21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Constrained Linear Quadratic Control in Networks with Limited Model-Information Sharing*

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Abstract— **The problem of finite-horizon linear quadratic control in a network of interconnected systems is studied. A gradient based algorithm is considered to solve the problem. Network structures are identified where the iterations can be calculated on a subsystem-by-subsystem basis in terms of model information from only a local subset of nodes in the network.**

I. INTRODUCTION

The study of large scale dynamical systems has gained much attention in recent years. Power networks, water distribution networks and transportation networks are examples of such systems. Particularly, an important problem of interest is to achieve a control objective with respect to a performance index. Many of the available results on controlling networks of interconnected systems do not assume any constraints on the availability of global model information among subsystems, see [?], [?], [?], [?]. However, in many cases it is desired to achieve the same objective while the information availability or communications among the subsystems are restricted by the structure of the system. This might arise in situations where the *privacy* of subsystems is of importance [?]. For example, situations where subsystems are willing to provide their model data to only a subset of other subsystems in the network or when they are not willing to provide their internal states to other systems, e.g. [?] can be considered as scenarios where privacy is of essence. This paper is another step in this direction.

We consider the problem of constrained quadratic control in a network of heterogenous coupled linear systems with constraints on the control variables. The coupling is modelled by a directed graph called the *interconnection graph*. Moreover, we call the undirected version of the interconnection graph, the *network graph*. The optimal control problem is formulated as a static quadratic program by stacking the variables over the finite horizon and is solved through the implementation of a projected gradient scheme. The corresponding iterates are structured in the sense that there is one distinct update calculation required for each sub-system control input sequence. The information required to calculate the update for each sub-system is modelled by an undirected graph called the *optimisation graph*. The main contribution of the paper is to identify the required model information and communications at each sub-system to be able to implement the iterations of the scheme locally for different interconnection graphs.

Taking advantage of the structure of the problem to solve optimisation problems is not new. However, in many of the existing methods, the computations carried out at each node either require some global model information about the system, or a local version of other decision variables need to be stored at each node. For example, the collection of all constraints when these are heterogeneous in [?], the collection of all decision variables from the previous iteration in [?], [?], or local copies of the decision variables of the neighbours [?]. This paper generalises the results presented in [?] and considers different levels of model information sharing between the subsystems that make up the network. In other words, the main objective of this paper is to identify the information exchange requirements for different interconnection graphs to solve problems without introducing any new variables, e.g. variables that enforce equality constraints along the links connecting subsystems to each other.

The outline of this paper is as follows. After introducing the notations used in this paper, Section **??** describes the problem of interest and provide all the required definitions. The solution to the problem introduced in Section **??** is presented in Section **??**, where those interconnections where iterations at each subsystem can be calculated via access to the model information of a subset of other subsystems are identified. Concluding remarks and future directions come in the end.

Notations.: In this paper capital letters are used to denote linear operators, small latin letter and greek letters represent vectors and scalars, and capital calligraphic letters represent sets, except for the well-known set $\mathbb{R}$ of real numbers. The cardinality of a set $\mathcal{A}$ is denoted by $|\mathcal{A}|$. Indices indicate a variable that is local to a sub-system with the same index. The superscript k is used to mean the value of the variable at the k-th iteration. Moreover, A_{ij} is the ij-th block of block matrix A, and a_{ij} is the j-th entry of vector a_i. For integer variables $i = a : b$ means the same as $i = a, a+1, \dots, b$ for integers a and b. We write $A > 0$ if A is symmetric positive definite, and we write $A^\top$ meaning the transpose of A. Moreover, $A \otimes B$ denotes the Kronecker product of A by B. By $x(t)$ and $x(t_1 : t_2)$ we mean the value of variable x at time t and the concatenation of values of variable x at times $t = t_1 : t_2$, respectively.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider n interconnected discrete-time systems in set $\mathcal{V} = \{1 : n\}$ where each system $i \in \mathcal{V}$ is described by

*This work is supported in part by a McKenzie Fellowship, a UoM Early Career Researcher Grant, and the Australian Research Council (DP110103778).

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$$x_i(t+1) = A_{ii}x_i(t) + B_{ii}u_i(t) + v_i(t) \tag{1}$$

where $x_i(t) \in \mathbb{R}^{p_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, $v_i(t) \in \mathbb{R}^{s_i}$, $A_{ii} \in \mathbb{R}^{p_i \times p_i}$, $B_{ii} \in \mathbb{R}^{p_i \times m_i}$, for some positive integers p_i, m_i, s_i, $x_i(t_0) = \xi_i$, and

$$v_i(t) = \sum_{l \in \mathcal{N}_i^-} B_{il}u_l(t) + \sum_{l \in \mathcal{N}_i^-} A_{il}x_l(t) \tag{2}$$

where B_{il} and A_{il} are coupling matrices of appropriate dimensions. The interconnection constraint (**??**) characterises the directed links between system i and systems in the set $\mathcal{N}_i^-$ and we assume that for any interconnection between i and l only one of B_{il} or A_{il} is nonzero. Additionally, let $\mathcal{N}_i^+ = \{l | i \in \mathcal{N}_l^-\}$.

Definition 1. *Let $\mathcal{C}$ be either a directed or undirected non-empty graph. A path is a non-empty graph $\mathcal{C}_P = (\mathcal{V}_P, \mathcal{E}_P) \subset \mathcal{C}$ of the form $\mathcal{V}_P = \{i\}_{i=1}^k$ and $\mathcal{E}_p = \{(j_i, j_{i+1})\}_{i=1}^{k-1}$, where $\{j_1, \cdots, j_k\}$ is a permutation of $\{1, \cdots, k\}$. The vertices $j_2, \cdots, j_{k-1}$ are the inner vertices of $\mathcal{C}_P$. An undirected (directed) graph is connected (strongly connected) if there is a (directed) path between any pair of vertices. Furthermore, every sequence of edges that form a closed path in $\mathcal{C}$ and do not visit the same node twice, except the start/end node, is called a (directed) cycle.*

Definition 2. *The network of systems described by* (**??**) *and* (**??**) *can be modelled as a directed* interconnection graph $\mathcal{I} = (\mathcal{V}, \mathcal{E}_\mathcal{I})$*, where the vertices of the graph are the (sub)systems, an edge connects two vertices if they are coupled, i.e. $\mathcal{E}_\mathcal{I} = \{(l,i) | l, i \in \mathcal{V}, l \in \mathcal{N}_i^-\}$, $\mathcal{N}_i^-$ is the set of in-neighbours of i and $\mathcal{N}_i^+$ is the set of out-neighbours of i.*

Definition 3. *The undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ such that $\mathcal{E} \triangleq \mathcal{E}_\mathcal{I} \bigcup \{(j,i) | (i,j) \in \mathcal{E}_\mathcal{I}\} \bigcup \{(i,i)\}_{i=1}^n$ is called the underlying* network graph of $\mathcal{I}$. *Moreover, we call the set $\mathcal{N}_i^\ell \subset \mathcal{V}$* the ℓ-hop neighbour set of node i *where $v \in \mathcal{N}_i^\ell$ if there is a path of length at most ℓ between i and v, particularly $\mathcal{N}_i^1 = \mathcal{N}_i^+ \cup \mathcal{N}_i^-$.*

Given a horizon length $\tau \geq 1$, for each system i and for $t = t_0 + 1 : t_0 + \tau$,

$$x_i(t_0+1 : t_0+\tau) = \begin{bmatrix} \mathbf{G}_i & \mathbf{H}_i & \mathbf{L}_i \end{bmatrix} \begin{bmatrix} u_i(t_0 : t_0+\tau-1) \\ v_i(t_0 : t_0+\tau-1) \\ \xi_i \end{bmatrix} \tag{3}$$

where $\mathbf{G}_i = \Xi_i(B_{ii})$, $\mathbf{H}_i = \Xi_i(I)$,

$$\Xi_i(\Delta) = \begin{bmatrix} \Delta & 0 & \dots & 0 \\ A_{ii}\Delta & \Delta & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ A_{ii}^{\tau-1}\Delta & A_{ii}^{\tau-2}\Delta & \dots & \Delta \end{bmatrix},$$

and $\mathbf{L}_i = \begin{bmatrix} A_{ii} \\ \vdots \\ A_{ii}^\tau \end{bmatrix}$. For ease of notation $\mathbf{x}_i$, $\mathbf{u}_i$, and $\mathbf{v}_i$ are used to denote $x_i(t_0+1 : t_0+\tau)$, $u_i(t_0 : t_0+\tau-1)$, and $v_i(t_0 : t_0+\tau-1)$, respectively. Thus, (**??**) becomes

$$\mathbf{x}_i = \begin{bmatrix} \mathbf{G}_i & \mathbf{H}_i & \mathbf{L}_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \\ \xi_i \end{bmatrix}. \tag{4}$$

Consider the aforementioned interconnected systems. Given $\tau \geq 1$, define

$$\begin{aligned} f(\mathbf{u}_1, \dots, \mathbf{u}_n) &= \frac{1}{2} \sum_{i=1}^n \sum_{t=t_0+1}^{t_0+\tau} f_i(x_i(t), u_i(t)) \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{t=t_0+1}^{t_0+\tau} x_i(t)^\top Q_i x_i(t) + u_i(t-1)^\top R_i u_i(t-1) \end{aligned} \tag{5}$$

for known $Q_i > 0$ and $R_i > 0$. Additionally, consider the following constraints

$$u_i(t) \in \mathcal{U}_i(t), \quad i = 1, \dots, n \tag{6}$$

where $\mathcal{U}_i(t)$ are convex polytopes[1]. Equivalently

$$\mathbf{u}_i \in \mathcal{U}_i, \quad i = 1, \dots, n \tag{7}$$

Let $\rho_0 > 0$ be such that $\nabla^2 f(\mathbf{u}_1, \dots, \mathbf{u}_n) \leq \rho_0^{-1}\mathbf{I}$. Then the following iterations converge to the minimiser of (**??**) subject to the constraints (**??**), see [**?**] for more details:

$$\mathbf{u}^{k+1} = \left[\mathbf{u}^k - \rho \nabla f(\mathbf{u}_1^k, \dots, \mathbf{u}_n^k)\right]^\mathcal{U} \tag{8}$$

where

$$[\bar{u}]^\mathcal{U} = \begin{array}{ll} \text{argmin} & \|\bar{u} - \mu\| \\ \text{s.t.} & \mu \in \mathcal{U}, \end{array} \tag{9}$$

and $0 < \sigma \leq \rho \leq 2\rho_0 - \sigma$ for some positive real scalar σ. Note that choosing ρ requires having some global knowledge about the cost function, namely an upper bound on the largest eigenvalue of the hessian. This is commonly assumed in the literature, e.g. see [**?**], [**?**].

We have the following definition.

Definition 4. *Given an interconnection graph $\mathcal{I}$, we denote the ℓ-hop model information available to i at time k by*

$$\begin{aligned} \mathcal{K}_i^\ell(k) \triangleq &\{A_{ij}, B_{ij}, Q_j, \xi_j\}_{j \in \mathcal{N}_i^\ell \cup \{i\}} \bigcup \{R_i, \mathcal{U}_i\} \\ &\bigcup \{\mathbf{u}_j^k\}_{j \in \mathcal{N}_i^\ell \cup \{i\}}. \end{aligned} \tag{10}$$

Problem 1. *Characterize the interconnection graphs $\mathcal{I}$ for which the iterates $\mathbf{u}_i^k$ in* (**??**) *can be computed using only $\mathcal{K}_i^\ell(k)$, given values of ℓ.*

III. Networked Constrained Finite-horizon Optimal Control

Initially in this section we consider the case where systems are coupled through their control signals, then we shift our focus to the case where the coupling is achieved via the state variables.

[1]Note that $\mathcal{U}_i$ being polytopes is not essential.

A. Coupling Through Controls

Let the polytopic constraint $\mathbf{u}_i \in \mathcal{U}_i$ for $t = t_0+1 : t_0+\tau$ and $i = 1 : n$ be characterised by

$$\mathbf{S}_i \mathbf{u}_i \leq \vartheta_i \tag{11}$$

for some known $\mathbf{S}_i$ and ϑ_i, $i = 1 : n$.

First, consider the scenario where the intersystem coupling is through the control inputs of the neighbouring systems; that is,

$$v_i(t) = \sum_{l \in \mathcal{N}_i^-} B_{il} u_l(t). \tag{12}$$

In this case, the objective function (**??**) can be rewritten as

$$f(\mathbf{u}) = \frac{1}{2}\mathbf{u}^\top \Pi \mathbf{u} + \mathbf{u}^\top \pi + c \tag{13}$$

where $\mathbf{u} = [\mathbf{u}_1^\top, \ldots, \mathbf{u}_n^\top]^\top$ and the matrix Π, the vector π, and the scalar c are given by

$$\begin{aligned} \Pi_{i,i} &= \mathbf{G}_i^\top \mathbf{Q}_i \mathbf{G}_i + \mathbf{R}_i + \sum_{j \in \mathcal{N}_i^+} \mathbf{B}_{ji}^\top \mathbf{H}_j^\top \mathbf{Q}_j \mathbf{H}_j \mathbf{B}_{ji}, \quad i = 1 : n \\ \Pi_{i,j} &= \Pi_{j,i}^\top = \mathbf{G}_i^\top \mathbf{Q}_i \mathbf{H}_i \mathbf{B}_{ij} + \sum_{l \in \{\bar{l} | i,j \in \mathcal{N}_{\bar{l}}^-\}} \mathbf{B}_{lj}^\top \mathbf{H}_l^\top \mathbf{Q}_l \mathbf{H}_l \mathbf{B}_{li}, \\ &\quad i,j = 1 : n,\ i \neq j \end{aligned} \tag{14}$$

$$\begin{aligned} \pi_i &= \sum_{j \in \mathcal{N}_i^+} \mathbf{B}_{ji} \mathbf{H}_j^\top \mathbf{Q}_j \mathbf{L}_j \xi_j + \mathbf{G}_i^\top \mathbf{Q}_i \mathbf{L}_i \xi_i, \quad i = 1 : n \\ c &= \frac{1}{2} \sum_{i=1}^n \xi_i^\top \mathbf{L}_i^\top \mathbf{Q}_i \mathbf{L}_i \xi_i, \quad i = 1 : n. \end{aligned} \tag{15}$$

with $\mathbf{B}_{ij} \triangleq \mathbf{I} \otimes B_{ij}$, $\mathbf{Q}_i \triangleq \mathbf{I} \otimes Q_i$, and $\mathbf{R}_i \triangleq \mathbf{I} \otimes R_i$.

Definition 5. *Let the undirected* optimization graph $\mathcal{O}(\Pi) = (\mathcal{V}, \mathcal{E}_\mathcal{O})$ *be such that* $(i,j) \in \mathcal{E}_\mathcal{O}$ *iff* $\Pi_{ij} \neq 0$ *in* (**??**). *Moreover,*

$$\begin{aligned} \mathcal{J}_i(k) \triangleq &\{A_{ij}, B_{ij}, Q_j, \xi_j\}_{j \in \mathcal{M}_i} \bigcup \{R_i, \mathcal{U}_i\} \\ &\bigcup \{\mathbf{u}_j^k\}_{j \in \mathcal{M}_i} \end{aligned} \tag{16}$$

where $\mathcal{M}_i \triangleq \{j : (i,j) \in \mathcal{E}_\mathcal{O}\}$.

The iteration (**??**) can be written as

$$\mathbf{u}_i^{k+1} = \left[\mathbf{u}_i^k - \rho \left(\sum_{\{j | (i,j) \in \mathcal{E}_\mathcal{O}\}} \Pi_{ij} \mathbf{u}_j^k + \pi_i \right) \right]^{\mathcal{U}_i} \quad i = 1 : n, \tag{17}$$

where

$$[\bar{u}]^{\mathcal{U}_i} = \begin{array}{ll} \text{argmin} & \|\bar{u} - \mu_i\| \\ \text{s.t.} & \mathbf{S}_i \mu_i \leq \vartheta_i. \end{array} \tag{18}$$

Addressing Problem **??** for the case where $\mathbf{u}_i^k$ is calculated via (**??**) and (**??**) involves identifying interconection graphs $\mathcal{I}$ such that both equations (**??**) and (**??**) can be evaluated using $\mathcal{K}_i^\ell(k)$.

The following result on the interconnection graph facilitates addressing Problem **??**.

Theorem 1. *Consider information sets* $\mathcal{K}_i^\ell(k)$, $\mathcal{J}_i(k)$, *and* Π *described by Definitions* **??**, **??**, *and* (**??**), *respectively. The following statements hold*

(i) For the case where $\ell = 0$, $\mathcal{J}_i(k) = \mathcal{K}_i^\ell(k)$ *coincide if and only if the edge set* $\mathcal{E}_\mathcal{I}$ *is empty.*

(ii) For the case where $\ell = 1$, $\mathcal{J}_i(k) = \mathcal{K}_i^\ell(k)$ *coincide if and only if for all pair* (i,j) *such that* $\{i,j\} \subset \mathcal{N}_l^-$, $l = 1 : n$, *then either* $(i,j) \in \mathcal{E}_\mathcal{I}$ *or* $(j,i) \in \mathcal{E}_\mathcal{I}$.

(iii) For the case where $\ell \geq 2$, $\mathcal{J}_i(k) \subseteq \mathcal{K}_i^\ell(k)$ *for all interconnection graphs* $\mathcal{I}$.

Proof: Note that $\mathcal{K}_i^\ell(k) = \mathcal{J}_i(k)$ if and only if $\mathcal{N}_i^\ell = \mathcal{M}_i$. Now we proceed by considering each case of ℓ. For $\ell = 0$, $\mathcal{N}_i^\ell = \mathcal{M}_i$ holds only in the case where $\mathcal{M}_i = \emptyset$. In light of (**??**), this only holds where $\mathcal{N}_i^- = \emptyset$ for all $i \in \mathcal{V}$.

For $\ell = 1$, by definition (i,j) and (j,i) are in $\mathcal{E}_\mathcal{O}$ if the corresponding block Π_{ij} is nonzero. From (**??**) it can be seen that Π_{ij} is generically nonzero if and only if i and j both lie in $\mathcal{N}_l^-$ for some $l \in \mathcal{V}$. Hence, (i,j) and (j,i) belong to $\mathcal{E}_\mathcal{O}$. On the other hand both edges (i,j) and (i,j) are in $\mathcal{E}$ if and only if either $(i,j) \in \mathcal{E}_\mathcal{I}$ or $(j,i) \in \mathcal{E}_\mathcal{I}$. Thus, $\mathcal{E}_\mathcal{O} = \mathcal{E}$ and consequently $\mathcal{O} = \mathcal{G}$.

Now note that generically

$$\mathcal{M}_i = \mathcal{N}_i \cup \{l \ : \ i,j \in \mathcal{N}_l^-\} \cup \{i\}.$$

Moreover, for any $\bar{l} \in \mathcal{M}_i$ there are three distinct possibilities: (1) there is path of length 0 if $\bar{l} = i$, (2) there is path of length 1 if $\bar{l} \in \mathcal{N}_i$, or (3) there is a path of length 2 if $\bar{l} \in \{l \ : \ i,j \in \mathcal{N}_l^-, j \in \mathcal{N}_i^-\}$. Thus, $\mathcal{M}_i \subseteq \mathcal{N}_i^2$. On the other hand, $\mathcal{N}_i^2 \subseteq \mathcal{N}_i^\ell$, for $\ell \geq 2$. Hence, $\mathcal{M}_i \subseteq \mathcal{N}_i^\ell$ for all $\ell \geq 2$. It follows immediately that $\mathcal{J}_i(k) \subseteq \mathcal{K}_i^\ell(k)$ for $\ell \geq 2$ and all interconnection graphs $\mathcal{I}$. □

Next we address Problem **??** for the case where the coupling is through control signals and only the control signals are constrained.

Corollary 1. *Consider* n *interconnected discrete-time systems with the interconnection graph* $\mathcal{I}$ *where each system* $i \in \mathcal{V}$ *is described by* (**??**), (**??**), *and* (**??**). *Each* u_i^k, $i = 1 : n$, *in* (**??**) *can be computed using* $\mathcal{K}_i^\ell(k)$ *if*

(i) $\ell = 0$, *and the edge set* $\mathcal{E}_\mathcal{I}$ *is empty.*

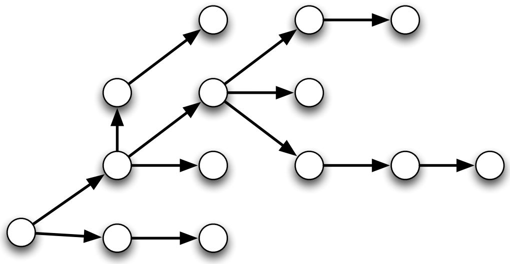
(ii) $\ell = 1$, *and for all pair* (i,j) *such that* $\{i,j\} \subset \mathcal{N}_l^-$, $l = 1 : n$, *then either* $(i,j) \in \mathcal{E}_\mathcal{I}$ *or* $(j,i) \in \mathcal{E}_\mathcal{I}$.

(iii) $\ell \geq 2$.

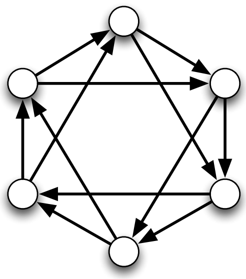
Proof: Note that (**??**) can be evaluated by i using only $\mathcal{U}_i$ for any $\bar{u}$. Thus, computing the update (**??**) for system i involves only $\mathcal{J}_i(k)$. Hence, (**??**) can be computed using $\mathcal{K}_i^\ell(k)$ if $\mathcal{K}_i^\ell(k) \subseteq \mathcal{J}_i(k)$. Applying Theorem **??** completes the proof. □

For the rest of this section we consider the case where $\ell = 1$. Specifically, we study the scenarios where Problem **??** can be solved using 1-hop neighbourhood information.

The following result identify well-known graph categories where Problem **??** can be solved when $\ell = 1$. Note that the conditions for $\mathcal{J}_i(k) = \mathcal{K}_i^1(k)$ are the same as the



(a)



(b)

Fig. 1. Examples for different interconnection graphs with satisfying the conditions of Theorem **??**.

conditions for $\mathcal{O} = \mathcal{G}$. In what follows, some well-known interconnection graphs that result in $\mathcal{O} = \mathcal{G}$ are presented.

Remark 1. *Consider graphs $\mathcal{G}$ and $\mathcal{O}$ described by Definitions* **??** *and* **??**, *respectively. The underlying network graph $\mathcal{G}$ and the optimisation graph $\mathcal{O}$ coincide for the following graphs: (1) $\mathcal{I}$ is any power of a rooted tree on $\mathcal{V}$, (2) $\mathcal{I}$ is any power of a simple directed cycle, (3) $\mathcal{G}$ is a complete graph, regardless of the orientation of the edges in $\mathcal{I}$, where a rooted tree is a directed graph such that the underlying undirected graph has no cycles and all the edges point away from the root, that is the vertex with in-degree equal to zero. The p-th power of a graph $\mathcal{C}$ is a graph with the same set of vertices as $\mathcal{C}$ and an edge between two vertices if and only if there is a path of length at most p between them in $\mathcal{C}$.*

Example interconnection graphs satisfying Theorem **??** are depicted in Fig. **??**. The graph of Fig. **??**(a) is a rooted tree and the one depicted in Fig. **??**(b) is the square of a directed cycle.

In light of Corollary **??**, it is possible to construct an algorithm to check if a given interconnection graph $\mathcal{I}$ admits a solution for Problem **??** under the coupling described by (**??**) where each iterate (**??**) can be calculated at each i for $\ell = 1$. The worst case computational complexity of such an algorithm based on Theorem **??** is of order n^3.

Moreover, it is possible to devise a method to add new couplings to a network with $\mathcal{I}$ satisfying the condition of Corollary **??**, i.e. adding new edge to $\mathcal{I}$, without compromising the ability to solve Problem **??** under the coupling described by (**??**) distributedly. Similarly one can describe how a coupling can be removed from a network, i.e. removing an edge from $\mathcal{I}$, such that Problem **??** under the coupling described by (**??**) admits a distributed solution for the resulting interconnection graph. Such edge additions and removals are characterised next in Algorithms **??** and **??**. And example for adding one node to a graph under Algorithm **??** is depicted in Fig. **??**.

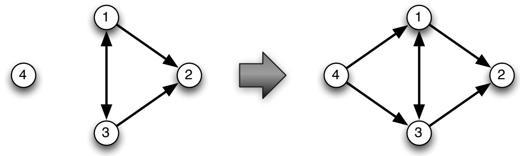


Fig. 2. Adding a system to a network under Algorithm **??**.

Algorithm 1 Adding a new edge (i, j) to the network graph $\mathcal{I}$.

Require: $\mathcal{G} = \mathcal{O}$, $\iota =$ **true** $\{\iota$ represents the possibility of solving the optimal control problem using local information only.$\}$
for $l \in \mathcal{N}_j^-$ **do**
  if $(l \notin \mathcal{N}_i^-$ **and** $l \notin \mathcal{N}_i^+)$ **then**
    $\iota \leftarrow$ **false**
    break
  end if
end for
if $\iota =$ **true then**
  $\mathcal{E}_\mathcal{I} \leftarrow \mathcal{E}_\mathcal{I} \bigcup \{(i, j)\}$ {edge (i, j) is added.}
end if

Algorithm 2 Removing an edge (i, j) from the network graph $\mathcal{I}$.

Require: $\mathcal{G} = \mathcal{O}$, $\iota =$ **true** $\{\iota$ represents the possibility of solving the optimal control problem using local information only.$\}$
for $l \in \mathcal{V}$ **do**
  if $(i \notin \mathcal{N}_l^-$ **and** $j \notin \mathcal{N}_l^-)$ **then**
    $\iota \leftarrow$ **false**
    break
  end if
end for
if $\iota =$ **true then**
  $\mathcal{E}_\mathcal{I} \leftarrow \mathcal{E}_\mathcal{I} \setminus \{(i, j)\}$ {edge (i, j) is removed.}
end if

It is possible to devise similar guidelines to connect or remove new systems to the network, i.e. adding new vertices to or removing vertices from $\mathcal{I}$. By applying Algorithm **??** to add edges and similar guidelines to add vertices to one of the graphs described in Theorem **??** it is possible to construct different $\mathcal{I}$ that results in $\mathcal{G}$ and $\mathcal{O}$ coinciding.

B. Coupling Through States

Now, the case where systems are coupled through states is considered, i.e.

$$v_i(t) = \sum_{l \in \mathcal{N}_i^-} A_{il} x_l(t). \tag{19}$$

The network can be described by

$$x(t+1) = \mathbf{A}x(t) + \mathbf{B}u(t) \tag{20}$$

where $x(t) = [x_1(t), \ldots, x_n(t)^\top]^\top$, $u(t) = [u_1(t), \ldots, u_n(t)^\top]^\top$, and

$$\mathbf{A} = \begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \ldots & A_{nn} \end{bmatrix}, \quad \mathbf{B} = \mathrm{diag}(B_{11}, \ldots, B_{nn}).$$

Let $M(t) \triangleq \mathbf{A}^t$. Then for $t = t_0 + 1 : t_0 + \tau$

$$x_i(t) = \sum_{j=1}^{n} M_{ij}(t-t_0)\xi_j + \sum_{k=t_0}^{t-1}\sum_{j=1}^{n} M_{ij}(k-t_0)B_j u_j(k-t_0).$$

In what follows we briefly study the structure of $\mathbf{A}$ and its powers. Note that the sparsity pattern of $\mathbf{A}$ is the same as the adjacency matrix of $\mathcal{I}$ with each node having self loops, i.e. the ij-th block is nonzero if $(j,i) \in \mathcal{E}_\mathcal{I}$ or if $i = j$. Consequently $M(t) = \mathbf{A}^t$ exhibits similar properties to those of the powers of the adjacency matrix of a graph for generic values of A_{ij}, $i,j = 1:n$. Particularly, the sufficient condition for the ij-th entry of $\mathbf{A}^t$ to nonzero is the existence of a path of the length t between systems i and j.

One can again rewrite (**??**) in the form

$$f(\mathbf{u}) = \frac{1}{2}\mathbf{u}^\top \overline{\Pi}\mathbf{u} + \mathbf{u}^\top \overline{\pi} + \overline{c}, \tag{21}$$

where $\overline{\Pi}$ is a positive definite matrix, $\overline{\pi}$ is a vector, and $\overline{c}$ is a scalar. However, the exact expression of $\overline{\Pi}$ and $\overline{\pi}$ depends on the interconnection graph and varies between networks with different interconnection graphs.

Theorem 2. *Consider the horizon length τ, model information sets $\mathcal{K}_i^\ell(k)$ and $\mathcal{J}_i(k)$ described by Definitions **??** and **??** respectively, and $\overline{\Pi}$ in (**??**). The following statements hold*

(i) For the case where $\ell = 0$, $\mathcal{J}_i(k) = \mathcal{K}_i^\ell(k)$ coincide if the edge set $\mathcal{E}_\mathcal{I}$ is empty.

(ii) For the case where $\ell = \tau - 1$, $\mathcal{J}_i(k) = \mathcal{K}_i^\ell(k)$ coincide if for all pair (i,j) such that $\{i,j\} \subset \mathcal{W}_l^\tau$, $l = 1:n$, where $\mathcal{W}_l^\ell$ is the set of all nodes with a path of length less than or equal to τ to l, then either $(i,j) \in \mathcal{E}_\mathcal{I}$ or $(j,i) \in \mathcal{E}_\mathcal{I}$.

(iii) For the case where $\ell \geq 2(\tau-1)$, $\mathcal{J}_i(k) \subseteq \mathcal{K}_i^\ell(k)$ for all interconnection graphs $\mathcal{I}$.

Proof: Note that $x_l^\top(t)Q_l x_l(t)$ is equal to (**??**). For $\overline{\Pi}_{ij}$ to be nonzero it is sufficient that the coefficient of the cross term $\mathbf{u}_i$ and $\mathbf{u}_j$ is nonzero. This in turn is true if there exists a t where the cross term of $u_i(t)$ and $u_j(t)$ is nonzero. From (**??**) this is true if there exists a $k \in \{t_0+1 : t_0+\tau\}$ such that either (1) $M_{ij}(k-t_0)$ is nonzero, or (2) for some l, $M_{li}(k-t_0)^\top Q_l M_{lj}(k-t_0)$ is nonzero. Now, for (1) to be true it is enough that $i \in \mathcal{N}^{\tau-1}$, additionally, (2) holds if there exists a node l such that both i and j are in $\mathcal{W}_l^{\tau-1}$. As a result if for all pair $\{i,j\} \subset \mathcal{W}_l^{\tau-1}$, there is an edge between i and j then $\mathcal{J}_i(k) = \mathcal{K}_i^\ell(k)$.

To prove *(iii)* we again note that $\mathcal{M}_i \subseteq \mathcal{N}_i^{2(\tau-1)} \cup \{i\}$ and consequently $\mathcal{M}_i \subseteq \mathcal{N}_i^{2\ell} \cup \{i\}$ for all $\ell \geq 2(\tau-1)$. □

As before we address Problem **??** for the case where the coupling is achieved via the states signals and only the control signals are constrained.

Corollary 2. *Consider n interconnected discrete-time systems with the interconnection graph $\mathcal{I}$ where each system $i \in \mathcal{V}$ is described by (**??**), (**??**), and (**??**). Each u_i^k, $i = 1:n$, in (**??**) can be computed using $\mathcal{K}_i^\ell(k)$ if*

(i) For the case where $\ell = 0$, $\mathcal{J}_i(k) = \mathcal{K}_i^\ell(k)$ coincide if the edge set $\mathcal{E}_\mathcal{I}$ is empty.

(ii) For the case where $\ell = \tau - 1$, $\mathcal{J}_i(k) = \mathcal{K}_i^\ell(k)$ coincide if for all pair (i,j) such that $\{i,j\} \subset \mathcal{W}_l^\ell$, $l = 1:n$, where $\mathcal{W}_l^\ell$ is the set of all nodes with a path of length less than or equal to τ to l, then either $(i,j) \in \mathcal{E}_\mathcal{I}$ or $(j,i) \in \mathcal{E}_\mathcal{I}$.

(iii) For the case where $\ell \geq 2(\tau-1)$, $\mathcal{J}_i(k) \subseteq \mathcal{K}_i^\ell(k)$ for all interconnection graphs $\mathcal{I}$.

For some values of τ, there are certain interconnections where graphs $\mathcal{G}$ and $\mathcal{O}$ coincide. One of such scenarios is when $\tau = 1$ for all $\mathcal{I}$ graphs. Moreover, for $\tau = 2$ if for all pair (i,j) such that $\{i,j\} \subset \mathcal{N}_l^-$, $l = 1:n$, then either $(i,j) \in \mathcal{E}_\mathcal{I}$ or $(j,i) \in \mathcal{E}_\mathcal{I}$ then $\mathcal{G} = \mathcal{O}$. Additionally, if the interconnection graph is a star graph regardless of the choice of τ then $\mathcal{G} = \mathcal{O}$. Note that this does not violate the statement of Theorem **??**, as $\mathcal{W}_i^\ell$ is the same for all the values of $\ell \geq 1$.

The aforementioned statement alludes to the ease of calculating one step-ahead policies when couplings are through state variables.

C. Coupling Through Controls and States

We conclude this section by considering the couplings of the form

$$v_i(t) = \sum_{l \in \mathcal{N}_i^-} A_{il}x_l(t) + \sum_{l \in \mathcal{N}_i^-} B_{il}u_l(t). \tag{23}$$

As the previous two cases (**??**) can be written as

$$f(\mathbf{u}) = \frac{1}{2}\mathbf{u}^\top \widetilde{\Pi}\mathbf{u} + \mathbf{u}^\top \widetilde{\pi} + \widetilde{c}, \tag{24}$$

for $\overline{\Pi}$ being a positive definite matrix, $\overline{\pi}$ being a vector, and $\overline{c}$ being a scalar. The following result characterises the generic situations where the iterations can be calculated at each subsystem with couplings described by (**??**).

Corollary 3. *Consider n interconnected discrete-time systems with the interconnection graph $\mathcal{I}$ where each system $i \in \mathcal{V}$ is described by (**??**), (**??**), and (**??**). Each u_i^k, $i = 1:n$, in (**??**) can be computed using $\mathcal{K}_i^\ell(k)$ if*

(i) For the case where $\ell = 0$, $\mathcal{J}_i(k) = \mathcal{K}_i^\ell(k)$ coincide if the edge set $\mathcal{E}_\mathcal{I}$ is empty.

(ii) For the case where $\ell = \tau - 1$, $\mathcal{J}_i(k) = \mathcal{K}_i^\ell(k)$ coincide if for all pair (i,j) such that $\{i,j\} \subset \mathcal{W}_l^\ell$, $l = 1:n$, where $\mathcal{W}_l^\ell$ is the set of all nodes with a path of length less than or equal to τ to l, then either $(i,j) \in \mathcal{E}_\mathcal{I}$ or $(j,i) \in \mathcal{E}_\mathcal{I}$.

(iii) For the case where $\ell \geq 2(\tau-1)$, $\mathcal{J}_i(k) \subseteq \mathcal{K}_i^\ell(k)$ for all interconnection graphs $\mathcal{I}$.

$$x_l^\top(t)Q_l x_l(t) = \left(\sum_{j=1}^{n} M_{lj}(t-t_0)\xi_j\right)^\top Q_l \sum_{j=1}^{n} M_{lj}(t-t_0)\xi_j + 2\left(\sum_{j=1}^{n} M_{lj}(t-t_0)\xi_j\right)^\top Q_l \sum_{k=t_0}^{t-1}\sum_{j=1}^{n} M_{lj}(k-t_0)B_j u_j(k-t_0) + \left(\sum_{k=t_0}^{t-1}\sum_{j=1}^{n} M_{lj}(k-t_0)B_j u_j(k-t_0)\right)^\top Q_l \sum_{k=t_0}^{t-1}\sum_{j=1}^{n} M_{lj}(k-t_0)B_j u_j(k-t_0). \tag{22}$$

As it can be seen the characterisation is identical to the case where couplings are only through state variables. The reason for it is that the existence of edges in the optimisation graph is influenced predominantly the state couplings.

IV. Concluding Remarks and Future Directions

In this paper the problem of constrained finite-horizon quadratic control for interconnected systems through implementing a projected gradient iterative scheme has been considered. Particularly, It was established when the iterations can be calculated via having access only to the model information of a subset of systems in the network it out to need any extra variables. Moreover, the relationship between such subsets and the interconnection graph of the network was identified. Studying the case where the state variables are constrained is a future research direction.

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