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The (weak) admissibility of the H^∞-calculus for semigroup generators

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Abstract— **The goal of the following is to use (infinite-dimensional) linear systems theory to approach the H^∞-functional calculus. This extends recent work by the authors using the notion of admissible observation operators to define $f(A)$ where f is bounded, analytic in the left half-plane and A generates an exponentially stable strongly continuous semigroup. Among other things it is shown that $f(A)$ is always weakly admissible and indeed coincides with classical approaches to H^∞-calculus.**

I. INTRODUCTION

Functional calculus, i.e. *making sense* of $f(A)$, where f is some (scalar-valued) function and A a, not necessarily bounded, operator, has been investigated and used extensively in the last decades. The particular case of bounded, analytic functions on the left half-plane and generators of strongly continuous semigroups will be considered here. Applications range from numerical analysis to questions about maximal regularity of PDEs.

The H^∞-calculus had its beginnings in the mid-eighties with the works of McIntosh, [1], and was investigated intensively in the following years. The framework is the one of *sectorial* operators. The book of Haase, [2], serves as a great collection of the result about this calculus.

However, as we are considering generators of (exponentially stable) semigroups the more suitable calculus to our setting is the well-known Hille-Phillips calculus or, more generally, the recently introduced calculus for *half-plane operators*, see [3].

A functional calculus is typically defined via some standard identity for scalar-valued functions. The approach we take is motivated by systems theory and an underlying one-to-one correspondence between *admissible output maps* and *admissible observation* operators for linear systems built up by the semigroup generator. As a direct consequence, we derive that the calculus is always *weakly admissible* and even *admissible* in the case of a Hilbert space.

Finally, we give sufficient conditions for a bounded H^∞-calculus by introducing the notion of *exact observability by direction* for general Banach spaces. In [4] the construction was done for Hilbert spaces, followed by the reflexive Banach space case in [5]. In the present work the assumption of reflexivity is not needed.

II. SETTING

In the following let X,Y denote Banach spaces, X' the dual space and $\langle\cdot,\cdot\rangle$ the duality brackets. By $\mathscr{B}(X,Y)$ we denote the bounded linear operators from X to Y, with $\mathscr{B}(X)=\mathscr{B}(X,X)$. For $M\geq 1$ and $\omega>0$ let $\mathscr{G}_{exp}(M,\omega)$ denote the set of exponentially stable semigroup generators A such that $\|e^{tA}\|\leq Me^{-t\omega}$, where e^{tA} denotes the semigroup. By X_1 we refer to the domain $D(A)$ equipped with the graph norm. By H^∞ we denote the Banach algebra of bounded analytic functions on the left half-plane $\mathbb{C}^-$ equipped with pointwise multiplication and the supremum norm $\|\cdot\|_\infty$. $L^2(I,X)$ is the classical Lebesgue space of X-valued functions from the interval I, with $L^2(I)=L^2(I,\mathbb{C})$.

In systems theory it is well-known that the Toeplitz operator $M_f: L^2(0,\infty)\to L^2(0,\infty)$ with symbol $f\in H^\infty$ maps exponentials to exponentials,

$$M_f(e^{at})=f(a)e^{at} \tag{1}$$

for fixed $a<0$. Obviously, $f\mapsto f(a)$ is a homomorphism from H^∞, the Banach algebra of bounded analytic functions on $\mathbb{C}^-$, to $\mathbb{C}$. Our idea is to replace the exponential by the strongly continuous semigroup e^{At} on the Banach space X. In fact, we show that the formally defined function

$$y(t)=M_f(e^{\cdot A}x_0)(t)$$

can be seen as the *output* of the linear system

$$\begin{aligned}\dot{x}(t)&=Ax(t), \qquad x(0)=x_0,\\ y(t)&=C_f x(t),\end{aligned}$$

for some (unbounded) operator C_f. Thus, formally $y(t)=C_f e^{tA}x_0$. This means that C_f takes the role of $f(a)$ in (1). Hence, the task is to find C_f given the *output mapping* $x_0\mapsto y(t)$. Incorporating the notion of *admissibility*, this can be done uniquely, see [6]. This construction yields a (in general) unbounded functional calculus which is shown to be an extension of the well-known *Hille-Phillips*-calculus for Laplace transforms of measures of bounded variation, see e.g. [2, Chapter 3.3]. This in turn gives that the derived definition of $f(A)$ coincides with the one from the H^∞-calculus for *half-plane operators*. The latter ones are, rather than sectorial operators, the suitable framework for generators of exponentially stable semigroups and bounded, analytic functions f on the left half-plane, see [3].

A. Admissibility

Definition 1: With respect to the semigroup e^{tA}, an operator $C\in\mathscr{B}(X_1,Y)$ is called

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- (infinite-time) *admissible*, if

  $$\Phi_C : X \to L^2((0,\infty),Y) : \ x \mapsto (t \mapsto Ce^{tA}x), \text{ is bounded,}$$

  in which case we define $\|C\|_{adm} := \|\Phi_C\|$,
- *weakly admissible*, if

  $$\forall y \in Y' : \quad y \circ \Phi_C : X \to \mathbb{C} \text{ is admissible, and}$$
  $$\|C\|_{w-adm} := \sup_{\|y\|=1} \|y \circ C\|_{adm} < \infty.$$

The map Φ_C, which is a-priori only defined on $D(A)$, is identified with its bounded extension on X.

III. MAIN RESULTS

A. (Weak) admissible H^∞-caclulus

Theorem 2: Let $A \in \mathscr{G}_{exp}(M,\omega)$, $f \in H^\infty$. Then, $f(A) \in \mathscr{B}(X_1,X)$ and is weakly admissible with

$$\|f(A)\|_{w-adm} \le \frac{M}{\sqrt{2\omega}} \cdot \|f\|_\infty. \tag{2}$$

Theorem 3 ([4]): If X is a Hilbert space, $A \in \mathscr{G}_{exp}(M,\omega)$ and $f \in H^\infty$, then $f(A)$ is admissible with

$$\|f(A)\|_{adm} \le \frac{M}{\sqrt{2\omega}} \cdot \|f\|_\infty. \tag{3}$$

The following result reflects the multiplicativity of the calculus, but is in fact more general.

Theorem 4: Let $C \in \mathscr{B}(X_1,Y)$ where Y is a Banach space, $f \in H^\infty$ and let A generate an exponentially stable semigroup on X.

1) If C is weakly admissible, then $Cf(A)$ can be (uniquely) extended to a weakly admissible operator in $\mathscr{B}(X_1,Y)$ with

$$\|Cf(A)\|_{w-adm} \le \|C\|_{w-adm}\|f\|_\infty. \tag{4}$$

2) If X is a Hilbert space and C admissible, then $Cf(A)$ can be (uniquely) extended to an admissible operator in $\mathscr{B}(X_1,Y)$,

$$\|Cf(A)\|_{adm} \le \|C\|_{adm}\|f\|_\infty. \tag{5}$$

B. Sufficient conditions for a bounded calculus

The question whether an operator A has a bounded H^∞-calculus, i.e. $f(A) \in \mathscr{B}(X)$ for all $f \in H^\infty$, has been of interest since the early days of the study of the calculus, [1]. In the framework of sectorial operators so-called *square function estimates* have turned out to be useful in characterizing it, see [1], [7]. It is known that those can be directly related to the admissibility of the square root of $-A$ in the Hilbert space, [8]. From our approach the following notion for general Banach spaces emerges naturally.

Definition 5: For $C \in \mathscr{B}(X_1,Y)$ the pair (C,A) is called *exactly observable by direction* if there exist $k,K > 0$ such that for every $x \in D(A)$ there is a $y_x \in Y'$ with $\|y_x\|_{Y'} = 1$ such that

$$k\|x\| \le \|\langle y_x, Ce^{\cdot A}x\rangle_{Y',Y}\|_{L^2(0,\infty)} \le K\|x\|. \tag{6}$$

Theorem 6: Let $C \in \mathscr{B}(X_1,Y)$ be such that (C,A) is exactly observable by direction. Then, $f \mapsto f(A)$ is a bounded H^∞-calculus with

$$\|f(A)\| \le \frac{K}{k}\|f\|_\infty, \tag{7}$$

where k,K are the constants from (6).
On Hilbert spaces the above mentioned square function estimates can be translated into the notion of *exact observability*. However, we point out that square function estimates in our situation of H^∞-functions imply that the semigroup is even analytic, whereas exact observability does not, hence is more general.

Definition 7: For an operator $C \in \mathscr{B}(X_1,Y)$ the pair (C,A) is called *exactly observable* if there exist $k,K > 0$ satisfying

$$k\|x\| \le \|Ce^{\cdot A}x\|_{L^2((0,\infty),Y)} \le K\|x\| \tag{8}$$

for all $x \in D(A)$.

The following result is the Hilbert space counterpart of Theorem 6, see [4].

Theorem 8 ([4]): If there exists an operator $C \in \mathscr{B}(X_1,Y)$ such that (C,A) is exactly observable, then the H^∞-calculus for A is bounded.
Condition (8) implies that the semigroup is similar to a contraction semigroup, which in turn implies that the corresponding calculus is bounded by von Neumann's inequality, which already proves Theorem 8.
By means of an example we can show that even on Hilbert spaces the notions of exact observability by direction and exact observability may differ.

IV. OUTLOOK & DISCUSSION

The following representation of the Toeplitz operator M_f,

$$M_f h = \Big(\mathscr{F}^{-1}\big(f(i.)\cdot[\mathscr{F}h](.)\big)\Big)|_{(0,\infty)},$$

$h \in L^2(0,\infty), f \in H^\infty$, where $\mathscr{F}$ denotes the Fourier transform shows the relation to *Fourier multipliers*. Using this, one derives that $f(A)$ is admissible on general Banach spaces if f is such that the Fourier multiplier is a bounded operator on $L^2((0,\infty),X))$, i.e. f lies in the *analytic multiplier algebra* $\mathscr{A}_2(X) \subset H^\infty$. Fourier multipliers have already been used recently by Haase and Rozendaal to study the H^∞-calculus for semigroup generators in [9] and [10]. In [10] estimates of the form

$$\|f(A)T(t)\| \le \kappa(t) \cdot \|f\|_\infty, \tag{9}$$

$t > 0$, $f \in \mathscr{A}_2(X)$ are derived, where $\kappa(t) \sim |\log(t)|$ for small t. This can also be used to prove that $f(A)$ is admissible for $f \in \mathscr{A}_2(X)$.
On the other hand, current work of the authors, [11], reveals that for exponentially stable semigroups that are analytic in addition, the H^∞-calculus is admissible in any Banach space. However, non-analytic semigroup generators on Banach spaces do not have an admissible calculus in general. Showing this gives an $f(A)$ which is not admissible, but nevertheless weakly admissible. Thus, provides another example for the failure of the *Weiss conjecture*, [12], with a comparatively nicely behaving operator.

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