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On the Existence of a Mean-square Stabilizing Solution to a Continuous-time Modified Algebraic Riccati Equation*

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Abstract— **In this paper, we investigate the existence of a mean-square stabilizing solution to a continuous-time modified algebraic Riccati equation (MARE). Such an MARE comes up in the linear-quadratic optimal control of a class of stochastic systems. In most existing research works, only sufficient conditions are given for the existence of a mean-square stabilizing solution in terms of stabilizability and observability (or detectability) of the underlying stochastic system. In some papers, numerical conditions are provided. However, such conditions do not have explicit interpretations with respect to the dynamical properties of the underlying stochastic system. Here, a necessary and sufficient condition is given directly in terms of the system parameters and has explicit interpretations with respect to the dynamical properties of the underlying stochastic system. It is shown that the common assumption or condition of observability or detectability of certain stochastic system is not necessary.**

I. Introduction

As one of the most fundamental and important problems in control theory, linear-quadratic (LQ) optimal control has been extensively studied, both for deterministic and stochastic systems. See [8], [17], [12], [7], [6], [2] and the references therein. In the solutions to the deterministic LQ optimal control problems, as well as in the deterministic optimal linear filtering problems, the well-known standard algebraic Riccati equations (AREs) arise and play an essential role. The solutions, properties and applications of AREs are studied in a large number of research works, see e.g. [9], [10], and [17].

In the study of stochastic LQ optimal control, a class of modified algebraic Riccati equations (MAREs), which play a similar role to AREs, appear. In general, there are many solutions to MAREs. Among them, the one called mean-square stabilizing solution is our concern. With the static state feedback controller associated with the mean-square stabilizing solution, the corresponding stochastic LQ problem is solvable, i.e., the cost function is minimized and the stochastic closed-loop system is mean-square stabilizing. Thus in this paper, we aim to study the mean-square stabilizing solution to a specific MARE and in particular, its existence issue. This MARE arises in a continuous-time stochastic LQ optimal control problem with multiple multiplicative white noises on both the system state and control input. Such an MARE represents a large class of MAREs and it may take different and simple forms in the various special cases. It has been shown that the mean-square stabilizing solution to an MARE, if it exists, is unique and coincides with the so-called maximal solution, see [2] and [6]. In most existing research works, only sufficient conditions are given for the existence of the mean-square stabilizing solution. These sufficient conditions are often given in terms of stabilizability and observability (or detectability) of certain stochastic systems [7], [3], [13], [2]. Note that the stabilizability of stochastic systems is usually defined in the mean-square sense while there are several ways to define the observability and detectability of stochastic systems from different view points in these papers. In [6], to ensure the existence of the mean-square stabilizing solution, together with mean-square stabilizability, only observability of the modes of a deterministic system on the imaginary axis is required, instead of detectability of this deterministic system. In the book [5], one numerical necessary and sufficient condition is given in terms of the feasibility of some linear matrix inequalities (LMIs). However, such a condition does not have explicit interpretations with respect to the dynamical properties of the underlying stochastic system.

In view of the current results, in this paper, we focus on investigating a necessary and sufficient condition ensuring the existence of the mean-square stabilizing solution to a continuous-time MARE. Moreover, we hope such a necessary and sufficient condition is given directly in terms of the system parameters and has explicit interpretations with respect to the dynamical properties of the underlying stochastic system. In [16], we develop an approach to obtain an explicit necessary and sufficient condition for the existence of the mean-square stabilizing solution to a discrete-time MARE, which arises in our previous work [14], [15], with the help of theory of positive operators. Inspired by the work in [16], we follow a similar approach to obtain a necessary and sufficient condition for the continuous-time MARE. It is shown that the observability or detectability of certain stochastic system is not necessary. It solves the question we have been asking whether there exists a necessary and sufficient condition which is analogous to the one ensuring the stabilizing solution to the standard definite ARE. We also apply this result to a special MARE under the framework of channel/controll co-design. In this case, the condition becomes simple and can be very easy to verified.

The remainder of this paper is organized as follows. The problem is stated in Section II. In Section III, a

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resolvent positive operator and its distinguished eigenvalues are defined to serve the next section. In Section IV, a necessary and sufficient condition for the existence of the mean-square stabilizing solution to a continuous-time MARE is obtained and several special cases are discussed. A conclusion follows in Section V.

The notation is rather standard in this paper. Denote the real vector space of $n\times n$ real symmetric matrices by $\mathcal{S}_n$. One proper cone of $\mathcal{S}_n$ is given by the subset of positive semi-definite matrices $\mathcal{P}_n \triangleq \{X \in \mathcal{S}_n : X \geq 0\}$. The symbol $\odot$ stands for Hadamard product.

II. Problem Formulation

In this paper, we investigate the mean-square stabilizing solution, which is defined later, to the following continuous-time MARE:

$$A'X + XA + Q + \sum_{i=1}^{N} A_i'XA_i - (XB + S + \sum_{i=1}^{N} A_i'XB_i) \times (R + \sum_{i=1}^{N} B_i'XB_i)^{-1}(XB + S + \sum_{i=1}^{N} A_i'XB_i)' = 0, \quad (1)$$

where A, $A_i \in \mathbb{R}^{n\times n}$, B, $B_i \in \mathbb{R}^{n\times m}$ for $i = 1, \ldots, N$, and $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \in \mathcal{P}_{n+m}$ with $R > 0$. It arises in the infinite-horizon continuous-time stochastic LQ optimal control problem described below.

The stochastic system is described by

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\mathrm{d}t + \sum_{i=1}^{N} [A_i x(t) + B_i u(t)]\mathrm{d}\omega_i(t), \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input which is generated by a finite-dimensional state feedback controller $\boldsymbol{K}$ with the following form

$$\begin{aligned} \dot{x}_K(t) &= A_K(t)x_K(t) + B_K(t)x(t), \\ u(t) &= C_K(t)x_K(t) + D_K(t)x(t), \end{aligned}$$

and $\omega_i(t)$, $i = 1, \ldots, N$, are mutually uncorrelated one-dimensional standard Wiener processes on a given probability space $(\Omega, \mathcal{F}, P)$. It is assumed that the initial state $x(0) \in \mathbb{R}^n$ is independent of $\omega_i(t)$. Denote

$$\boldsymbol{A} = \begin{bmatrix} A' & A_1' & \ldots & A_N' \end{bmatrix}', \boldsymbol{B} = \begin{bmatrix} B' & B_1' & \ldots & B_N' \end{bmatrix}'.$$

The stochastic system (2) can be denoted by $\begin{bmatrix} \boldsymbol{A} \mid \boldsymbol{B} \end{bmatrix}$ for simplicity. The corresponding autonomous stochastic system is described by

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + \sum_{i=1}^{N} A_i x(t)\mathrm{d}\omega_i(t). \quad (3)$$

Definition 1: The stochastic system (3) is said to be mean-square stable, if for any initial state $x(0)$, $\boldsymbol{E}[x(t)x'(t)]$ is well-defined for any $t > 0$ and $\lim_{t\to\infty} \boldsymbol{E}[x(t)x'(t)] = 0$.

When $u(t)$ is generated by a controller $\boldsymbol{K}$, denote the overall system state by $\hat{x}(t) = \begin{bmatrix} x'(t) & x_K'(t) \end{bmatrix}'$. The closed-loop system can be written as

$$\mathrm{d}\hat{x}(t) = \begin{bmatrix} A + BD_K(t) & BC_K(t) \\ B_K(t) & A_K(t) \end{bmatrix} \hat{x}(t)\mathrm{d}t + \sum_{i=1}^{N} \begin{bmatrix} A_i + B_i D_K(t) & B_i C_K(t) \\ 0 & 0 \end{bmatrix} \hat{x}(t)\mathrm{d}\omega_i(t). \quad (4)$$

Then $\begin{bmatrix} \boldsymbol{A} \mid \boldsymbol{B} \end{bmatrix}$ is said to be mean-square stabilizable if there exists a controller $\boldsymbol{K}$ such that the closed-loop system (4) is mean-square stable. And this controller $\boldsymbol{K}$ is said to be mean-square stabilizing.

With the LQ cost function defined as

$$J(x(0), u(\cdot)) = \boldsymbol{E}\left[\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t\right],$$

the aim of the stochastic LQ optimal control is to find an optimal mean-square stabilizing state feedback controller to minimize $J(x(0), u(\cdot))$. It is shown in [7] that this stochastic LQ optimal control problem is solvable if and only if the MARE (1) has the unique mean-square stabilizing solution $X \in \mathcal{P}_n$, i.e., the associated static state feedback gain

$$F = -(R + \sum_{i=1}^{N} B_i'XB_i)^{-1}(XB + S + \sum_{i=1}^{N} A_i'XB_i)', \quad (5)$$

is mean-square stabilizing. Then F is exactly the optimal state feedback controller we are looking for with the minimal cost $x'(0)Xx(0)$.

Thus researchers are interested in the existence of the mean-square stabilizing solution to the MARE (1). As far as we know, in most existing research works, only sufficient conditions are provided. In this paper, we are devoted to seeking an explicit necessary and sufficient condition directly in terms of the system parameters. Note that most of the forthcoming lemmas have been derived by other researchers. We employ them and the theory of positive operators to get a desired necessary and sufficient condition.

III. Distinguished Eigenvalues

Let $X(t) \triangleq \boldsymbol{E}[x(t)x'(t)]$, where $x(t)$ is the state of the autonomous stochastic system (3). By Itô's formula, we have that

$$\dot{X}(t) = AX(t) + X(t)A' + \sum_{i=1}^{N} A_i X(t) A_i'. \quad (6)$$

Hence the following associated operator is defined:

$$\mathcal{T}_{\boldsymbol{A}} : X \in \mathcal{S}_n \mapsto AX + XA' + \sum_{i=1}^{N} A_i X A_i' \in \mathcal{S}_n.$$

Denote the complex spectrum of $\mathcal{T}_{\boldsymbol{A}}$ by $\sigma(\mathcal{T}_{\boldsymbol{A}})$. The spectral abscissa is defined as $\beta(\mathcal{T}_{\boldsymbol{A}}) \triangleq \max_{\lambda\in\sigma(\mathcal{T}_{\boldsymbol{A}})} \Re(\lambda)$. It is shown in [3] that the stochastic system (3) is mean-square stable if and only if $\beta(\mathcal{T}_{\boldsymbol{A}}) < 0$.

Let us take a close look at the operator $\mathcal{T}_{\boldsymbol{A}}$. We know that, a linear operator $\mathcal{T}:\mathcal{S}_n\to\mathcal{S}_n$ is called positive if $\mathcal{T}(\mathcal{P}_n)\subset\mathcal{P}_n$. It is called resolvent positive if there exists an $\alpha_0\in\mathbb{R}$, such that for any $\alpha>\alpha_0$, the operator $\alpha\mathcal{I}-\mathcal{T}$ has a positive inverse. Evidently the operator $X\in\mathcal{S}_n\mapsto A_iXA_i'\in\mathcal{S}_n$ is positive. It is easy to check that, as in [3] and [7], the continuous-time Lyapunov operator

$$\mathcal{L}_A: X\in\mathcal{S}_n\mapsto AX+XA'\in\mathcal{S}_n$$

is resolvent positive. Therefore $\mathcal{T}_{\boldsymbol{A}}$ is also resolvent positive due to the fact that the sum of a resolvent positive operator and a positive operator is still resolvent positive. Moreover, $\beta(\mathcal{T}_{\boldsymbol{A}})$ is an eigenvalue of $\mathcal{T}_{\boldsymbol{A}}$ with some eigenvector $X\in\mathcal{P}_n\backslash\{0\}$, i.e., $\mathcal{T}_{\boldsymbol{A}}(X)=\beta(\mathcal{T}_{\boldsymbol{A}})X$. These facts can be found in [1] and [3].

Since it holds $X(t)\geq 0$ and $X(t)$ satisfies the differential equation or linear system (6), we conjecture that the positive semi-definite eigenvectors of the operator $\mathcal{T}_{\boldsymbol{A}}$ are essential. Indeed, the spectrum of $\mathcal{T}_{\boldsymbol{A}}$ restricted to $\mathcal{P}_n$, defined as follows, play an important role in our problem:

$$\sigma_{\mathcal{P}}(\mathcal{T}_{\boldsymbol{A}})=\{\lambda\in\sigma(\mathcal{T}_{\boldsymbol{A}}):\mathcal{T}_{\boldsymbol{A}}(X)=\lambda X, X\in\mathcal{P}_n, X\neq 0\}.$$

It is easy to show that $\sigma_{\mathcal{P}}(\mathcal{T}_{\boldsymbol{A}})\subset\mathbb{R}$. However, $\sigma_{\mathcal{P}}(\mathcal{T}_{\boldsymbol{A}})$ does not include all the real eigenvalues of $\sigma(\mathcal{T}_{\boldsymbol{A}})$. From the above discussion, it is easy to see that

$$\beta(\mathcal{T}_{\boldsymbol{A}})\in\sigma_{\mathcal{P}}(\mathcal{T}_{\boldsymbol{A}}),$$

which implies

$$\beta(\mathcal{T}_{\boldsymbol{A}})=\max\sigma_{\mathcal{P}}(\mathcal{T}_{\boldsymbol{A}}).$$

We call the members in $\sigma_{\mathcal{P}}(\mathcal{T}_{\boldsymbol{A}})$ the distinguished eigenvalues of $\mathcal{T}_{\boldsymbol{A}}$ or system (6). Let $\lambda\in\sigma_{\mathcal{P}}(\mathcal{T}_{\boldsymbol{A}})$. If $\lambda<0$, it is said to be stable, otherwise, it is unstable.

Define another positive operator

$$\mathcal{O}_C: X\in\mathcal{S}_n\mapsto CXC'\in\mathcal{S}_p,$$

where $C\in\mathbb{R}^{p\times n}$. Then observability of the distinguished eigenvalues is defined as follows.

Definition 2: A distinguished eigenvalue λ of $\mathcal{T}_{\boldsymbol{A}}$ is said to be observable with respect to $\mathcal{O}_C$ if for the associated eigenvector $X\in\mathcal{P}_n\backslash\{0\}$, we have that

$$\mathcal{O}_C(X)\neq 0.$$

We can simply say that λ is an observable distinguished eigenvalue of $\left[\dfrac{\mathcal{T}_A}{\mathcal{O}_C}\right]$. Otherwise, it is said to be unobservable.

Among all the distinguished eigenvalues, the one at 0 plays a crucial role in our problem. For the continuous-time Lyapunov operator $\mathcal{L}_A$, one property of its distinguished eigenvalues at 0, which will be used in the later development, is stated below. Its proof is shown in Appendix. Note that the proof is similar to the one for Lemma 4 in [4] with some corrections.

Lemma 1: $\left[\dfrac{A}{C}\right]$ has no unobservable eigenvalues on the imaginary axis if and only if $\left[\dfrac{\mathcal{L}_A}{\mathcal{O}_C}\right]$ has no unobservable distinguished eigenvalue at 0.

IV. A NECESSARY AND SUFFICIENT CONDITION

Since the optimal mean-square stabilizing state feedback controller, if it exists, must be a static one, we only concern the static state feedback controller in what follows.

When $u(t)$ is generated by a static state feedback gain $F\in\mathbb{R}^{m\times n}$, the closed-loop system (4) is given by

$$\mathrm{d}x(t)=(A+BF)x(t)\mathrm{d}t+\sum_{i=1}^{N}(A_i+B_iF)x(t)\mathrm{d}\omega_i(t). \quad (7)$$

Then the associated resolvent positive operator is given by

$$\begin{aligned}\mathcal{T}_{\boldsymbol{A+BF}}: X\in\mathcal{S}_n\mapsto&(A+BF)X+X(A+BF)'\\&+\sum_{i=1}^{N}(A_i+B_iF)X(A_i+B_iF)'\in\mathcal{S}_n.\end{aligned}$$

Since $\mathcal{S}_n$ is a Hilbert space endowed with the inner product $\langle X,Y\rangle=\mathrm{tr}XY$, the adjoint operator $\mathcal{T}^*_{\boldsymbol{A+BF}}$ satisfying

$$\mathrm{tr}\mathcal{T}_{\boldsymbol{A+BF}}(X)Y=\mathrm{tr}X\mathcal{T}^*_{\boldsymbol{A+BF}}(Y),\ \forall X,Y\in\mathcal{S}_n$$

is given by:

$$\begin{aligned}\mathcal{T}^*_{\boldsymbol{A+BF}}: X\in\mathcal{S}_n\mapsto&(A+BF)'X+X(A+BF)\\&+\sum_{i=1}^{N}(A_i+B_iF)'X(A_i+B_iF)\in\mathcal{S}_n.\end{aligned}$$

It is clear that a solution $X\in\mathcal{P}_n$ to the MARE (1) is mean-square stabilizing if $\beta(\mathcal{T}_{\boldsymbol{A+BF}})<0$ where F is the associated controller given by (5) and it is said to be a strong solution if $\beta(\mathcal{T}_{\boldsymbol{A+BF}})\leq 0$.

Denote

$$\begin{aligned}\mathcal{M}(X)&\triangleq A'X+XA+Q+\sum_{i=1}^{N}A_i'XA_i,\\ \mathcal{L}(X)&\triangleq XB+S+\sum_{i=1}^{N}A_i'XB_i,\\ \mathcal{N}(X)&\triangleq R+\sum_{i=1}^{N}B_i'XB_i,\end{aligned}$$

and define the following set

$$\Gamma=\left\{X\in\mathcal{S}_n\left|\begin{bmatrix}\mathcal{M}(X)&\mathcal{L}(X)\\\mathcal{L}'(X)&\mathcal{N}(X)\end{bmatrix}\geq 0, \mathcal{N}(X)>0\right.\right\}.$$

We know that a solution $X\in\mathcal{S}_n$ to the MARE (1) is said to be a maximal solution if $X\geq\tilde{X}$ for any $\tilde{X}\in\Gamma$.

The following lemma states the relationship between the maximal solution and the stochastic LQ problem.

Lemma 2: [3] When $\begin{bmatrix}\boldsymbol{A}\mid\boldsymbol{B}\end{bmatrix}$ is mean-square stabilizable, there exists the unique maximal solution $X_+\in\mathcal{P}_n$ to the MARE (1) and the minimal cost function is given by $x'(0)X_+x(0)$. The maximal solution is a strong solution. Moreover, the mean-square stabilizing solution, if it exists, coincides with the maximal solution.

The maximal solution can be numerically computed by solving a convex optimization problem [2] or applying an iterative procedure, e.g., [3], [7]. We are more interested in finding an explicit condition under which the maximal solution is indeed the mean-square stabilizing solution.

To this end, an associated stochastic system is defined as follows:

$$\mathrm{d}x(t)=(A-BR^{-1}S')x(t)\mathrm{d}t+\sum_{i=1}^{N}(A_i-B_iR^{-1}S')x(t)\mathrm{d}\omega_i(t),$$
$$z(t)=(Q-SR^{-1}S')^{1/2}x(t).$$

The corresponding resolvent positive operator is exactly $\mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}$. Before we state the main theorem, we state a lemma that will be used in the proof of the theorem. The lemma is interesting in its own sake.

Lemma 3: Given a solution $X \in \mathcal{P}_n$ to the MARE (1), any unstable distinguished eigenvalue of $\mathcal{T}_{\boldsymbol{A}+\boldsymbol{B}F}$ where F is the associated controller of X given by (5) is an unobservable distinguished eigenvalue of $\left[\dfrac{\mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}}{\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}}\right]$.

Now we are ready to state the desired necessary and sufficient condition.

Theorem 1: There exists a mean-square stabilizing solution to the MARE (1) if and only if

i) $[\boldsymbol{A} \mid \boldsymbol{B}]$ is mean-square stabilizable;

ii) $\left[\dfrac{\mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}}{\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}}\right]$ has no unobservable distinguished eigenvalue at 0.

The proofs of Lemma 3 and Theorem 1 are shown in Appendix. The above main theorem solves the question we have been asking whether there exists a necessary and sufficient condition which is analogous to the one ensuring the stabilizing solution to the standard definite ARE [17]. There may be no too much computational advantage in verifying this necessary and sufficient condition, especially for such a general MARE. However, this result can cover or imply most of the existing conditions ensuring the existence of the mean-square stabilizing solution to MAREs. Moreover, in the special cases provided below, our condition can be very easy to verify.

One is the standard definite continuous-time ARE:

$$A'X + XA + Q - (XB + S)R^{-1}(B'X + S') = 0, \quad (8)$$

with $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \triangleq \begin{bmatrix} C' \\ D' \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}$ and D has full column rank. Then the following corollary is derived by directly applying Theorem 1 with $A_i = 0$, $B_i = 0$ for all $i \in \{1, 2, \ldots, N\}$.

Corrollary 1: A necessary and sufficient condition for the existence of the stabilizing solution to the ARE (8) is given by

i) $[A|B]$ is stabilizable;

ii) $\left[\dfrac{\mathcal{L}_{A-BR^{-1}S'}}{\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}}\right]$ has no unobservable distinguished eigenvalue at 0.

By Lemma 1 together with some computations, condition ii) is equivalent to that $\left[\dfrac{A-BR^{-1}S'}{C-DR^{-1}S'}\right]$ has no unobservable eigenvalues on the imaginary axis, which is the result shown in [17].

Now consider the following MARE

$$A'X + XA + C'C - (XB + C'D) \quad (9)$$
$$\times\, [\mathrm{SNR}^{-1} \odot (B'XB) + W \odot (D'D)]^{-1}(B'X + D'C) = 0,$$

with the associated controller given by

$$F = -M^{-1}[\mathrm{SNR}^{-1} \odot (B'XB) + W \odot (D'D)]^{-1}(B'X + D'C).$$

The MARE (9) arises in our previous work [2] on the LQ optimal control problem for continuous-time LTI systems with random input gains as shown in Fig.1.

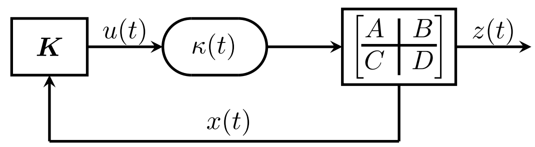

Fig. 1. LQ optimal control
for continuous-time LTI systems with random input gains

The stochastic system is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\kappa(t)u(t), \\ z(t) &= Cx(t) + D\kappa(t)u(t), \end{aligned} \quad (10)$$

The random input gain $\kappa(t)$ is given by a diagonal random matrix $\mathrm{diag}\{\kappa_1(t), \ldots, \kappa_m(t)\}$, whose diagonal elements $\kappa_i(t), i = 1, \ldots, m$ are mutually uncorrelated i.i.d random processes with mean $\mu_i = \boldsymbol{E}[\kappa_i(t)] \neq 0$ and variance $\sigma_i^2 = \boldsymbol{E}\left[(\kappa_i(t) - \mu_i)^2\right]$, respectively. The signal-to-noise ratio of the i-th input channel is denoted by $\mathrm{SNR}_i \triangleq \dfrac{\mu_i^2}{\sigma_i^2}$. Denote

$$M \triangleq \mathrm{diag}\{\mu_1, \ldots, \mu_m\}, \quad \Sigma^2 \triangleq \mathrm{diag}\{\sigma_1^2, \ldots, \sigma_m^2\},$$
$$\mathrm{SNR} \triangleq \mathrm{diag}\{\mathrm{SNR}_1, \ldots, \mathrm{SNR}_m\},$$
$$W \triangleq \begin{bmatrix} 1+\mathrm{SNR}_1^{-1} & 1 & \cdots & 1 \\ 1 & 1+\mathrm{SNR}_2^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1+\mathrm{SNR}_m^{-1} \end{bmatrix}.$$

The channel capacity of the i-th input channel is defined as

$$\mathfrak{C}_i = \frac{1}{2}\mathrm{SNR}_i,$$

while the overall channel capacity is given by $\mathfrak{C} = \sum_{i=1}^{m} \mathfrak{C}_i$.

The LQ cost function is defined as

$$J(x(0), u(\cdot)) = \boldsymbol{E}\left[\int_0^\infty z'(t)z(t)\mathrm{d}t\right]$$
$$= \boldsymbol{E}\left[\int_0^\infty \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix}' \begin{bmatrix} C'C & C'D \\ D'C & W \odot (D'D) \end{bmatrix} \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix} \mathrm{d}t\right].$$

Let $v(t) = Mu(t)$. Then

$$J(x(0), u(\cdot))$$
$$= \boldsymbol{E}\left[\int_0^\infty \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \begin{bmatrix} C'C & C'D \\ D'C & W \odot (D'D) \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \mathrm{d}t\right].$$

Hence

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \triangleq \begin{bmatrix} C'C & C'D \\ D'C & W \odot (D'D) \end{bmatrix}.$$

Also assume that $R > 0$, replacing the stronger assumption that D has full column rank.

In [2], we put this LQ problem under the framework of channel/controller co-design, i.e., the controller designer also has the freedom to design the channel by allocating the given overall channel capacity to the individual channels. Under this framework, the mean-square stabilization problem becomes analytically solvable, i.e., system (10) is mean-square stabilizable if and only if $\mathfrak{C} > H_c(A)$, where

$$H_c(A) \triangleq \sum_{\substack{\lambda_i \in \sigma(A) \\ \Re(\lambda_i) > 0}} \lambda_i$$

is the topological entropy of system (10).

Define the following stochastic system

$$\begin{aligned} \dot{x}(t) &= [A - B\kappa(t) M^{-1} R^{-1} S'] x(t), \\ z(t) &= (Q - SR^{-1}S')^{1/2} x(t). \end{aligned}$$

The associated resolvent positive operator is given by $\mathcal{T}_{\tilde{\boldsymbol{A}}}$ with

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} A - BR^{-1}S' \\ \sigma_1 b_1 (-M^{-1}R^{-1}S')_1 \\ \vdots \\ \sigma_m b_m (-M^{-1}R^{-1}S')_m \end{bmatrix},$$

where b_i is the i-th column of B and $(-M^{-1}R^{-1}S')_i$ is the i-th row of $-M^{-1}R^{-1}S'$. Then a necessary and sufficient condition for the existence of the mean-square stabilizing solution to the MARE (9) under a given overall channel capacity $\mathfrak{C}$ is shown below by applying Theorem 1.

Corrollary 2: Under the framework of channel/controller co-design, there exists a mean-square stabilizing solution to the MARE (9) if and only if

i) $\mathfrak{C} > H_c(A)$.
ii) $\left[\frac{\mathcal{T}_{\tilde{\boldsymbol{A}}}}{\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}}\right]$ has no unobservable distinguished eigenvalue at 0.

Obviously, condition i) is very easy to verify. It takes some steps to verify condition ii). If we further assume that $S = 0$, then condition ii) becomes that $\left[\frac{A}{C}\right]$ has no unobservable eigenvalues on the imaginary axis by Lemma 1. Then in this case, condition ii) is very easy to computed.

V. CONCLUSION

An explicit necessary and sufficient condition ensuring the existence of the mean-square stabilizing solution to the MARE (1) is obtained. It generalizes the one for the standard definite ARE (8). We hope more insight can be gained by this result in a follow-up study.

APPENDIX

Proof of Lemma 1

Suppose that $\left[\frac{\mathcal{L}_A}{\mathcal{O}_C}\right]$ has no unobservable distinguished eigenvalue at 0. Assume that $\left[\frac{A}{C}\right]$ has an unobservable eigenvalue on the imaginary axis, i.e., there exists $\lambda \in \mathbb{C}$ with $\Re(\lambda) = 0$ and $v \in \mathbb{C}^n \backslash \{0\}$ such that

$$\begin{aligned} Av &= \lambda v, \\ Cv &= 0. \end{aligned}$$

It is trivial when the imaginary part of λ is zero, i.e., $\lambda = 0$. When the imaginary part of λ is not zero, we have

$$\begin{aligned} A\bar{v} &= \lambda^* \bar{v}, \\ C\bar{v} &= 0, \end{aligned}$$

since the complex eigenvalues of a real matrix always come in conjugate pairs whose eigenvectors are also conjugate. Let

$$X = vv^* + \bar{v}\bar{v}^* = vv^* + \overline{vv^*} \in \mathcal{P}_n.$$

Then it is easy to verify that

$$\begin{aligned} \mathcal{L}_A(X) &= 0, \\ \mathcal{O}_C(X) &= 0, \end{aligned}$$

which causes a contradiction.

Vice versa assume that there exists $X \in \mathcal{P}_n \backslash \{0\}$ such that $\mathcal{L}_A(X) = 0$ and $\mathcal{O}_C(X) = 0$. As shown in [11], X has a decomposition $X = \sum_{j=1}^r x_j x_j^*$ with

$$\begin{aligned} Ax_j x_j^* + x_j x_j^* A' &= 0, \\ Cx_j x_j^* C' &= 0, \end{aligned} \tag{11}$$

where $x_j \in \mathbb{C}^n$ and $r = \text{rank} X \geq 1$.

Note that $\sigma(\mathcal{L}_A) \subset \sigma(\tilde{\mathcal{L}}_A)$ where

$$\tilde{\mathcal{L}}_A : X \in \mathbb{R}^{n \times n} \mapsto AX + XA' \in \mathbb{R}^{n \times n}.$$

It is well known that

$$\sigma(\tilde{\mathcal{L}}_A) = \{\lambda + \bar{\mu} | \lambda, \mu \in \sigma(A)\},$$

and $\nu\tau^*$ is the eigenvector of $\tilde{\mathcal{L}}_A$ corresponding to $\lambda + \bar{\mu}$ where ν, τ are the eigenvectors of A corresponding to λ, μ, respectively.

Therefore by viewing the equation (11), at least there exists λ with $\Re(\lambda) = 0$ such that $Ax_1 = \lambda x_1$ and $Cx_1 = 0$. This causes a contradiction with that $\left[\frac{A}{C}\right]$ has no unobservable eigenvalues on the imaginary axis. □

Proof of Lemma 3

With the solution X, the MARE (1) can be rewritten as

$$\mathcal{T}^*_{\boldsymbol{A}+\boldsymbol{B}F}(X) + \begin{bmatrix} I \\ F \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} = 0. \tag{12}$$

Suppose that $\lambda \geq 0$ is an unstable distinguished eigenvalue of $\mathcal{T}_{\boldsymbol{A}+\boldsymbol{B}F}$ with an eigenvector $Y \in \mathcal{P}_n\backslash\{0\}$ such that $\mathcal{T}_{\boldsymbol{A}+\boldsymbol{B}F}(Y) = \lambda Y$. Then we have

$$\begin{aligned} 0 \leq& \text{tr} \begin{bmatrix} I \\ F \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} Y = -\text{tr}\mathcal{T}^*_{\boldsymbol{A}+\boldsymbol{B}F}(X)Y \\ =& -\text{tr}X\mathcal{T}_{\boldsymbol{A}+\boldsymbol{B}F}(Y) = -\lambda\text{tr}XY \leq 0, \end{aligned}$$

which implies $\text{tr} \begin{bmatrix} I \\ F \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} Y = 0$. On the other hand,

$$\begin{aligned} &\begin{bmatrix} I \\ F \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} \\ &= Q - SR^{-1}S' + (R^{-1}S' + F)'R(R^{-1}S' + F), \end{aligned}$$

hence

$$\text{tr}[Q - SR^{-1}S' + (R^{-1}S' + F)'R(R^{-1}S' + F)]Y = 0.$$

Since $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$ and $R > 0$, it follows that the Schur complement $Q - SR^{-1}S' \geq 0$. Therefore

$$\begin{aligned} &\text{tr}(Q - SR^{-1}S')Y = 0, \\ &\text{tr}(R^{-1}S' + F)'R(R^{-1}S' + F)Y = 0. \end{aligned}$$

With the assumption $R > 0$, we have $(R^{-1}S' + F)Y = 0$, i.e., $FY = -R^{-1}S'Y$. Therefore

$$\mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}(Y) = \mathcal{T}_{\boldsymbol{A}+\boldsymbol{B}F}(Y) = \lambda Y,$$

i.e., λ is a distinguished eigenvalue of $\mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}$. At the same time,

$$\text{tr}\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}(Y) = \text{tr}(Q - SR^{-1}S')Y = 0,$$

which implies $\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}(Y) = 0$. Hence λ is an unobservable distinguished eigenvalue of $\left[\dfrac{\mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}}{\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}}\right]$. □

Proof of Theorem 1

We first show the sufficiency. By Lemma 2, the maximal solution $X_+ \in \mathcal{P}_n$ exists and $\beta(\mathcal{T}_{\boldsymbol{A}+\boldsymbol{B}F_+}) \leq 0$ with the associated controller F_+. Since 0 is not an unobservable distinguished eigenvalue of $\left[\dfrac{\mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}}{\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}}\right]$, by Lemma 3 $\beta(\mathcal{T}_{\boldsymbol{A}+\boldsymbol{B}F_+}) < 0$. Therefore the maximal solution is exactly the mean-square stabilizing solution.

Evidently, the mean-square stabilizability of $[\boldsymbol{A} \mid \boldsymbol{B}]$ is necessary. Then the necessity is proved by showing that when $\left[\dfrac{\mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}}{\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}}\right]$ has an unobservable distinguished eigenvalue at 0, for any $X \in \mathcal{P}_n$ satisfying the MARE (1), it is not mean-square stabilizing. For simplicity, denote

$$\begin{aligned} T =& \begin{bmatrix} T_{11} & T_{12} \\ T'_{12} & T_{22} \end{bmatrix} \\ \triangleq& \begin{bmatrix} I & -SR^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \mathcal{N}(X) \end{bmatrix} \begin{bmatrix} I & 0 \\ -R^{-1}S' & I \end{bmatrix}, \end{aligned}$$

with

$$\begin{aligned} &T_{11} = \mathcal{T}^*_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}(X) + Q - SR^{-1}S', \\ &T_{12} = XB + \sum_{i=1}^{N} A'_i X B_i - \sum_{i=1}^{N} SR^{-1}B'_i X B_i, \\ &T_{22} = \mathcal{N}(X). \end{aligned}$$

With some computations, we can get that

$$\mathcal{M}(X) - \mathcal{L}(X)\mathcal{N}(X)^{-1}\mathcal{L}'(X) = T_{11} - T_{12}T_{22}^{-1}T'_{12}.$$

Since $\left[\dfrac{\mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}}{\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}}\right]$ has an unobservable distinguished at 0, there exists $Y \geq 0$ such that

$$\begin{aligned} &\mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}(Y) = 0 \\ &\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}(Y) = 0. \end{aligned}$$

It follows that

$$\begin{aligned} \text{tr}T_{11}Y &= \text{tr}\mathcal{T}^*_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}(X)Y + \text{tr}(Q - SR^{-1}S')Y \\ &= \text{tr}X\mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}(Y) + \text{tr}\mathcal{O}_{(Q-SR^{-1}S')^{1/2}}(Y) \\ &= 0. \end{aligned}$$

Then for any solution $X \in \mathcal{P}^n$ to the MARE (1),

$$\begin{aligned} 0 &= \text{tr}[\mathcal{M}(X) - \mathcal{L}(X)\mathcal{N}(X)^{-1}\mathcal{L}'(X)]Y \\ &= \text{tr}(T_{11} - T_{12}T_{22}^{-1}T'_{12})Y \\ &= -\text{tr}T_{12}T_{22}^{-1}T'_{12}Y, \end{aligned}$$

which implies $T_{12}T_{22}^{-1}T'_{12}Y = 0$ due to $T_{22} > 0$ and $Y \geq 0$. It follows that $T'_{12}Y = 0$. On the other hand,

$$\begin{aligned} &(R^{-1}S' + F)Y \\ =& \left[R^{-1}S' - \mathcal{N}(X)^{-1}(B'X + S' + \sum_{i=1}^{N} B'_i X A_i)\right] Y \\ =& \mathcal{N}(X)^{-1} \left[\sum_{i=1}^{N} B'_i X B_i R^{-1}S' - B'X - \sum_{i=1}^{N} B'_i X A_i\right] Y \\ =& -\mathcal{N}(X)^{-1}T'_{12}Y \\ =& 0, \end{aligned}$$

which implies

$$FY = -R^{-1}S'Y.$$

Therefore

$$\mathcal{T}_{\boldsymbol{A}+\boldsymbol{B}F}(Y) = \mathcal{T}_{\boldsymbol{A}-\boldsymbol{B}R^{-1}S'}(Y) = 0,$$

i.e., $0 \in \sigma_{\mathcal{P}}(\mathcal{T}_{\boldsymbol{A}+\boldsymbol{B}F})$, then $\beta(\mathcal{T}_{\boldsymbol{A}+\boldsymbol{B}F}) \geq 0$. Hence the closed-loop system (7) is not mean-square stabilizing with the controller F, i.e., X is not mean-square stabilizing. This shows the necessity. □

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