2)$ of (22) and the upper bound of (34) for example 2.

V. CONCLUSIONS

New results for various properties of a class of difference Riccati equations are derived by exploiting a closed-form representation of the solution to the DRE from the max-plus primal space fundamental solution. These results demonstrate that the max-plus primal space fundamental solution, besides its demonstrated advantages in computation, is a particularly powerful tool in studying the DREs.

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