

Input-to-state stability for Lur'e systems*

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Abstract—We present input-to-state stability results for forced Lur'e systems. The main result shows that conditions very similar to those in the complex Aizerman conjecture (see [4]) guarantee input-to-state stability of the Lur'e system. As a corollary of the main result, we obtain a sufficient condition for input-to-state stability which is reminiscent of the circle criterion.

I. INTRODUCTION

Let $\mathbb{F} = \mathbb{C}$ or \mathbb{R} . We consider forced Lur'e systems of the following form

$$\dot{x}(t) = Ax(t) + B(f(Cx(t)) + d(t)), \quad x(0) = x^0 \in \mathbb{F}^n, \quad (1)$$

where $(A, B, C) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m} \times \mathbb{F}^{p \times n}$, $f: \mathbb{F}^p \rightarrow \mathbb{F}^m$ is locally Lipschitz continuous and $d: \mathbb{R}_+ \rightarrow \mathbb{F}^m$ is locally essentially bounded. The forcing term d can be interpreted as a disturbance or as an input (a reference signal, for example). Lur'e systems are a common and important class of nonlinear systems and there is a large body of work on the absolute stability theory of these systems: see, for example, [2], [7], [8], [15]. Traditionally, Lyapunov approaches to the stability theory of systems of the form (1) consider unforced Lur'e systems (that is, $d = 0$ in (1)), whilst Lur'e systems with forcing have been studied using the input-output framework initiated by Sandberg and Zames in the 1960s, see, for example, [15]. More recently, forced Lur'e systems have been analyzed in the context of input-to-state stability (ISS) theory, see [1], [5], [6] (and [11] for discrete-time systems). The current paper makes a contribution to the ISS theory of Lur'e systems.

The following theorem on unforced Lur'e systems, which is of particular relevance in our context, is an immediate consequence of a result in [3], [4]. It shows in particular, that the well-known Aizerman conjecture is true over the complex field.

Theorem I.1. *Let $(A, B, C) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m} \times \mathbb{F}^{p \times n}$ and let $r > 0$. If $A + BFC$ is Hurwitz for all complex matrices $F \in \mathbb{C}^{m \times p}$ with $\|F\| < r$, then the unforced Lur'e system*

$$\dot{x}(t) = Ax(t) + Bf(Cx(t)), \quad x(0) = x^0, \quad (2)$$

is globally asymptotically stable for all locally Lipschitz continuous $f: \mathbb{F}^p \rightarrow \mathbb{F}^m$ satisfying

$$\|f(\xi)\| < r \|\xi\| \quad \forall \xi \in \mathbb{F}^p, \quad \xi \neq 0.$$

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We recall the definition of the structured stability radius $r_{\mathbb{F}}(A; B, C)$ for a Hurwitz matrix $A \in \mathbb{F}^{n \times n}$:

$$r_{\mathbb{F}}(A; B, C) = \inf\{\|F\| : F \in \mathbb{F}^{m \times p}, \sigma(A + BFC) \not\subseteq \mathbb{C}_-\},$$

where \mathbb{C}_- denotes the open left-half plane. It is well known that the complex stability radius satisfies

$$r_{\mathbb{C}}(A; B, C) = \frac{1}{\|G\|_{H^\infty}},$$

where $G(s) := C(sI - A)^{-1}B$ is the transfer function of the linear state-space system (A, B, C) , see, for example, [4]. Therefore, assuming that A is Hurwitz, it follows from Theorem I.1 that the unforced Lur'e system (2) is globally asymptotically stable for every locally Lipschitz continuous $f: \mathbb{F}^p \rightarrow \mathbb{F}^m$ such that

$$\frac{\|f(\xi)\|}{\|\xi\|} \|G\|_{H^\infty} < 1 \quad \forall \xi \in \mathbb{F}^p, \quad \xi \neq 0. \quad (3)$$

Obviously, (3) can be viewed as a small-gain condition, but we emphasize that Theorem I.1 allows for f which have the property that

$$\limsup_{\xi \rightarrow 0} \frac{\|f(\xi)\|}{\|\xi\|} = \frac{1}{\|G\|_{H^\infty}}$$

or/and

$$\limsup_{\|\xi\| \rightarrow \infty} \frac{\|f(\xi)\|}{\|\xi\|} = \frac{1}{\|G\|_{H^\infty}}.$$

The main result of this paper, see Theorem II.2, is very similar in spirit to that of Theorem I.1, but it guarantees ISS of (1).

The following corollary shows that, if the underlying linear system is real and “non-negative”, then the Aizerman conjecture holds over the real field. We recall that a matrix $A \in \mathbb{R}^{n \times n}$ is said to be Metzler if all off-diagonal entries of A are non-negative.

Corollary I.2. *Let $A \in \mathbb{R}^{n \times n}$ be Metzler, let $B \in \mathbb{R}_+^{n \times m}$ and $C \in \mathbb{R}_+^{p \times n}$ be non-negative matrices, and let $r > 0$. If $A + BFC$ is Hurwitz for all real matrices $F \in \mathbb{R}^{m \times p}$ with $\|F\| < r$, then the unforced Lur'e system (2) is globally asymptotically stable for all locally Lipschitz continuous $f: \mathbb{R}^p \rightarrow \mathbb{R}^m$ satisfying*

$$\|f(\xi)\| < r \|\xi\| \quad \forall \xi \in \mathbb{R}^p, \quad \xi \neq 0.$$

Corollary I.2 follows from Theorem I.1 because under the assumption on (A, B, C) imposed in the corollary, $r_{\mathbb{C}}(A; B, C) = r_{\mathbb{R}}(A; B, C)$, see [12].

II. INPUT-TO-STATE STABILITY OF LUR'E SYSTEMS

We recall the definitions of certain classes of comparison functions. Let \mathcal{K} be the set of all continuous function $\alpha: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\alpha(0) = 0$ and α is strictly increasing. We define $\mathcal{K}_\infty \subset \mathcal{K}$ by $\mathcal{K}_\infty := \{\alpha \in \mathcal{K} : \lim_{s \rightarrow \infty} \alpha(s) = \infty\}$. Let \mathcal{KL} denote the set of functions in two variables $\beta: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with the following properties: $\beta(\cdot, t) \in \mathcal{K}$ for all $t \geq 0$, and $\beta(s, \cdot)$ is nonincreasing with $\lim_{t \rightarrow \infty} \beta(s, t) = 0$ for all $s \geq 0$. ISS can be defined for a wide class of systems, but we will define it here only for Lur'e systems of the form (1).

Definition II.1. *The Lur'e system (1) is said to be input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that*

$$\|x(t; x^0, d)\| \leq \beta(\|x^0\|, t) + \gamma\left(\sup_{s \in [0, t]} \|d(s)\|\right),$$

for all $t \geq 0$, for all $x^0 \in \mathbb{F}^n$ and all $d \in L_{\text{loc}}^\infty(\mathbb{R}_+, \mathbb{F}^m)$, where $x(\cdot; x^0, d)$ denotes the unique solution of the initial-value problem (1).

For an overview of ISS theory see [14]. We are now in the position to state the our main result.

Theorem II.2. *Let $(A, B, C) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m} \times \mathbb{F}^{p \times n}$, and let $r > 0$. If $A + BFC$ is Hurwitz for all complex matrices $F \in \mathbb{C}^{m \times p}$ with $\|F\| < r$, then the Lur'e system (1) is ISS for all locally Lipschitz continuous functions $f: \mathbb{F}^p \rightarrow \mathbb{F}^m$ for which there exists $\alpha \in \mathcal{K}_\infty$ such that*

$$\|f(\xi)\| \leq r \|\xi\| - \alpha(\|\xi\|) \quad \forall \xi \in \mathbb{F}^p. \quad (4)$$

We stress that, in the statement of Theorem II.2, $A + BFC$ has to be Hurwitz for all complex matrices F , while the Lur'e system (2) could be defined in terms of real data only (that is, $\mathbb{F} = \mathbb{R}$).

As the following example shows, Theorem II.2 does not remain true if the condition on α is relaxed to $\alpha \in \mathcal{K}$.

Example. Define $\alpha \in \mathcal{K} \setminus \mathcal{K}_\infty$ by $\alpha(s) := 1 - e^{-s}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(\xi) := \xi - \text{sgn}(\xi)\alpha(|\xi|)$. Consider the one-dimensional SISO linear system $(-1, 1, 1)$ with associated forced Lur'e system

$$\dot{x}(t) = -x(t) + f(x(t)) + d(t).$$

Obviously, $-1 + F$ is Hurwitz for all $F \in \mathbb{C}$ with $|F| < 1$. Moreover, if $d(t) = 1 + \varepsilon$ for some positive ε , then $\dot{x}(t) \geq \varepsilon$ for all $t \geq 0$ and hence the forced Lur'e system is not ISS.

The following simple example provides an illustration of Theorem II.2

Example. Consider the forced nonlinear oscillation described by

$$\ddot{z}(t) + 4\dot{z}(t) + 5z(t) + f(z(t), \dot{z}(t)) + d(t) = 0, \quad (5)$$

which can be written in the form (1) with

$$A = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The transfer function $G(s) = C(sI - A)^{-1}B$ is given by

$$G(s) = \frac{1}{s^2 + 4s + 5} \begin{pmatrix} 1 \\ s \end{pmatrix}.$$

A calculation reveals that $\|G\|_{H^\infty} = 3.598$ (see Example 5.3.12 in [4]). Consequently, by Theorem II.2, the forced nonlinear oscillation is ISS provided that there exists $\alpha \in \mathcal{K}_\infty$ such that

$$|f(\xi, \eta)| \leq 3.598 \times \|(\xi, \eta)\| - \alpha(\|(\xi, \eta)\|) \quad \forall (\xi, \eta) \in \mathbb{R}^2.$$

As a special case consider the forced Liénard system

$$\ddot{z}(t) + g(z(t))\dot{z}(t) + 5z(t) + d(t) = 0, \quad (6)$$

which is (5) with $f(\xi, \eta) = (g(\xi) - 4)\eta$. In particular, it follows that (6) is ISS if $|g(\xi) - 4| \leq 3.59$ for all $\xi \in \mathbb{R}$.

The next result is the ISS counterpart of Corollary I.2.

Corollary II.3. *Let $A \in \mathbb{R}^{n \times n}$ be Metzler, let $B \in \mathbb{R}_+^{n \times m}$ and $C \in \mathbb{R}_+^{p \times n}$ be non-negative matrices, and let $r > 0$. If $A + BFC$ is Hurwitz for all real matrices $F \in \mathbb{R}^{m \times p}$ with $\|F\| < r$, then the Lur'e system (1) is ISS for all locally Lipschitz continuous functions $f: \mathbb{R}^p \rightarrow \mathbb{R}^m$ for which there exists $\alpha \in \mathcal{K}_\infty$ such that*

$$\|f(\xi)\| \leq r \|\xi\| - \alpha(\|\xi\|) \quad \forall \xi \in \mathbb{R}^p.$$

Theorem II.2 has another interesting corollary, namely a result which is reminiscent of the circle criterion.

Corollary II.4. *Let $(A, B, C) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m} \times \mathbb{F}^{p \times n}$ be stabilizable and detectable, let $K_1, K_2 \in \mathbb{F}^{m \times p}$ and assume that $(I - K_2G)(I - K_1G)^{-1}$ is positive real, where $G(s) = C(sI - A)^{-1}B$. Then the Lur'e system (1) is ISS for all locally Lipschitz continuous functions $f: \mathbb{F}^p \rightarrow \mathbb{F}^m$ for which there exists $\alpha \in \mathcal{K}_\infty$ such that*

$$\text{Re} \langle f(\xi) - K_1\xi, f(\xi) - K_2\xi \rangle \leq -\|\xi\| \alpha(\|\xi\|) \quad \forall \xi \in \mathbb{F}^p.$$

Corollary II.4 can be viewed as a version of the circle criterion which guarantees ISS and contains a number of results in [5], [6] as special cases.

III. COMMENTS ON THE PROOF OF THE MAIN RESULT

The proof of Theorem II.2 is facilitated by the construction of a suitable ISS-Lyapunov function, which combines two components. The first one is a quadratic form obtained via a suitable version of the bounded real lemma (see [10]).

Proposition III.1. *Consider the Lur'e system (1) and let the assumptions of Theorem II.2 hold. Then there exist $\alpha_1, \gamma_1 \in \mathcal{K}_\infty$ and a continuously differentiable $U: \mathbb{F}^n \rightarrow \mathbb{R}_+$ such that*

$$\frac{d}{dt} U(x(t)) \leq -\|Cx(t)\| \alpha_1(\|Cx(t)\|) + \gamma_1(\|d(t)\|)$$

for all $t \geq 0$, $x^0 \in \mathbb{F}^n$ and $d \in L_{\text{loc}}^\infty(\mathbb{R}_+, \mathbb{F}^m)$, where $x(t) := x(\cdot; x^0, d)$.

The second ingredient is the following proposition.

Proposition III.2. *Under the assumptions of Theorem II.2 there exist positive semi-definite $P = P^* \in \mathbb{F}^{n \times n}$ and $\delta > 0$ such that the function $V(\xi) := \langle P\xi, \xi \rangle$ satisfies*

$$\frac{d}{dt}V(x(t)) \leq -\delta \|x(t)\|^2 + \|Cx(t)\|^2 + \|d(t)\|^2$$

for all $t \geq 0$, $x^0 \in \mathbb{F}^n$ and $d \in L_{\text{loc}}^\infty(\mathbb{R}_+, \mathbb{F}^m)$, where $x(t) := x(\cdot; x^0, d)$.

Moreover, for every $\alpha_2 \in \mathcal{K}_\infty$, there exist a continuously differentiable function $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\eta, \gamma_2 \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \frac{d}{dt}(h \circ V \circ x)(t) \\ \leq -\eta(\|x(t)\|) + \|Cx(t)\| \alpha_2(\|Cx(t)\|) + \gamma_2(\|d(t)\|) \end{aligned}$$

for all $t \geq 0$, $x^0 \in \mathbb{F}^n$ and $d \in L_{\text{loc}}^\infty(\mathbb{R}_+, \mathbb{F}^m)$.

By combining Propositions III.1 and III.2 (with $\alpha_2 = \alpha_1$) it can be established that $U + h \circ V$ is an ISS-Lyapunov function for the Lur'e system (1). The proof of Theorem II.2 is then completed by noting that the existence of an ISS-Lyapunov function implies ISS, see e.g. [13] (in fact, the converse is true as well, see e.g. [9]).

Detailed proofs of Theorem II.2, Corollary II.4, Proposition III.1 and Proposition III.2 can be found in [10].

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