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Robust Stability in Anisotropy-based Theory with Non-zero Mean of Input Sequence

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***Abstract*— One of the problems for the anisotropy-based analysis with non-zero mean of input sequence is considered. The closed-loop system robust stability conditions on anisotropic norm of the system transfer function and the uncertainty transfer function are obtained for linear discrete time-invariant systems. For the systems with anisotropic norm bounded uncertainty, the robust stability criterion is derived.**

***Index Terms*— Discrete-time systems, small gain theorem, robust stability**

I. INTRODUCTION

During the last thirty years the controller design method, called H_∞–control theory [1], [2], has been developed. The standard H_∞– control problem is to find controller $\hat{K}$, which minimizes the H_∞–norm of the closed-loop transfer function from input disturbances to controllable output over internally stabilizing controllers K. It is assumed that input the disturbances are square-summable signals. In that theory some criteria of robust stability, based on Small Gain Theorem [3], are derived. The Small Gain Theorem is a corollary from the submultiplicative property of induced norms and its sense is that system is robust stable relative to uncertainty if the product of the H_∞–norms of the transfer function of the plant and uncertainty is strictly less than 1.

For the last twenty years, the anisotropy-based control theory has been derived [4]–[8]. In the anisotropy-based theory the statistical characteristics of input disturbances are not known precisely. This theory is based on using the stochastic norm in performance criteria. The stochastic norm measures the sensitivity of the system output to random input disturbances whose probability distribution is not precisely known. The stochastic norm is the induced one. The a-anisotropic norm of a system is a particular case of the stochastic norm [6] defined as the supremum of the ratio of the root mean square value of the system output to that of the input over all stationary Gaussian inputs, the mean anisotropy of which is bounded above by a nonnegative parameter a.

Since there is a notion of robust stability for the H_∞–norm-bounded uncertainties, then it is natural to use the a-anisotropic norm for getting the conditions corresponding to the Small Gain Theorem. In this case we can expect, that ability to vary the parameter a allows to reduce a priori requests to model parameter uncertainty and to get less conservative requirements to system and uncertainties then the same in the H_∞–control theory.

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The results of this paper generalize the results of [5] to the case when the input random sequence has non-zero expectation.

II. PROBLEM STATEMENT

Let us consider the nominal plant M, which defines input-output relations as follows

$$\begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} q \\ w \end{pmatrix}. \qquad (1)$$

Here $q \in R^m$, $w \in R^p$ are the system inputs, $p \in R^m$, $z \in R^q$ are the system outputs, q may be non observable, M_{ij} is the transfer function from the i-th input to j-th output, $i, j = 1, 2$.

Fig. 1 represents the idealized scheme of the plant with uncertainty. The uncertainty is described by the transfer function Δ. The inaccuracy in plant parameters is considered as a noise incoming over input of M and depending on output q. If Δ is the block-diagonal matrix then such uncertainty is called the structured one. The matrix Δ describes the parametric uncertainty in the case when blocks are scalar; in the case of complete real blocks it describes the matrix uncertainty and in the case of complete complex blocks it describes the frequency uncertainty [11].

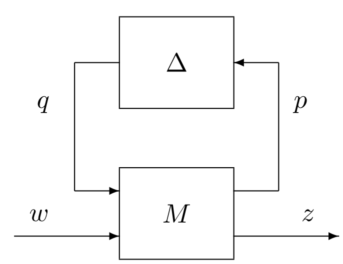

Fig. 1. Closed loop system with nominal plant M and uncertainty Δ.

The problem is to find the range for changing of the parameter uncertainty Δ for given nominal plant M, which is measured by the anisotropic norm, in order to guarantee internal stability of plant with uncertainty. It means the robust stability of the plant M with respect to the uncertainty Δ.

Further we give some essential terms of anisotropic analysis, which are the basic ones for this paper. The details can be found in [8].

III. BACKGROUND

A. Definition and properties of mean anisotropy

The anisotropy $\mathbf{A}(W)$ of an m-dimensional random vector W with probability density function (pdf) f is defined [4]

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as

$$\mathbf{A}(W) = \min_{\lambda>0} \mathbf{D}(f||p_{m,\lambda}), \tag{2}$$

where

$$\mathbf{D}(f||p_{m,\lambda}) = \mathbf{E}_f\left[\ln\left(\frac{f}{p_{m,\lambda}}\right)\right] \tag{3}$$

is the Kullback-Leibler divergence of f with respect to the Gaussian pdf

$$p_{m,\lambda}(x) = (2\pi\lambda)^{-m/2}\exp\left\{-\frac{x^{\mathrm{T}}x}{2\lambda}\right\},$$

and $\mathbf{E}_f[\,\cdot\,]$ denotes the mathematical expectation in sense of f. Thus, one can interpret the anisotropy $\mathbf{A}(W)$ as the measure of closeness of the certain vector W to the set of Gaussian vectors with pdf $p_{m,\lambda}$, parameterized by λ.

If the vector W is also the Gaussian and its pdf has the form

$$f(x) = ((2\pi)^m|S|)^{-1/2}\exp\left\{-\frac{1}{2}x^{\mathrm{T}}S^{-1}x\right\},$$

where $|\cdot|$ denotes the determinant of matrix and S is positive define matrix, then

$$\mathbf{D}(f||p_{m,\lambda}) = -h(W) - \frac{m}{2}\ln(2\pi\lambda) + \frac{\mathrm{tr}S}{2\lambda}, \tag{4}$$

where $h(W)$ stands for the differential entropy of W defined as

$$h(W) = -\mathbf{E}_f[\ln f],$$

$\mathrm{tr}(\,\cdot\,)$ means the trace of a matrix. From (2) and (4) we have got the following formula [8]

$$\mathbf{A}(W) = -\frac{1}{2}\ln\frac{|S|}{\left(\frac{1}{m}\mathrm{tr}S\right)^m}.$$

One of the methods to form the colored sequence $\{w_k\}$ is to represent this sequence as an output of the following discrete time-invariant system called the shaping filter:

$$\left\{\begin{array}{l} x_{k+1} = Ax_k + Bv_k, \\ w_k = Cx_k + Dv_k, \end{array}\right. \qquad x_0 = 0, \tag{5}$$

where x_k denotes the filter internal state; the input $\{v_k\}$ is the Gaussian white noise. The mean anisotropy $\overline{\mathbf{A}}(W)$ of the stationary ergodic sequence $W = \{w_k\}$ is defined as

$$\overline{\mathbf{A}}(W) = \lim_{N\to\infty}\frac{\mathbf{A}(W_{0:N-1})}{N}, \tag{6}$$

where

$$W_{0:N-1} = \left[\begin{array}{c} w_0 \\ \dots \\ w_{N-1} \end{array}\right].$$

is an extended vector.

Let Σ be the covariance matrix of the vector $w_k|_{k\to\infty}$, so $\Sigma = \mathrm{cov}(w_k|_{k\to\infty})$. The matrix Σ is connected with the solution $P > 0$ of the Lyapunov equation

$$P = APA^T + BB^T. \tag{7}$$

by following expression

$$\Sigma = CPC^{\mathrm{T}} + DD^{\mathrm{T}}.$$

Let Ψ be the covariance matrix of the prediction error vector $\tilde{w}_k|_{k\to\infty} > 0$, $\widetilde{w}_k = w_k - \mathbf{E}[w_k|\{w_j\}_{j<k}]$ as $k\to\infty$.

Define

$$\Xi = \Psi - \Sigma.$$

It is shown in [8] that the expression for the mean anisotropy (6) by virtue of (5) can be written as

$$\overline{\mathbf{A}}(W) = -\frac{1}{2}\ln\frac{|\Sigma+\Xi|}{\left(\frac{1}{m}\mathrm{tr}\Sigma\right)^m}, \tag{8}$$

$$\Xi = CRC^{\mathrm{T}},$$

where R is the solution of the Riccati equation

$$R = ARA^T + \Lambda(\Sigma+\Xi)^{-1}\Lambda^T, \tag{9}$$

with the notation

$$\Lambda = BD^{\mathrm{T}} + A(P+R)C^{\mathrm{T}}.$$

These formulas allow to compute the mean anisotropy of the random sequence W generated by the shaping filter (5) in terms of the second-order moments of W on the base of solutions to the respective algebraic Riccati and Lyapunov equations. One can read about the anisotropy-based theory in detail in [9].

Let us consider the Gaussian random vector W with the nonzero mean value μ and covariance matrix S, i.e. with the pdf

$$f(W) = ((2\pi)^m|S|)^{-1/2}\exp\left\{-\frac{1}{2}(W-\mu)^{\mathrm{T}}S^{-1}(W-\mu)\right\}.$$

According to (3), its relative entropy takes the form

$$\mathbf{D}(f||p_{m,\lambda}) = -h(W) - \frac{m}{2}\ln(2\pi\lambda) + \frac{\mathrm{tr}S + \|\mu\|_2^2}{2\lambda},$$

where the expectation μ appears in contrast to (4), and therefore the anisotropy of the vector W is written as

$$\mathbf{A}(W) = -\frac{1}{2}\ln\frac{|S|}{\left(\frac{1}{m}\left(\mathrm{tr}S + \|\mu\|_2^2\right)\right)^m}. \tag{10}$$

To generate a sequence $\{w_k\}$ of Gaussian random vectors with the nonzero mean value μ, it is required to rewrite (5) in the form

$$\left\{\begin{array}{l} x_{k+1} = Ax_k + Bv_k, \\ w'_k = Cx_k + Dv_k + \mu, \end{array}\right. \qquad x_0 = 0, \tag{11}$$

where as before x_k denotes the filter internal state, and the input sequence $\{v_k\}$ is the Gaussian white noise.

The application of the mean anisotropy definition (6) to the sequence $W' = \{w'_k\}$ gives

$$\overline{\mathbf{A}}(W') = \lim_{N\to\infty}\frac{\mathbf{A}(W'_{0:N-1})}{N},$$

or, equivalently,

$$\overline{\mathbf{A}}(W') = -\frac{1}{2}\ln\frac{\lim\limits_{N\to\infty}|\Sigma'_{0:N-1}|^{1/N}}{\lim\limits_{N\to\infty}\left(\frac{1}{mN}(\mathrm{tr}\Sigma'_{0:N-1} + \|\mu'_{0:N-1}\|_2^2)\right)^m}, \tag{12}$$

where

$$\mu'_{0:N-1} = \begin{bmatrix} \mu \\ \dots \\ \mu \end{bmatrix},$$

and

$$\Sigma'_{0:N-1} = \mathrm{cov}(W'_{0:N-1})$$

stands for the covariance matrix of the vector $W'_{0:N-1}$, which is equal to the covariance matrix of the vector $W_{0:N-1}$, i.e. $\Sigma'_{0:N-1} = \Sigma_{0:N-1}$. It is easy to see that

$$\Sigma_{0:N-1} = \begin{bmatrix} DD^{\mathrm{T}} & \dots & D(CA^{N-2}B)^{\mathrm{T}} \\ \vdots & \ddots & \vdots \\ * & \dots & DD^{\mathrm{T}} + \sum\limits_{k=0}^{N-2} CA^kB(CA^kB)^{\mathrm{T}} \end{bmatrix}$$

and therefore

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{tr}\Sigma_{0:N-1} = \mathrm{tr}\Sigma,$$

where $\Sigma = \mathrm{cov}(w_k)\,|_{k\to\infty}$.

Comparing (8) and (12) by virtue of the above formula, one can conclude that the ergodic property of the considered sequence W leads to the existence of the limit

$$\lim_{N\to\infty} |\Sigma_{0:N-1}|^{1/N} = |\Sigma + \Xi|$$

for the case when $\mu = 0$. Such property also extends to the sequence W' because $\Sigma'_{0:N-1} = \Sigma_{0:N-1}$. Then on the strength of

$$\lim_{N\to\infty} \frac{1}{N} \|\mu'_{0:N-1}\|_2^2 = \|\mu\|_2^2,$$

the next formula for the mean anisotropy of W' arises:

$$\overline{\mathbf{A}}(W') = -\frac{1}{2} \ln \frac{|\Sigma + \Xi|}{\left(\frac{1}{m}\left(\mathrm{tr}\Sigma + \|\mu\|_2^2\right)\right)^m}, \qquad (13)$$

Further we will use the notation $\overline{\mathbf{A}}_\mu(W)$ for the mean anisotropy of the sequence W with nonzero mean μ.

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$$\overline{\mathbf{A}}(W') = -\frac{1}{2} \ln \frac{|\Sigma + \Xi|}{\left(\frac{1}{m}\left(\mathrm{tr}\Sigma + \|\mu\|_2^2\right)\right)^m}, \qquad (14)$$

Further we will use the notation $\overline{\mathbf{A}}_\mu(W)$ for the mean anisotropy of the sequence W with nonzero mean μ.

Theorem 1. The formula (14) for the mean anisotropy $\overline{\mathbf{A}}_\mu(W)$ of the sequence W generated by the filter

$$G: \begin{cases} x_{k+1} = Ax_k + Bv_k, \\ w_k = Cx_k + Dv_k + \mu, \end{cases}$$

can be written as

$$\overline{\mathbf{A}}_\mu(W) = \overline{\mathbf{A}}_0(W) + \frac{m}{2} \ln \frac{\|G\|_2^2 + \|\mu\|_2^2}{\|G\|_2^2},$$

where $\|G\|_2$ denotes the H_2-norm of the matrix transfer function $G(z) = D + C(zI_m - A)^{-1}B$.

B. Definition and properties of anisotropic norm

Consider an linear discrete-time system written in a state-space representation

$$M \sim \begin{cases} x_{k+1} = Ax_k + Bw_k, \\ z_k = Cx_k + Dw_k \end{cases} \qquad (15)$$

where $x_k \in R^{n_1}$ is the state, $w_k \in R^m$ and $z_k \in R^p$ are input and output signals, respectively. A, B, C, D are constant real matrices of appropriate dimensions. $W = \{w_k\}$ is the stationary Gaussian sequence of m-dimensional random vectors with a given mean anisotropy level $\overline{\mathbf{A}}(W) = a \geq 0$ and known non-zero mean $\mathbf{E}[W] = \mathcal{M}$, $|\mathcal{M}| < \infty$.

For a given system M with the input signal $W = \{w_k\}$ the mean-square gain is defined as [9]

$$Q(M,W) = \frac{\|z\|_{\mathcal{P}}}{\|w\|_{\mathcal{P}}} = \sqrt{\lim_{N\to\infty} \frac{\frac{1}{N}\sum\limits_{k=0}^{N-1} \mathbf{E}|z_k|^2}{\frac{1}{N}\sum\limits_{k=0}^{N-1} \mathbf{E}|w_k|^2}} \qquad (16)$$

where

$$\|y\|_{\mathcal{P}} = \sqrt{\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{E}|y_k|^2}$$

is the power norm of the signal $\{y_k\}$.

Let the sequence $\{w_k\}$ be represented in the form

$$w_k = C_g x_k + D_g(v_k + \mu) \qquad (17)$$

where x_k is the state of the system (15), and μ is a known vector. Using (17), we obtain the filter

$$G \sim \begin{cases} x_{k+1} = (A + BC_g)x_k + BD_g(v_k + \mu), \\ w_k = C_g x_k + D_g(v_k + \mu). \end{cases} \qquad (18)$$

The power norms of outputs of the systems (15) and (18) are written as

$$\begin{aligned} \|w\|_{\mathcal{P}}^2 &= \lim_{k\to\infty} \left(\mathrm{tr\ cov}(w_k) + |\mathbf{E}w_k|^2\right) \\ &= \|G\|_2^2 + |\mathcal{M}|^2, \\ \|z\|_{\mathcal{P}}^2 &= \lim_{k\to\infty} \left(\mathrm{tr\ cov}(z_k) + |\mathbf{E}z_k|^2\right) \\ &= \|MG\|_2^2 + |\mathcal{F}\mathcal{M}|^2 \end{aligned}$$

where

$$\mathcal{M} = M(1) = D + C(E - A)^{-1}B.$$

The mean-square gain (16) for the system with non-zero mean input signal is given by the following expression:

$$Q(M,W) = Q(M,G) = \sqrt{\frac{\|MG\|_2^2 + |\mathcal{F}\mathcal{M}|^2}{\|G\|_2^2 + |\mathcal{M}|^2}}. \qquad (19)$$

Finally, the anisotropic norm of the system is defined as [8]

$$\|\!|M|\!\|_a = \sup_{G:\overline{\mathbf{A}}(G)\leq a} Q(M,G). \tag{20}$$

Let us notice one important property of anisotropic norm:

Lemma 1. For two filters $F \in H^{p\times m}$ and $G \in H^{m\times m}$ the following inequality holds true

$$\|\!|FG|\!\|_a \leq \|\!|F|\!\|_b \|\!|G|\!\|_a, \tag{21}$$

where

$$b = \overline{\mathbf{A}}(G) + a + \frac{m}{2}\ln\left(\frac{m\|\!|G|\!\|_a^2}{\|G\|_2^2 + |\mathcal{M}|^2}\right), \tag{22}$$

where $\mathcal{M}$ is the expectation of the signal on the input of G.

The proof of the Lemma 1 can be done according to the proof of result in [8], where the Lemma above was formulated without the expectation $\mathcal{M}$.

IV. ROBUST STABILITY SUFFICIENT CONDITIONS IN ANISOTROPY-BASED THEORY

In this section, we derive the main the result stating robust stability sufficient conditions of a system with parametric uncertainty.

The uncertainty Δ represented at Fig. 1 is called feasible for M, if $\Delta \in RH_\infty^{m\times m}$ and $\mathcal{F}_u(M,\Delta)$ are internally stable. Here $\mathcal{F}_u(M,\Delta)$ is the upper linear fractional transformation of M and Δ.

Theorem 2. Consider the system with transfer function $\mathcal{F}_u(M,\Delta)$ represented on Fig. 1, where Δ and M are causal linear systems and input-output relationships are given by (1). If

- The uncertainty $\Delta \in RH_\infty^{m\times m}$ is contained in the class
$$D_a(\varepsilon,\mu) \equiv \left\{\Delta : \|\!|\Delta|\!\|_a \leq \varepsilon, \operatorname*{ess\,inf}_\omega \underline{\sigma}(\hat{\Delta}(\omega)) \geq \mu\right\},$$
for any real $\varepsilon > 0$ and $\mu > 0$, $\underline{\sigma}(\Delta) \equiv \sqrt{\lambda_{min}(\Delta^*\Delta)}$ is minimal singular value of Δ.
- The anisotropy level a is defined by formula
$$a = -\frac{1}{2}\ln\det\frac{m(\Sigma+\Xi)}{\operatorname{Tr}\Sigma + |\mathcal{M}|^2} - m\ln\frac{\varepsilon}{\mu}, \tag{23}$$
where $\mathcal{M}$ is the mean of $\{w_k\}$, $\Sigma = (I_m - qM_{11}^*M_{11})^{-1}$, and parameter $q \in [0, \|M_{11}\|_\infty^{-2})$ is the solution of the inequality
$$\operatorname{Tr}\left[\left(I_m - \varepsilon^2 M_{11}^*M_{11}\right)\left(I_m - qM_{11}^*M_{11}\right)^{-1}\right] \leq 0. \tag{24}$$
where $\Sigma = CPC^T + DD^T$ and $\Xi = CRC^T$, P and R are the solutions of the Lyapunov (7) and Riccati (9) equations correspondingly,
- M is stable and
$$\|\!|M_{11}|\!\|_c < \varepsilon^{-1}, \tag{25}$$
where $c = a + m\ln\frac{\varepsilon}{\mu}$,

then for all $\Delta \in D_a(\varepsilon,\mu)$ the system $\mathcal{F}_u(M,\Delta)$ is internally stable.

Proof: Let us denote

$$\hat{\Delta}(\omega) \equiv \lim_{r\to 1-0} \Delta\left(re^{i\omega}\right), \ \omega \in [-\pi;\pi]. \tag{26}$$

The equations of the close-loop system can be written as

$$\begin{aligned} p(s) &= M_{11}q(s) + M_{12}w(s), \\ q(s) &= \Delta p(s), \end{aligned} \tag{27}$$

or equivalently

$$(I - M_{11}\Delta)p = M_{12}w. \tag{28}$$

The matrix $(I - M_{11}\Delta)$ is invertible in conditions of theorem 2. Contrary $\det(I - M_{11}\Delta) = 0$, and hence $M_{11}\Delta$ has the eigenvalue equal to one: $\lambda_i(M_{11}\Delta) = 1$. In according with definition

$$\begin{aligned} \|M_{11}\Delta\|_\infty &= \sup_{|z|<1}\sigma(M_{11}\Delta) = \\ \sup_{|z|<1} &\sqrt{\lambda((M_{11}\Delta)^*(M_{11}\Delta))} \geq 1. \end{aligned} \tag{29}$$

It is the well-known fact [8] that the next inequality holds true

$$\|M_{11}\Delta\|_\infty \geq \|\!|M_{11}\Delta|\!\|_a \tag{30}$$

for any non-negative a. It means that there is the positive a, the following inequality is right

$$\|M_{11}\Delta\|_\infty \geq \|\!|M_{11}\Delta|\!\|_a \geq 1. \tag{31}$$

Since M and Δ belong appropriative Hardy spaces, we can use Lemma 1 and inequality (21) can be written as

$$1 \leq \|\!|M_{11}\Delta|\!\|_a \leq \|\!|M_{11}|\!\|_b \|\!|\Delta|\!\|_a, \tag{32}$$

where

$$b = a + \overline{\mathbf{A}}(\Delta) + \frac{m}{2}\ln\frac{m\|\!|\Delta|\!\|_a^2}{\|\Delta\|_2^2 + |\mathcal{M}_q|^2},$$

$\mathcal{M}_q$ is the expectation of $\{q_k\}$. One can get the estimation of b as

$$\begin{aligned} b &= a + \overline{\mathbf{A}}(\Delta) + \frac{m}{2}\ln\frac{m\|\!|\Delta|\!\|_a^2}{\|\Delta\|_2^2 + |\mathcal{M}_q|^2} \\ &= a - \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det\frac{m\hat{\Delta}(\omega)\hat{\Delta}^*(\omega)}{\|\Delta\|_2^2 + |\mathcal{M}_q|^2}d\omega \\ &\quad + \frac{m}{2}\ln\frac{m\|\!|\Delta|\!\|_a^2}{\|\Delta\|_2^2 + |\mathcal{M}_q|^2} \\ &= a - \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det(\hat{\Delta}(\omega)\hat{\Delta}^*(\omega))d\omega \\ &\quad + \frac{m}{2}\ln\frac{\|\Delta\|_2^2 + |\mathcal{M}_q|^2}{m} + \frac{m}{2}\ln\frac{m\|\!|\Delta|\!\|_a^2}{\|\Delta\|_2^2 + |\mathcal{M}_q|^2} \\ &= a - \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det(\hat{\Delta}(\omega)\hat{\Delta}^*(\omega))d\omega + \frac{m}{2}\ln\|\!|\Delta|\!\|_a^2. \end{aligned}$$

The integral in the last line of the previous formula is the invariant of the unitary transformation [9]. It means if we take a matrix U containing the eigenvectors of Δ, the value of this integral would not be change. The matrix $U^T\Delta U = \widetilde{\Delta}$ is the diagonal one with the eigenvalues of Δ standing on the main diagonal of the matrix. Now we can get the estimation

$$\det(\hat{\widetilde{\Delta}}(\omega)\hat{\widetilde{\Delta}}^*(\omega)) = \prod_{k=1}^{m} \lambda_k(\Delta^*\Delta) \geq (\lambda_{\min}(\Delta^*\Delta))^m.$$

We can continue the estimation of b with the help of the last inequality as

$$\begin{aligned} b &= a - \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det(\hat{\widetilde{\Delta}}(\omega)\hat{\widetilde{\Delta}}^*(\omega))d\omega + \frac{m}{2}\ln |\!|\!|\widetilde{\Delta}|\!|\!|_a^2 \\ &\leq a + \frac{m}{2}\ln\frac{|\!|\!|\Delta|\!|\!|_a^2}{\operatorname*{ess\,inf}_{-\pi\leq\omega\leq\pi} \lambda_{\min}(\Delta^*\Delta)} \\ &= a + \frac{m}{2}\ln\frac{|\!|\!|\Delta|\!|\!|_a^2}{\operatorname*{ess\,inf}_{-\pi\leq\omega\leq\pi} \underline{\sigma}^2(\Delta)} \leq a + m\ln\frac{\varepsilon}{\mu} = c. \end{aligned}$$

where the last inequality holds true because of Δ belongs to $D_a(\varepsilon,\mu)$. Since $|\!|\!|M_{11}|\!|\!|_c |\!|\!|\Delta|\!|\!|_a \geq 1$, $\Delta \in RH_\infty^{m\times m}$, and $|\!|\!|\Delta|\!|\!|_a < \varepsilon$ then

$$|\!|\!|M_{11}|\!|\!|_c \geq \frac{1}{\varepsilon}, \tag{33}$$

but it is in conflict with (25) and, consequently, $(I - M_{11}\Delta)$ is invertible. Let us show the existence of a satisfying (32). One can get

$$\begin{aligned} |\!|\!|M_{11}|\!|\!|_c &= \sup_{G\in\mathbb{G}_a} \frac{\|M_{11}G\|_2^2 + |\mathcal{F}\mathcal{M}|^2}{\|G\|_2^2 + |\mathcal{M}|^2} \\ &\leq \sup_{G\in\mathbb{G}_a} \frac{\|M_{11}G\|_2^2}{\|G\|_2^2} + \frac{|\mathcal{F}\mathcal{M}|^2}{|\mathcal{M}|^2} \end{aligned}$$

and, according to (33), we have

$$\sup_{G\in\mathbb{G}_a} \frac{\|M_{11}G\|_2^2}{\|G\|_2^2} \geq \frac{1}{\varepsilon^2} - \frac{|\mathcal{F}\mathcal{M}|^2}{|\mathcal{M}|^2} = \frac{1}{\delta^2}.$$

The upper bound of the last expression will be on

$$\Sigma = GG^* = (I - qM_{11}^*M_{11})^{-1},$$

and q can be found as the solution of the inequality

$$\mathrm{Tr}\left[(I - \delta^2 M_{11}^*M_{11})(I - qM_{11}^*M_{11})^{-1}\right] \leq 0.$$

From (28) we can get that for any $\Delta \in D_a(\varepsilon) = \{\Delta \in RH_\infty^{m\times m} : |\!|\!|\Delta|\!|\!|_a < \varepsilon\}$ there exist a unique solution of the system (27):

$$\begin{aligned} p &= (I - M_{11}\Delta)^{-1}M_{12}w, \\ q &= \Delta p, \\ z &= M_{21}\Delta p + M_{22}w. \end{aligned} \tag{34}$$

We can obtain that the system $\mathcal{F}_u(M,\Delta)$ is internally stable.

■

Remark 1. In the case when $\Delta \equiv 0$ the condition (25) is transformed to another one $|\!|\!|M_{11}|\!|\!|_a < \infty$ for any nonnegative a and additive condition $\ln\frac{\varepsilon}{\mu} = \ln 1 = 0$. It is always true due to stability of M_{11}.

Remark 2. If $\operatorname*{ess\,inf}_{-\pi\leq\omega\leq\pi} \underline{\sigma}(\hat{\Delta}(\omega)) = 0$, $\Delta \not\equiv 0$ then the condition $|\!|\!|M_{11}|\!|\!|_b |\!|\!|\Delta|\!|\!|_a < 1$ is converted to $\|M_{11}\|_\infty \|\Delta\|_\infty < 1$. This result is well–known in the H_∞ theory as the Small Gain theorem.

Remark 3. If the uncertainty satisfies

$$\|\Delta\|_2 < \sqrt{m}\,\|\Delta\|_\infty, \tag{35}$$

then the inequality $|\!|\!|\Delta|\!|\!|_a \leq \|\Delta\|_\infty$ holds true for any $a \in R^+$. It is well-known that [11]

$$\frac{1}{\operatorname*{ess\,inf}_{-\pi\leq\omega\leq\pi} \underline{\sigma}(\hat{\Delta}(\omega)} = \operatorname*{ess\,sup}_{-\pi\leq\omega\leq\pi} \overline{\sigma}(\hat{\Delta}(j\omega)^{-1}).$$

Then the condition $b = a + m\ln(\varepsilon/\mu)$ can be rewritten as follows:

$$\begin{aligned} b &= a + m\ln\|\Delta\|_\infty\|\Delta^{-1}\|_\infty \\ &= a + m\ln \operatorname*{ess\,sup}_{-\pi\leq\omega\leq\pi} \mathrm{cond}\,(\Delta(j\omega)), \end{aligned}$$

where $\mathrm{cond}\,(\Delta)$ is the singular condition number of Δ. The existence of Δ^{-1} follows from stability of Δ. If it is not true for $c \equiv \infty$.

To determine stability of $\mathcal{F}_u(M,\Delta)$ it is sufficient to know the bounds of the singular spectrum of Δ. The operator Δ can be unknown.

Remark 4. The condition $c = a$ is fulfilled iff the upper and lower bounds of the singular spectrum of the linear operator Δ coincide. It is so if the uncertainty is

$$\Delta = \begin{pmatrix} \lambda(s) & 0 & \dots & 0 \\ 0 & \lambda(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda(s) \end{pmatrix}, \quad s \in \mathbb{C}. \tag{36}$$

In this case the condition (25) has symmetric shape

$$|\!|\!|M|\!|\!|_a |\!|\!|\Delta|\!|\!|_a < 1. \tag{37}$$

Theorem 2 gives the sufficient conditions for internal stability of the systems which have anisotropy norm-bounded uncertainty and degree of separability from zero for this uncertainty is known. This theorem allows to reduce the conservative condition $\|M_{11}\|_\infty < 1/\varepsilon$ of the Small Gain theorem by changing it to the condition (25). In this case anisotropy level c shows how the conditions of the Small Gain Theorem can be decreased without robust stability loss. It is the well-known fact that the H_2-norm is not induced and it is not possible to obtain conditions like (25), which guarantee the robust stability. But from $\|\Delta\|_2 = \sqrt{m}\lim_{a\to 0} |\!|\!|\Delta|\!|\!|_a$ we see that there are plants with the uncertainty (36) and condition $\|M_{11}\|_2\|\Delta\|_2 < 1$. This condition means internal stability of $\mathcal{F}_u(M,\Delta)$.

Now we show how to find the finite value of the mean anisotropy a such that the internal stability holds under the condition (25). Consider the state space realization of the nominal plant

$$M \sim \left[\begin{array}{c|cc} A & B & B_2 \\ \hline C & D & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

We have $M_{11} \sim [A, B, C, D]$. The formula for calculation of the anisotropic norm in the state space is given by [8]

$$|||M_{11}|||_c = \left(\frac{\mathrm{Tr}\,\Upsilon - m + q|\mathcal{F}\mathcal{M}|^2}{q\mathrm{Tr}\,\Upsilon + q|\mathcal{M}|^2} \right)^{1/2}, \qquad (38)$$

where the matrices L and Υ are derived from the solution R of the algebraic Riccati equation

$$R = A^T RA + qC^T C + L^T \Pi^{-1} L\,, \qquad (39)$$

$$\Pi \equiv \left(I_m - qD^T D - B^T RB \right)^{-1}, \qquad (40)$$

$$L \equiv \Pi \left(B^T RA + qD^T C \right). \qquad (41)$$

And the matrix Υ is defined as

$$\Upsilon = L\tilde{P}L^T + \Pi\,, \qquad (42)$$

where $\tilde{P}$ is the controllability gramian, which is the solution of Lyapunov equation

$$\tilde{P} = (A + BL)\,\tilde{P}\,(A + BL)^T + B\Pi B^T. \qquad (43)$$

The mean anisotropy level is calculated by formula

$$c = a + m \ln \frac{\varepsilon}{\mu} =$$

$$-\frac{1}{2} \ln\det \left(\frac{m\,\Pi}{\mathrm{Tr}\,\Upsilon + |\mathcal{M}|^2} \right). \qquad (44)$$

Taking into account (38) the condition (32) leads to

$$\frac{1}{\varepsilon^2} - \frac{\mathrm{Tr}\,\Upsilon - m + q|\mathcal{F}\mathcal{M}|^2}{q\mathrm{Tr}\,\Upsilon + q|\mathcal{M}|^2} \le 0. \qquad (45)$$

Calculation of the acceptable anisotropy level leads to optimal problem: find $\max q$ under the constraints $q \in [0, \|F\|_\infty^{-2})$ and (32). Since the performance function is linear; the desired value of parameter q is on the boundary. Therefore q is obtained as the solution of

$$\frac{1}{\varepsilon^2} - \frac{\mathrm{Tr}\,\Upsilon - m + q|\mathcal{F}\mathcal{M}|^2}{q\mathrm{Tr}\,\Upsilon + q|\mathcal{M}|^2} = 0 \qquad (46)$$

over the semiopen interval $[0, \|M_{11}\|_\infty^{-2})$.

V. ROBUST STABILITY CRITERION

For any real $\varepsilon > 0$ define the uncertainty class

$$D_a(\varepsilon) \equiv \left\{ \Delta \in RH_\infty^{m\times m} \quad : \quad |||\Delta|||_a < \varepsilon \right\}.$$

For a given system with additive uncertainty [1] the following theorem allows to calculate possible uncertainty radius. The theorem is an analogue of Glover-McFarlane theorem [12].

Theorem 3. Consider the system S represented at Fig. 2 such that it consists of the additive uncertainty, generalized plant, controller, where Δ, G, K are linear causal systems. Let the maximal condition singular number of uncertainty $\psi = \operatorname*{ess\,sup}_{-\pi\le\omega\le\pi} \mathrm{cond}\,(\Delta)$ is known. The controller K stabilizes $\mathcal{F}_u(G, \Delta)$ if:

1) it stabilizes the nominal, e.g. system with transfer function $\mathcal{F}_l(G, K)$ is stable,

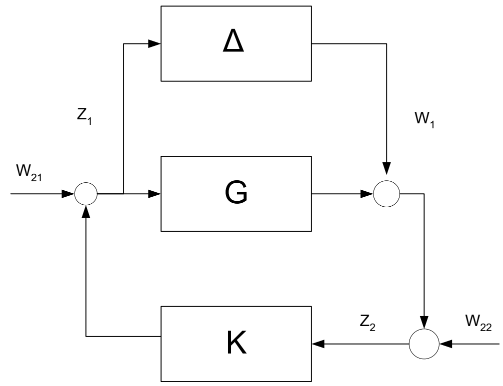

Fig. 2. Closed-loop system with nominal plant M, uncertainty Δ and controller K.

2) $\Delta \in D_a\left(|||K(I - GK)^{-1}|||_c^{-1} \right)$, where $c = a + m \ln \psi$.

Proof: The system S can be transformed to the form as at Fig. 1 by transformation $w = \begin{pmatrix} w_{21} & w_{22} \end{pmatrix}^T$. Direct computation gives the expression for $M_{11} = K(I - GK)^{-1}$. Taking into account that K stabilizes the nominal plant, we see that the system G is stable. Application of Theorem 2 completes the proof. ■

Remark 5. Using Lower Linear Fractional Transformation

$$\mathcal{F}_l(M, K) = M_{11} + M_{12}K(I - M_{22}K)^{-1}M_{21},$$

the condition 2) from Theorem 3 can be replaced by $\Delta \in D_a\left(|||\,\mathcal{F}_l(M, K)|||_c^{-1} \right)$.

In general case the application of Theorem 3 is rather difficult by the reason that the information about ψ is needed. This difficulty disappears in the case of parametric uncertainty (36) because of in this case $\psi = 0$.

If $\psi \neq 0$ then condition (2) of Theorem 3 should be considered as the algebraic inclusion

$$\Delta \in D_a\left(|||\,\mathcal{F}_l(G, K)|||_{a+m\psi}^{-1} \right),$$

where D_a is the map $D_a : RH_\infty^{m\times m} \to RH_\infty^{m\times m}$.

The robust criteria based on Theorem 2 for other uncertainty types can be obtained in the same way.

VI. CONCLUSION

The sufficient conditions of robust stability for the linear system with non-zero mean of input sequence and bounded anisotropic norm uncertainty are found. These conditions are relaxation of the Small Gain Theorem conditions. The way of calculation of additive uncertainty range that keep stability of closed loop system with given controller is shown.

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