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# REAL ALGEBRAIC GEOMETRY FOR MATRICES OVER COMMUTATIVE RINGS

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Real algebraic geometry studies sets of the form

$$K_{\{g_1,\ldots,g_m\}} = \{x \in \mathbb{R}^d \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\},$$

where $d \in \mathbb{N}$ and $g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_d]$, and the corresponding preorderings

$$T_{\{g_1,\ldots,g_m\}} = \{ \sum_{\varepsilon \in \{0,1\}^m} c_\varepsilon g_1^{\varepsilon_1} \cdots g_m^{\varepsilon_m} \mid c_\varepsilon \in \sum \mathbb{R}[x_1, \ldots, x_d]^2 \}.$$

Its most fundamental result is due to Krivine and Stengle:

**Theorem A.** *For every $f, g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_d]$ we have that:*

(1) $K_{\{g_1,\ldots,g_m\}} = \emptyset$ *iff* $-1 \in T_{\{g_1,\ldots,g_m\}}$.
(2) $f(x) > 0$ *for every* $x \in K_{\{g_1,\ldots,g_m\}}$ *iff there exist* $t, t' \in T_{\{g_1,\ldots,g_m\}}$ *such that* $ft = 1 + t'$.
(3) $f(x) \geq 0$ *for every* $x \in K_{\{g_1,\ldots,g_m\}}$ *iff there exist* $t, t' \in T_{\{g_1,\ldots,g_m\}}$ *and* $k \in \mathbb{N}$ *such that* $ft = f^{2k} + t'$.
(4) $f(x) = 0$ *for every* $x \in K_{\{g_1,\ldots,g_m\}}$ *iff there exists* $k \in \mathbb{N}$ *such that* $-f^{2k} \in T_{\{g_1,\ldots,g_m\}}$.

Assertion (2) is called the Positivstellensatz, assertion (3) the Nichtnegativstellensatz and assertion (4) the semialgebraic Nullstellensatz.

We would like to extend this theory to matrix polynomials of fixed size $n \in \mathbb{N}$, i.e. from the ring $\mathbb{R}[x_1, \ldots, x_d]$ to the ring $\mathcal{M}_n(\mathbb{R}[x_1, \ldots, x_d])$ of all $n \times n$ matrices with entries from $\mathbb{R}[x_1, \ldots, x_d]$. We will consider sets of the form

$$K_{\{\mathbf{G}_1,\ldots,\mathbf{G}_m\}} = \{x \in \mathbb{R}^d \mid \mathbf{G}_1(x) \succeq 0, \ldots, \mathbf{G}_m(x) \succeq 0\}$$

where $\mathbf{G}_1, \ldots, \mathbf{G}_m$ belong to the set $\mathcal{S}_n(\mathbb{R}[x_1, \ldots, x_d])$ of all symmetric matrices from $\mathcal{M}_n(\mathbb{R}[x_1, \ldots, x_d])$ and "$\succeq 0$" means "is positive semidefinite". We will define the corresponding preordering $T_{\{\mathbf{G}_1,\ldots,\mathbf{G}_m\}} \subseteq \mathcal{S}_n(\mathbb{R}[x_1, \ldots, x_d])$ as the smallest quadratic module in $\mathcal{M}_n(\mathbb{R}[x_1, \ldots, x_d])$ which contains $\mathbf{G}_1, \ldots, \mathbf{G}_m$ and whose intersection with with the set $\mathbb{R}[x_1, \ldots, x_d] \cdot \mathbf{I}_n$ (where $\mathbf{I}_n$ is the identity matrix) is closed for multiplication. Then we will prove the following generalization of the Krivine-Stengle theorem:

**Theorem B.** *For every* $\mathbf{F}, \mathbf{G}_1, \ldots, \mathbf{G}_m \in \mathcal{S}_n(\mathbb{R}[x_1, \ldots, x_d])$ *we have that:*

(1) $K_{\{\mathbf{G}_1,\ldots,\mathbf{G}_m\}} = \emptyset$ *iff* $-\mathbf{I}_n \in T_{\{\mathbf{G}_1,\ldots,\mathbf{G}_m\}}$.
(2) $\mathbf{F}(x)$ *is positive definite for every* $x \in K_{\{\mathbf{G}_1,\ldots,\mathbf{G}_m\}}$ *iff there exist* $\mathbf{B}, \mathbf{B}' \in T_{\{\mathbf{G}_1,\ldots,\mathbf{G}_m\}}$ *such that* $\mathbf{FB} = \mathbf{BF} = \mathbf{I}_n + \mathbf{B}'$.
(3) $\mathbf{F}(x) \succeq 0$ *for every* $x \in K_{\{\mathbf{G}_1,\ldots,\mathbf{G}_m\}}$ *iff there exist* $\mathbf{B}, \mathbf{B}' \in T_{\{\mathbf{G}_1,\ldots,\mathbf{G}_m\}}$ *and* $k \in \mathbb{N}$ *such that* $\mathbf{FB} = \mathbf{BF} = \mathbf{F}^{2k} + \mathbf{B}'$.
(4) $\mathbf{F}(x) = 0$ *for every* $x \in K_{\{\mathbf{G}_1,\ldots,\mathbf{G}_m\}}$ *iff there exists* $k \in \mathbb{N}$ *such that* $-\mathbf{F}^{2k} \in T_{\{\mathbf{G}_1,\ldots,\mathbf{G}_m\}}$.

*The element* $\mathbf{B}$ *from assertion (2) can always be chosen from the set* $\mathbb{R}[x_1, \ldots, x_d] \cdot \mathbf{I}_n$ *while the element* $\mathbf{B}$ *from assertion (3) cannot.*

The main step in the proof of Theorem B is the reduction to the case where $\mathbf{G}_1, \ldots, \mathbf{G}_m$ belong to $\mathbb{R}[x_1, \ldots, x_d] \cdot \mathbf{I}_n$.

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